KNIZHNIK-ZAMOLODCHIKOV EQUATIONS IN DELIGNE CATEGORIES

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Abstract. We derive integral formulas for the solutions of the Knizhnik-Zamolodchikov equations in the setting of Deligne Categories.

Contents

1. Introduction

The Knizhnik-Zamolodchikov (KZ) connection is an important object in representation theory of affine Lie algebras and quantum groups. Namely, for an arbitrary simple Lie algebra $\mathfrak{g}$ we may consider a connection ∇_{KZ} on a base space

$$\mathbb{C}^r \setminus \bigcup_{1\le i<j\le r} \{z \in \mathbb{C}^r | z_i - z_j\} \tag{1.1}$$

given by

$$\nabla_{KZ} = d - \hbar \sum_{1\le i<j\le r} \frac{d(z_i - z_j)}{z_i - z_j}\Omega_{ij}, \tag{1.2}$$

where $\Omega_{ij} \in U(\mathfrak{g})^{\otimes r}$ is equal to

$$\Omega_{ij} = \sum_{1\le a\le \dim(\mathfrak{g})} 1^{(1)}\otimes \dots 1^{(i-1)}\otimes e_a^{(i)}\otimes 1^{(i+1)}\otimes\cdots\otimes\dots 1^{(j-1)}\otimes e^{a(j)}\otimes 1^{(j+1)}\otimes\cdots\otimes 1^{(r)} \tag{1.3}$$

and e_a, e^a are dual bases in $\mathfrak{g}$. The Knizhnik-Zamolodchikov connection admits an obvious generalization to the case of general linear groups.

One may consider an analogous vector bundle with the fiber equal to the tensor product of $\mathfrak{g}$ modules $V_1, \dots, V_r$. We may choose a root system and the Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{g}$. Then since the action of $\mathfrak{g}$ on $V_1 \otimes \cdots \otimes V_r$ commutes with Ω_{ij} and since the operators

Ω_{ij} have weight 0 with respect to $\mathfrak{g}$, it makes sense to restrict the connection to a weight space

$$V_1 \otimes \cdots \otimes V_r[\mu] \subset V_1 \otimes \cdots \otimes V_r, \quad \mu \in \mathfrak{h}^*. \tag{1.4}$$

If we slightly deform the KZ connection and look for the flat sections of ∇_{KZ} on (1.4), one write a system of compatible dynamical equations and integral solutions which satisfy both the KZ and dynamical equations [6].

The Deligne category $\underline{\mathrm{Rep}}(GL_t)$ for a parameter $t \in \mathbb{C}$ is a certain interpolation of the representation category of the classical algebraic Lie group GL_n [1]. It is possible to produce a pencil of KZ connections in the setting of Deligne categories depending on the parameter t [5]. Therefore we may look for the flat sections of the KZ equations in this case as well. A direct application of the approach in [6] fails since there are no weight spaces (1.4) in Deligne categories. Nevertheless, it is possible to find a certain $(\mathfrak{gl}_t, \mathfrak{gl}_r)$-duality which allows us to write integral formulas for the solutions to the KZ equations for all non-integer and large enough integer t.

The paper is structured as follows. Chapter 2 contains preliminaries. In Chapter 3 we produce the duality under which the KZ connection maps to the dynamical connection on a certain simple module for the dual general linear algebra. Finally, the integral formulas for solutions to the dynamical equations from Chapter 3 are presented in Chapter 4.

2. Preliminaries

2.1. **Kac-Moody Lie algebras.** Suppose we are given an integer square matrix A of size n and rank l, such that

$$a_{ii} = 2, \quad a_{ij} \le 0 \text{ if } i \ne j, \quad a_{ij} = 0 \Rightarrow a_{ji} = 0. \tag{2.1}$$

It is called a generalized Cartan matrix. Let $\mathfrak{h}$ be a vector space of dimension $2n - l$ with independent simple co-roots $\Pi^\vee = \{h_1^\vee, \dots, h_n^\vee\}$ in $\mathfrak{h}$ and let Π be a set of independent simple roots $\{\alpha_1, \dots, \alpha_n\}$ in $\mathfrak{h}^*$, such that

$$\langle h_i^\vee, \alpha_j \rangle = a_{ij}. \tag{2.2}$$

Then there exists a Lie algebra $\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, such that $\mathfrak{n}_+$ is generated by elements $e_1, \dots, e_n$ and $\mathfrak{n}_-$ is generated by elements $f_1, \dots, f_n$ with relations

$$[e_i, f_j] = \delta_{ij} h_i^\vee, \quad [h, h'] = 0, \quad [h, e_i] = \alpha_i(h) e_i, \quad [h, f_i] = -\alpha_i(h) f_i \tag{2.3}$$

for $h, h' \in \mathfrak{h}$ and

$$ad_{e_i}^{1-a_{ij}}(e_j) = 0, \quad ad_{f_i}^{1-a_{ij}}(f_j) = 0. \tag{2.4}$$

Those are called the Chevalley-Serre generators and relations respectively. The constructed Lie algebra is called the Kac-Moody Lie algebra associated to the generalized Cartan matrix A [7].

2.2. **General linear groups and algebras.** The general linear group $GL_n(\mathbb{C})$ is the group of invertible matrices of size n over $\mathbb{C}$. The standard choice of a maximal torus T_n of $GL_n(\mathbb{C})$ is the subgroup of diagonal matrices and the standard choice of a Borel subgroup B_n is the subgroup of upper-triangular matrices. This yields the following description of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ of $GL_n(\mathbb{C})$ and its root system:

$$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \quad \mathfrak{n}_- = \mathrm{span}\langle E_{ij} \rangle_{i>j}, \quad \mathfrak{n}_+ = \mathrm{span}\langle E_{ij} \rangle_{i<j}, \quad \mathfrak{h} = \mathrm{span}\langle E_{ii} \rangle, \tag{2.5}$$

$$\mathfrak{h}^* = \mathrm{span}\langle \theta_i \rangle, \quad \theta_i(E_{jj}) = \delta_{ij}, \quad R = \{\theta_i - \theta_j | i \ne j\}, \quad R^+ = \{\theta_i - \theta_j | i < j\}, \tag{2.6}$$

$$\Pi = \{\theta_i - \theta_{i+1}\}, \quad \Pi^\vee = \{E_{i,i} - E_{i+1,i+1}\}. \tag{2.7}$$

All irreducible representatios of $GL_n(\mathbb{C})$ (or, equivalently, integrable irreducible representations of $\mathfrak{gl}_n(\mathbb{C})$) are parameterized by an n-tuples of integers $(\lambda_1, \dots, \lambda_n)$ such that $\lambda_i \geq \lambda_{i+1}$. If V is the tautological representation of $GL_n(\mathbb{C})$ then any such representation can be tensored with the one-dimensional representation $\Lambda^n V$ several times, so that λ becomes a partition of length not greater than n. The resulting representation may be realized via a Schur functor $\mathbb{S}^\lambda$ applied to V.

2.3. **The Deligne category.** The Deligne category $\underline{\mathrm{Rep}}(GL, T)$ for a formal variable T and a field $\mathbb{C}$ of characteristic 0 is the Karoubi closure of the additive closure of the free rigid monoidal $\mathbb{C}[T]$-linear category generated by an object V of dimension T. For non-negative integers n, m the endomorphism algebra of an object $V^{\otimes n} \otimes V^{*\otimes m}$ is the walled Brauer algebra $Br_{n,m}(T)$ over $\mathbb{C}[T]$ [1].

For any element t of $\mathbb{C}$ we may specialize the category $\underline{\mathrm{Rep}}(GL, T)$ to $T = t$. The resulting $\mathbb{C}$-linear category $\underline{\mathrm{Rep}}(GL_t)$ is also usually called a Deligne category [3]. If t is not an integer then $\underline{\mathrm{Rep}}(GL_t)$ is abelian and semisimple [1]. For integer t it is only Karoubian [1].

Indecomposable objects $L_{[\lambda,\mu]}$ of $\underline{\mathrm{Rep}}(GL_t)$ are parameterized by bi-partitions (λ, μ) and are obtained by applying appropriate idempotents to $V^{\otimes|\lambda|} \otimes V^{*\otimes|\mu|}$. For any positive integer t the category $\underline{\mathrm{Rep}}(GL_t)$ admits a full tensor functor F to $\mathrm{Rep}(GL_t)$ which sends V to the tautological representation of GL_t and $L_{[\lambda,\mu]}$ to the simple representation in $V^{\otimes|\lambda|} \otimes V^{*\otimes|\mu|}$ with the largest (w.r.t. the standard partial order on the root lattice) highest weight if $l(\lambda) + l(\mu) \leq t$. If $l(\lambda) + l(\mu) > t$, then $F(L_{[\lambda,\mu]}) = 0$.

The group GL_t is the fundamental group of $\underline{\mathrm{Rep}}(GL_t)$ [2, 4]. The Lie algebra $\mathfrak{gl}_t$ (or $\mathfrak{gl}(V)$) of GL_t is

$$\mathfrak{gl}_t = V \otimes V^*. \tag{2.8}$$

Note that $\mathfrak{gl}_t$ is an associative algebra via the evaluation map, therefore it is also a Lie algebra [4].

3. KZ equations and dynamical differential equations

3.1. **Knizhnik-Zamolodchikov equations.** Consider the category $\underline{\mathrm{Rep}}(GL_t)$ for a complex t. For integer $m, n \geq 0$ we may consider the Casimir operators

$$\Omega_{ij} : V^{*\otimes n} \otimes V^{\otimes m} \to V^{*\otimes n} \otimes V^{\otimes m}, \quad \Omega_{ij} = \Omega_{ji} \tag{3.1}$$

which act in i, j tensor components via a flip if $i, j \leq n$ or $i, j > n$, and via $-\mathrm{coev} \circ \mathrm{ev}$ for other i, j. Here ev: $V \otimes V^* \to \mathbb{1}$ and coev: $\mathbb{1} \to V \otimes V^*$ are the evaluation and coevaluation maps.

One may write the Knizhnik-Zamolodchikov connection on $\mathbb{C}^{m+n} \setminus \{\text{diagonals}\}$ with values in $\mathrm{Hom}_{\underline{\mathrm{Rep}}(GL_t)}(V_{\lambda,\mu}, V^{*\otimes n} \otimes V^{\otimes m})$

$$\nabla_{KZ}(\hbar) = d - \hbar \sum_{i<j} \frac{dz_i - dz_j}{z_i - z_j} \Omega_{ij}, \tag{3.2}$$

where the action of Ω_{ij} on $V^{*\otimes n} \otimes V^{\otimes m}$ is extended to endomorphisms of $\mathrm{Hom}_{\underline{\mathrm{Rep}}(GL_t)}(V_{\lambda,\mu}, V^{*\otimes n} \otimes V^{\otimes m})$. We may assume $|\lambda| + n = |\mu| + m$, otherwise

$$\mathrm{Hom}_{\underline{\mathrm{Rep}}(GL_t)}(V_{\lambda,\mu}, V^{*\otimes n} \otimes V^{\otimes m}) = 0 \tag{3.3}$$

Example 3.1.1 ([5]). *In the case when both* $\lambda, \mu = 0$ *and* $m = n$ *we can describe* Ω_{ij} *explicitly: note that* $\mathrm{Hom}_{\underline{Rep}(GL_t)}(\mathbb{1}, V^{*\otimes m} \otimes V^{\otimes m}) = \mathbb{C}[S_m]$, *so for* $1 \leq i < j \leq 2m$ *and* $\sigma \in S_m$ *we have*

$$\Omega_{ij}\sigma = \begin{cases} (i,j) \circ \sigma, & i,j \leq m \\ \sigma \circ (i-m, j-m), & i,j > m \\ -t\sigma, & \sigma(j-m) = i, i \leq m < j \\ -(i, \sigma(j-m)) \circ \sigma, & \sigma(j-m) \neq i, i \leq m < j \end{cases} \tag{3.4}$$

Since the vector space of homomorphisms $\mathrm{Hom}_{\underline{\mathrm{Rep}}(GL_t)}(V_{\lambda,\mu}, V^{*\otimes n} \otimes V^{\otimes m})$ has the same dimension for all non-integer and all large enough integer $t = \dim V$, it is sufficient for us to consider the setup for $\mathfrak{gl}_t, t \in \mathbb{N}$ - for large t we have an isomorphism

$$F : \mathrm{Hom}_{\underline{\mathrm{Rep}}(GL_t)}(V_{\lambda,\mu}, V^{*\otimes n} \otimes V^{\otimes m}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{gl}_t}(V_{\lambda,\mu}, V^{*\otimes n} \otimes V^{\otimes m}). \tag{3.5}$$

Where $V_{\lambda,\mu}$ is the irreducible representation of $\mathfrak{gl}_t$ weight

$$(\lambda_1, \lambda_2, \ldots, 0, \ldots, 0, \ldots, -\mu_2, -\mu_1), \tag{3.6}$$

where the first coordinates are the coordinates of λ, the last coordinates are the coordinates of $-\mu$ and the coordinates in between are all zeros.

For large positive integer t we have

$$\mathrm{Hom}_{\mathfrak{gl}_t}(V_{\lambda,\mu}, V^{*\otimes n} \otimes V^{\otimes m}) \cong \mathrm{Hom}_{\mathfrak{gl}_t}(V_{\lambda,\mu} \otimes (\Lambda^t V)^{\otimes n}, (\Lambda^{t-1}V)^{\otimes n} \otimes V^{\otimes m}). \tag{3.7}$$

3.2. $(\mathfrak{gl}_t, \mathfrak{gl}_{m+n})$ **duality.** Let t be a positive integer and $\mathfrak{gl}_t$ the corresponding general linear Lie algebra. In this section we derive a duality between the KZ equations for $\mathfrak{gl}_t$ and dynamical differential equations for $\mathfrak{gl}_{m+n}$, via the joint action of $\mathfrak{gl}_t$ and $\mathfrak{gl}_{m+n}$ on the space $\Lambda^\bullet(V \otimes W)$, where V, W are the tautological representations for $\mathfrak{gl}_t$ and $\mathfrak{gl}_{m+n}$ respectively. The derivation is similar to [11].

The space in (3.7) can be given the structure of a weight space of a $\mathfrak{gl}(n+m)$ module. Namely, consider the space $\Lambda^\bullet(V \otimes W)$, which inherits the action of $\mathfrak{gl}(V) \oplus \mathfrak{gl}(W)$. The skew-Howe duality states that as a $\mathfrak{gl}(V) \oplus \mathfrak{gl}(W)$-module

$$\Lambda^\bullet(V \otimes W) = \bigoplus_{\delta,\ l(\delta) \leq t, l(\delta^\top) \leq m+n} V_\delta \otimes W_{\delta^\top}, \tag{3.8}$$

where $V_\delta, W_{\delta^\top}$ are the irreducible representations of $\mathfrak{gl}(V)$ and $\mathfrak{gl}(W)$ of weights δ and $\delta^\top$ respectively. The sum is over all partitions δ satisfying the written conditions. Also, given a choice of basis for W, we have an embedding $(\Lambda^{t-1}V)^{\otimes n} \otimes V^{\otimes m} \hookrightarrow \Lambda^\bullet(V \otimes W)$, whose image is the subspace of $\mathfrak{gl}(n+m)$ weight

$$\beta := (\underbrace{t-1, \ldots, t-1}_{n \text{ times}}, \underbrace{1, \ldots, 1}_{m \text{ times}}). \tag{3.9}$$

Therefore, if $\mu_1 \leq n, \lambda_1 \leq m$ (otherwise the space (3.7) is 0) we have an embedding

$$\mathrm{Hom}_{\mathfrak{gl}_t}(V_{\lambda,\mu} \otimes (\Lambda^t V)^{\otimes n}, (\Lambda^{t-1}V)^{\otimes n} \otimes V^{\otimes m}) \hookrightarrow \mathrm{Hom}_{\mathfrak{gl}_t}(V_{\lambda,\mu} \otimes (\Lambda^t V)^{\otimes n}, \Lambda^\bullet(V \otimes W)) \cong W_{\gamma^\top}, \tag{3.10}$$

with $W_{\gamma^\top}[\beta]$ being the image of the embedding. Here γ is the highest weight of the $\mathfrak{gl}_t$-module $V_{\lambda,\mu} \otimes (\Lambda^t V)^{\otimes n}$,

$$\gamma := (\underbrace{n + \lambda_1, n + \lambda_2, \ldots, n - \mu_2, n - \mu_1}_{t \text{ entries}}). \tag{3.11}$$

The upshot is that we will identify the space $\mathrm{Hom}_{\underline{\mathrm{Rep}}(GL_t)}(V_{\lambda,\mu}, V^{*\otimes n} \otimes V^{\otimes m})$ with the weight space $W_{\gamma^\top}[\beta]$ (Notice that the transpose $\gamma^\top$ can be interpolated to generic t;

Lemma 3.2.3 will extend the isomorphism to generic t).

Let us take a basis $x_{a,i}, 1 \leq a \leq t, 1 \leq i \leq m+n$ of $V \otimes W$. As a $\mathfrak{gl}_t$-module, the space $\Lambda^\bullet(V \otimes W)$ is isomorphic to

$$\Lambda^\bullet[x_{1,1}, \ldots, x_{t,1}] \otimes \cdots \otimes \Lambda^\bullet[x_{1,n+m}, \ldots, x_{t,n+m}]. \tag{3.12}$$

The $\mathfrak{gl}_t$ Casimir operators Ω_{ij} (as in (3.1)) act on this space as

$$\Omega_{ij} = \sum_a (e_a)_{(i)}(e^a)_{(j)} \tag{3.13}$$

where $\{e_a\}, \{e^a\}$ are dual bases of $\mathfrak{gl}_t$, and the outside subscripts (i) indicate action on the i-th factor of the tensor product. Meanwhile, as a $\mathfrak{gl}_{m+n}$-module, we have action by the operators κ_{ij} for $1 \leq i, j \leq m+n$, $i \neq j$ defined by

$$\kappa_{ij} := e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha \tag{3.14}$$

where α is the root $\theta_i - \theta_j$ of $\mathfrak{gl}_{m+n}$ and $e_{\pm\alpha}$ are the corresponding root vectors from $\mathfrak{g}_\alpha \subset \mathfrak{gl}_{m+n}$ normalized by $\mathrm{Tr}(e_\alpha e_{-\alpha}) = 1$.

Let E_{ij}, $1 \leq i, j \leq m+n$, be the standard basis of $\mathfrak{gl}_{m+n}$.

Lemma 3.2.1. *For any $1 \leq i < j \leq m+n$, the equality*

$$2\Omega_{ij} = -\kappa_{ij} + E_{ii} + E_{jj} \tag{3.15}$$

holds as operators on $\Lambda^\bullet(V \otimes W)$.

Proof. The action of $\tilde{\Omega}_{ij}$ on $\Lambda^\bullet(V \otimes W)$ can be written as

$$\sum_{1 \leq a,b \leq t} x_{a,i}\partial_{b,i}x_{b,j}\partial_{a,j} \tag{3.16}$$

where $x_{r,c}$ and $\partial_{r,c}$ are the operators of multiplication and differentiation by $x_{r,c}$ (with appropriate powers of -1). Similarly, the action of κ_{ij} is

$$\sum_{1 \leq a,b \leq t} x_{a,i}\partial_{a,j}x_{b,j}\partial_{b,i} + x_{b,j}\partial_{b,i}x_{a,i}\partial_{a,j}. \tag{3.17}$$

In view of the anticommutation relation $x_{a,i}\partial_{b,j} + x_{b,j}\partial_{a,i} = \delta_{a,b}\delta_{i,j}$, we have

$$\kappa_{ij} = \sum_{1 \leq a,b \leq t} x_{a,i}(-x_{b,j}\partial_{a,j} + \delta_{a,b})\partial_{b,i} + x_{b,j}(-x_{a,i}\partial_{b,i} + \delta_{a,b})\partial_{a,j} \tag{3.18}$$

$$= -2\Omega_{ij} + \sum_{1 \leq a \leq t} (x_{a,i}\partial_{a,i} + x_{a,j}\partial_{a,j}) = -2\Omega_{ij} + E_{ii} + E_{jj} \tag{3.19}$$

as desired. □

Let $M_{\alpha,\beta}$ be the subspace of $\Lambda^\bullet(V \otimes W)$ with $\mathfrak{gl}_t$-weight α and $\mathfrak{gl}_{m+n}$-weight β. As a consequence of the above lemma, we have the following theorem.

Theorem 3.2.2. *A function $f : \{(z_1, \cdots, z_{m+n}) \in \mathbb{C}^{m+n} \mid z_i \neq z_j\} \to M_{\lambda,\mu}$ is a flat section of the KZ connection*

$$\nabla_{KZ} = d - \hbar \sum_{1 \leq i < j \leq m+n} \frac{dz_i - dz_j}{z_i - z_j}\Omega_{ij} \tag{3.20}$$

if and only if the function $g = f \cdot \prod_{1 \leq i < j \leq m+n}(z_i - z_j)^{-(\beta_i + \beta_j)\hbar/2}$ is a flat section of the connection

$$\nabla_\kappa := d + \frac{\hbar}{2} \sum_{1 \leq i < j \leq m+n} \frac{dz_i - dz_j}{z_i - z_j}\kappa_{ij}. \tag{3.21}$$

Additionally, by using the gauge transformation $\nabla_\kappa \to h\nabla_\kappa h^{-1}$ *where*

$$h = \exp\left(\frac{\hbar}{2}\sum_{1\le i<j\le m+n}(\beta_i-\beta_j)\log(z_i-z_j)\right) \tag{3.22}$$

we can change the ∇_κ *connection to the dynamical connection* ∇_D *as in* [6]*:*

$$\nabla_D = d+\hbar\sum_{1\le i<j\le m+n}\frac{dz_i-dz_j}{z_i-z_j}e_{-\alpha}e_\alpha \tag{3.23}$$

Proof. A straightforward computation. For the second part, note that

$$e_\alpha e_{-\alpha}+e_{-\alpha}e_\alpha = 2e_{-\alpha}e_\alpha + h_\alpha^\vee \tag{3.24}$$

and $h_\alpha^\vee$ acts on $M_{\alpha,\beta}$ by $\beta_i-\beta_j$, where $\alpha=\theta_i-\theta_j$. □

We also need the following lemma.

Lemma 3.2.3. *For all non-integer and large enough integer* t *we have an isomorphism*

$$\phi := \mathrm{Hom}_{\underline{Rep}(GL_t)}(V_{\lambda,\mu}, V^{*\otimes n}\otimes V^{\otimes m}) \cong L_{\gamma^\top}[\beta], \tag{3.25}$$

where $L_{\gamma^\top}$ *is the unique irreducible* $\mathfrak{gl}_{m+n}$*-module of the highest weight* $\gamma^\top$*.*

Proof. Note that for the specified t the dimension of the LHS is the same as the dimension of the same space for some large enough integer t. The dimension of the RHS is constant for the aforementioned t due to the BGG resolution.

For the choice of the LHS basis let us consider a projection

$$\pi : \mathrm{Hom}_{\underline{\mathrm{Rep}}(GL_t)}(V^{\otimes|\lambda|}\otimes V^{*\otimes|\mu|}, V^{*\otimes n}\otimes V^{\otimes m}) \twoheadrightarrow \mathrm{Hom}_{\underline{\mathrm{Rep}}(GL_t)}(V_{\lambda,\mu}, V^{*\otimes n}\otimes V^{\otimes m}) \tag{3.26}$$

and let us fix a set B from the spanning set of $(w,w')-$diagrams from the bigger space as in [1] for t from a Zariski open set $U\subset\mathbb{C}$ such that the projection of B is a basis.

Let us fix a basis of the corresponding weight space from the PBW-spanning set on the RHS. Note that the relations on the PBW vectors from the RHS are independent on t. Indeed, otherwise it would mean that we have a singular vector above β in the Verma module $M_{\gamma^\top}$ whose coefficients necessarily depend on t. This in turn would imply the same fact for all large integer t, but with this assumption all the relations on the PBW vectors are independent on t due to the consideration of the embedding below. In particular, it is clear that if a subset from the set of spanning PBW vectors is a basis for some t as in the lemma, then it will also be a basis for the same weight space for all non-integer or large enough integer t because the weights of the singular vectors of the corresponding Verma module sitting above β are all the same for such t.

For a large integer t we may associate the space (3.7) with the space of $\mathfrak{gl}(V)$ highest weight vectors of the $\mathfrak{gl}(V)\oplus\mathfrak{gl}(W)$-weight (γ,β) in $\Lambda^\bullet(V\otimes W)$. We may embed both spaces for a large integer t into the (γ,β)-weight space of $\Lambda^\bullet(V\otimes W)$. In turn, it can be viewed as the space

$$\Lambda^{\gamma_1}W\otimes\cdots\otimes\Lambda^{\gamma_t}W[\beta]. \tag{3.27}$$

The highest weight vector of $\Lambda^{\gamma_1}W\otimes\cdots\otimes\Lambda^{\gamma_t}W$ is already $\mathfrak{gl}(V)$-singular when it is embedded back into $\Lambda^\bullet(V\otimes W)$, so if we want to get the image of LHS/RHS in (3.27), it is sufficient for us to apply chains of $\mathfrak{gl}(W)$ lowering operators to this vector, so that we arrive in the correct weight space β. The space in (3.27) has a basis

$$w_I = f_{i_1^1}\dots f_{i_{s_1}^1}w_1\otimes\cdots\otimes f_{i_1^t}\dots f_{i_{s_t}^t}w_t, \tag{3.28}$$

where w_i are the highest weight vectors in $\Lambda^{\gamma_i}W$ and

$$\mathrm{wt}(f_{i_1^1}\dots f_{i_{s_1}^1}\dots f_{i_1^t}\dots f_{i_{s_t}^t}) = \gamma^\top - \beta. \tag{3.29}$$

The coefficients in terms of (3.28) of the basis from the LHS will be rational in t and the coefficients of the basis from the RHS will be constant. We want to produce the matrix of the basis change from RHS to the LHS. However, when $t \to +\infty$ the image of the LHS/RHS spaces lies in the subspace of a fixed (independent on t) finite dimension:

$$\Lambda^{\gamma_1}W \otimes \cdots \Lambda^{\gamma_{|\lambda|}}W \otimes (\Lambda^n W \otimes \cdots \otimes \Lambda^n W)^{S_{t-|\lambda|-|\mu|}} \otimes \Lambda^{t-|\mu|+1}W \otimes \cdots \otimes \Lambda^t W[\beta], \tag{3.30}$$

where $S_{t-|\lambda|-|\mu|}$ acts by permutations of the tensor factors. Therefore, the matrix of the basis change has fixed rational coefficients in t and we can identify the spaces from (3.25) for $t \in U$.

To describe the isomorphism for all non-integer and all large enough integer t we may simply choose a different set B' instead of B with the new supporting set U' for t'. It is clear that the isomorphisms agree over the intersection $U \cap U'$. $\square$

We have the following consequence of this lemma.

Theorem 3.2.4. *The isomorphism ϕ in (3.25) identifies the dynamical connection on $L_{\gamma^\top}[\beta]$ and the KZ connection on $\mathrm{Hom}_{\underline{Rep}(GL_t)}(V_{\lambda,\mu}, V^{*\otimes n}\otimes V^{\otimes m})$. In particular, if*

$$f : \{(z_1, \cdots, z_{m+n}) \in \mathbb{C}^{m+n} \mid z_i \neq z_j\} \to L_{\gamma^\top}[\beta] \tag{3.31}$$

is a flat section of the dynamical connection, then

$$\prod_{1\le i<j\le m+n} (z_i - z_j)^{-(\beta_i+\beta_j)\hbar/2} \cdot h \cdot \phi^{-1}(f) = \prod_{1\le i<j\le m+n} (z_i - z_j)^{-\beta_j\hbar} \cdot \phi^{-1}(f) \tag{3.32}$$

is a flat section of ∇_{KZ}.

Proof. Since we know that $e_{-\alpha}e_\alpha$ act as truncated Casimirs on $L_{\gamma^\top}[\beta]$ for all sufficiently large integers t and this action is polynomial in t (in terms of a PBW basis), it follows that $e_{-\alpha}e_\alpha$ will still act as truncated Casimirs for all non-integer and large enough integer t. $\square$

4. Solutions to dynamical differential equations

4.1. **Integral formulas.** Due to Theorem 3.2.2 it suffices to find flat sections of the dynamical connection (3.23) for the Lie algebra $\mathfrak{gl}_{n+m}$ and the weight space $L_{\gamma^\top}[\beta]$. Explicitly, we are looking for solutions $u : \{(z_1,\dots,z_{m+n}) \in \mathbb{C}^{m+n} \mid z_i \neq z_j\} \to L_{\gamma^\top}[\beta]$ to the equations

$$du = -\hbar \sum_{1\le i<j\le n+m} \frac{dz_i - dz_j}{z_i - z_j} e_{-\alpha}e_\alpha u \tag{4.1}$$

where α is the root $\theta_i - \theta_j$ of $\mathfrak{gl}_{n+m}$ and $e_{\pm\alpha}$ are the corresponding normalized root vectors.

From [6] we have integral solutions to these equations, which we will now describe. Let $f_i = E_{i,i+1} \in \mathfrak{gl}_{n+m}$ for $1 \le i \le n+m-1$ be the standard lowering operators; associated with them are the simple roots $\alpha_i = \theta_i - \theta_{i+1}$. Write $\gamma^\top - \beta$ as a sum of simple roots $\lambda = \sum_{i=1}^{n+m-1} m_i\alpha_i$ for some $m_i \in \mathbb{Z}_{\ge 0}$ (note that $\gamma^\top - \beta$ stabilizes for generic t, so the m_i do too). Let $\overline{m} = \sum_{i=1}^{n+m-1} m_i$, and let c be the unique non-decreasing function from $\{1,\dots,\overline{m}\} \to \{1,\dots,n+m-1\}$ such that $|c^{-1}(i)| = m_i$ for all $1 \le i \le n+m-1$.

For permutations $\sigma \in S_{\overline{m}}$ define the differential $\overline{m}$-form $\omega_\sigma = d\log(t_{\sigma(1)} - t_{\sigma(2)}) \wedge \cdots \wedge d\log(t_{\sigma(\overline{m}-1)} - t_{\sigma(\overline{m})}) \wedge d\log(t_{\sigma(\overline{m})})$; also define the operator $f_{c(\sigma)}v := f_{c(\sigma(1))}\cdots f_{c(\sigma(\overline{m}))}$ Let

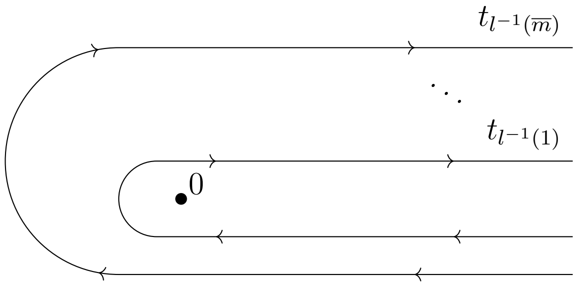
v denote the highest weight vector in $L_{\gamma^\top}$. The $L_{\gamma^\top}[\beta]$-valued differential $\overline{m}$-form ω is defined as

$$\omega(t_1,\dots,t_{\overline{m}}) = \sum_{\sigma\in\Sigma_{\overline{m}}} (-1)^{|\sigma|}\omega_\sigma f_{c(\sigma)}v. \tag{4.2}$$

Theorem 4.1.1. *Let us fix an ordering $l : \{1,\dots,\overline{m}\} \to \{1,\dots,\overline{m}\}$ of the set $\{1,\dots,\overline{m}\}$. The sections of the form*

$$u_l := \int_{\Gamma_l} \exp\left(\hbar\sum_{i=1}^{\overline{m}}\left(z_{c(i)} - z_{c(i)+1}\right)t_i\right)\Phi^{-\hbar}\omega \tag{4.3}$$

span the space of solutions of the dynamical equations in $L_{\gamma^\top}[\beta]$. Here Γ_l is a cycle given by the picture below.



Pic. 1. Integration contours for Γ_l.

Φ is the master function

$$\Phi(t_1,\dots,t_{\overline{m}}) := \prod_{1\le i\le\overline{m}} t_i^{-(\alpha_{c(i)},\gamma^\top)}\prod_{i<j}(t_i-t_j)^{(\alpha_{c(i)},\alpha_{c(j)})} \tag{4.4}$$

and ω is a $L_{\gamma^\top}[\beta]$-valued differential $\overline{m}$-form which has the combinatorial description (4.2). The integrals converge in the region $\Re(\hbar(z_i - z_{i+1})) < 0$.

Proof. From [6] we know that the sections

$$u(z_1,\dots,z_{m+n},x) := \int_\Gamma \exp\left(\hbar\left(\sum_{i=1}^{\overline{m}}\left(z_{c(i)} - z_{c(i)+1}\right)t_i - \langle\gamma^\top, z\rangle x\right)\right)\tilde{\Phi}^{-\hbar}(x)\omega(t,x) \tag{4.5}$$

for any appropriate cycle Γ satisfy both the trivial Knizhnik-Zamolodchikov connection (on the single variable x)

$$\frac{d}{dx} + \hbar z \tag{4.6}$$

where we view $z = (z_1,\dots,z_{m+n})$ as an element of the standard Cartan subalgebra of $\mathfrak{gl}_{m+n}$ and the dynamical equations for

$$\nabla'_D := d_z + \hbar\left(\sum_{i=1}^{m+n}\beta_i x dz_i + \sum_{1\le i<j\le m+n}\frac{dz_i - dz_j}{z_i - z_j}e_{-\alpha}e_\alpha\right). \tag{4.7}$$

Here we have

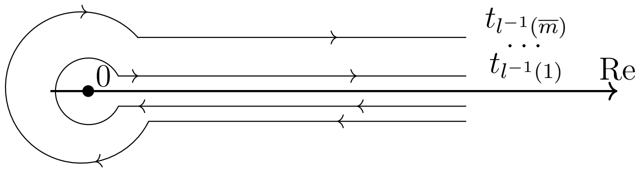
$$\tilde{\Phi}(x) := \prod_{1\le i\le\overline{m}}(t_i - x)^{-(\alpha_{c(i)},\gamma^\top)}\prod_{i<j}(t_i-t_j)^{(\alpha_{c(i)},\alpha_{c(j)})} \tag{4.8}$$

and $\omega(t,x)$ is as in (4.2) but with ω_σ replaced by

$$\omega_\sigma(t,x) := d\log(t_{\sigma(1)} - t_{\sigma(2)})\wedge\cdots\wedge d\log(t_{\sigma(\overline{m}-1)} - t_{\sigma(\overline{m})})\wedge d\log(t_{\sigma(\overline{m})} - x), \quad d = d_t. \tag{4.9}$$

If we assume that $|x|$ is sufficiently small and make a suitable linear change of variables for t_i we may expand the solutions in x. From the form of the (4.7) we deduce that the first term of this expansion, i.e. (4.3), is a flat section for ∇_D.

The sections u_l in (4.3) span the space of flat sections of ∇_D. Indeed, consider a limit $\hbar = \epsilon$, $z_i = z'_i/\epsilon$, $\epsilon \to 0$ so that $\Re(z'_i - z'_{i+1}) < 0$. By deforming the contours of integration we may assume that both "tails" of each individual contour are close to the real line.



Pic. 2. Deformation of the cycle Γ_l.

We may note that the integral $u_l(z'_i, \epsilon)$ converges absolutely in ϵ, so it is entire in ϵ and we may consider $u_l(z'_i, 0)$. Assume for simplicity that we are working with only one term ω_σ of ω. When we let $\epsilon = 0$ the function under the integral (4.3) becomes entire on $\mathbb{C}^{\overline{m}} \setminus (\bigcup_{i<j}\{t_i = t_j\} \cup \bigcup_i \{t_i = 0\})$. If we look at the function $u_l(z'_i, 0)$ without the integral over $t_{l^{-1}(1)}$, the resulting function $f_1(t_{l^{-1}(1)}, z'_i)$ is meromorphic in $t_{l^{-1}(1)}$ with the only pole at $t_{l^{-1}(1)} = 0$. Therefore, if we perform the missing integration in $t_{l^{-1}(1)}$ and pinch two tails of integration from $a + i0+$ to $+\infty + i0+$ and back from $+\infty - i0+$ to $a - i0+$ where $a \in \mathbb{R}_{>0}$ they will cancel each other out. Thus the resulting integral computes the residue of $f_1(t_{l^{-1}(1)}, z'_i)$ at $t_{l^{-1}(1)} = 0$. If ω_σ does not have a pole at $t_{l^{-1}(1)} = 0$ the integral is zero.

This argument shows that we may algebraically compute the residue of a function

$$\exp\big(\sum_{i=1}^{\overline{m}} (z'_{c(i)} - z'_{c(i)+1})t_i\big) \frac{1}{(t_{\sigma(1)} - t_{\sigma(2)})(t_{\sigma(2)} - t_{\sigma(3)}) \dots (t_{\sigma(\overline{m}-1)} - t_{\sigma(\overline{m})}) t_{\sigma(\overline{m})}} \tag{4.10}$$

at $t_{l^{-1}(1)} = 0$ and then perform the other $\overline{m} - 1$ integrations. Then we can consider the same argument for the next variable $t_{l^{-1}(2)}$. This time the function will look differently, namely we'll have (assume $\sigma(\overline{m}) = l^{-1}(1)$)

$$(-2\pi i) \cdot \exp\big(\sum_{i=1, i\neq l^{-1}(1)}^{\overline{m}} (z'_{c(i)} - z'_{c(i)+1})t_i\big)\Big[\frac{1}{(t_{\sigma(1)} - t_{\sigma(2)})(t_{\sigma(2)} - t_{\sigma(3)}) \dots t_{\sigma(\overline{m}-1)}} - \tag{4.11}$$

$$-(z'_{c(\overline{m})} - z'_{c(\overline{m})+1}) \frac{1}{(t_{\sigma(1)} - t_{\sigma(2)})(t_{\sigma(2)} - t_{\sigma(3)}) \dots (t_{\sigma(\overline{m}-2)} - t_{\sigma(\overline{m}-1)})}\Big], \tag{4.12}$$

but the last term does not contribute to the integral, because the degree of the denominator is smaller than the number of integrations. If we compute the integral over the last term by the trick above, eventually we will have a function without any poles in some variable t_i which will give us 0 after the integration.

From this we see that

$$\int_{\Gamma_l} \exp\big(\sum_{i=1, i\neq l^{-1}(1)}^{\overline{m}} (z'_{c(i)} - z'_{c(i)+1})t_i\big)\omega_\sigma = (-2\pi i)^{\overline{m}} \delta_{\sigma, l^{-1}\circ w}, \tag{4.13}$$

where $w(i) = \overline{m} + 1 - i, 1 \le i \le \overline{m}$. Then we have

$$u_l(z'_i, 0) = \int_{\Gamma_l} \sum_{\sigma \in S_{\overline{m}}} \omega_\sigma f_{c(\sigma)} v = (-2\pi i)^{\overline{m}} f_{c(l^{-1}\circ w)} v. \tag{4.14}$$

The vectors $f_{c(\sigma)} v$ span $L_{\gamma^\top}[\beta]$, so the solutions (4.3) span the space of all solutions in $L_{\gamma^\top}[\beta]$. $\square$

Remark 4.1.2. For large integer t there is a natural embedding $\varphi : L_{\gamma^\top} \to \Lambda^{\gamma_1} W \otimes \cdots \otimes \Lambda^{\gamma_t} W$ which sends the highest-weight vector of $L_{\gamma^\top}$ to the product of highest-weight vectors. One might also try to write solutions for large integer t using Theorem 3.1 in [6] on the $\mathfrak{gl}_{n+m}$ weight space $(\Lambda^{\gamma_1} W \otimes \cdots \otimes \Lambda^{\gamma_t} W)[\beta]$. However, we can show that the solutions obtained in this way actually lie in the image of φ, and are in fact the same as the solutions obtained in Theorem 4.1.1. Explicitly, the "new" solutions are described as follows: let P be the set of sequences $\sigma = (i_1^1, \dots, i_{s_1}^1; \dots; i_1^t, \dots, i_{s_t}^t)$ consisting of the numbers $1, \dots, \overline{m}$ arranged into t rows. For each such sequence, define the differential form $\omega_\sigma = \omega_{i_1^1,\dots,i_{s_1}^1} \wedge \cdots \wedge \omega_{i_1^t,\dots,i_{s_t}^t}$ where $\omega_{i_1,\dots,i_s} := d\log(t_{i_1} - t_{i_2}) \wedge \cdots \wedge d\log(t_{i_{s-1}} - t_{i_s}) \wedge d\log(t_{i_s})$. Also define the vector $f_\sigma v := f_{c(i_1^1)} \cdots f_{c(i_{s_1}^1)} v_1 \otimes \cdots \otimes f_{c(i_1^t)} \cdots f_{c(i_{s_t}^t)} v_t$ where v_j is the highest-weight vector in $\Lambda^{\gamma_j} W$. Then the "new" solutions are given by

$$u_l = \int_{\Gamma_l} \exp\left(\hbar \sum_{i=1}^{\overline{m}} \left(z_{c(i)} - z_{c(i)+1}\right) t_i\right) \Phi^{-\hbar} \tilde{\omega} \tag{4.15}$$

where

$$\tilde{\omega} := \sum_{\sigma \in P} (-1)^{|\sigma|} \omega_\sigma f_\sigma v. \tag{4.16}$$

By repeatedly using Lemma 7.4.4 from [10] and the formula for the action of a Lie algebra $\mathfrak{g}$ on a tensor product of $\mathfrak{g}$-modules we can re-arrange terms in (4.15). This way we can see that the solutions (4.15) are the same as (4.3).

Example 4.1.3. *As in Example 3.1.1, consider the case when $\lambda, \mu = 0$ and $m = n$ so we have $\mathrm{Hom}_{\underline{Rep}(GL_t)}(\mathbb{1}, V^{*\otimes m} \otimes V^{\otimes m}) \cong \mathbb{C}[S_m]$. This is also identified with the $\mathfrak{gl}_{2m}$-weight space $L_{\gamma^\top}[\beta]$ where*

$$\gamma^\top = (\underbrace{t, \dots, t}_{m \text{ times}}, \underbrace{0, \dots, 0}_{m \text{ times}}). \tag{4.17}$$

The difference $\gamma^\top - \beta$ is written as the sum of simple roots $\sum_{i=1}^{2m-1} m_i \alpha_i$ where $m_i = m - |m - i|$, so our solutions involve $\overline{m} = m^2$ integrations.

4.2. **Bethe ansatz.** The problem of the Bethe ansatz is to simultaneously diagonalize the Gaudin operators $H_i = \sum_{j \neq i} \Omega_{ij}/(z_i - z_j)$ which appear on the right hand side of the KZ equations. We can obtain such eigenvectors from KZ solutions by taking the limit $\hbar \to 0$. Explicitly, if the spectrum of H_i is simple, the Bethe vectors for H_i acting on the RHS of (3.2.3) can be extracted from KZ solutions (4.3) by using WKB method. For that we would need to find all the critical points of the function

$$\exp\left(\hbar \sum_{i=1}^{\overline{m}} (z_{c(i)} - z_{c(i)+1}) t_i\right) \Phi^{-\hbar} \tag{4.18}$$

However, it turns out that this function has many critical points which are difficult to analyze.

Nevertheless, we may still prove that the spectrum is indeed simple.

Proposition 4.2.1. *The common spectrum of the Gaudin hamiltonians H_i on*

$$\mathrm{Hom}_{\underline{Rep}(GL_t)}(V_{\lambda,\mu}, V^{*\otimes n} \otimes V^{\otimes m}) \tag{4.19}$$

is simple for generic t, z_i.

Proof. The simplicity of the spectrum is a Zariski open condition on parameters t, z_i, so it is sufficient for us to prove it for a special t and generic z_i. The latter can be proved by taking a sufficiently large integer t. In this case we have isomorphisms (3.5), (3.7) and (3.25), so the space (4.19) can be identified with the space $\mathrm{Sing}(\Lambda^\bullet(V\otimes W)_{\gamma,\beta})$ of all $\mathfrak{gl}(V)$-singular vectors of $\mathfrak{gl}(V)\oplus\mathfrak{gl}(W)$ weight space (γ,β) in $\Lambda^\bullet(V\otimes W)$. However, from [8] we know that Gaudin hamiltonians H_i separate the Bethe vectors basis in $\mathrm{Sing}(\Lambda^\bullet(V\otimes W)_{\gamma,\beta})$, thus we have the proposition. $\square$

Appendix A. Hypergeometric solutions for $\lambda,\mu=0$ and $m=n=2$

In this appendix we will describe explicit solutions in terms of hypergeometric functions for the special case when $\lambda,\mu=0$ and $m=n=2$. In this case we are working in the space $\mathrm{Hom}_{\underline{\mathrm{Rep}}(GL_t)}(\mathbb{1},V^{*\otimes 2}\otimes V^{\otimes 2})\cong\mathbb{C}[S_2]$, and the Casimirs Ω_{ij} act as in (3.4). Then letting $e,(12)$ be the two permutations in $\mathbb{C}[S_2]$ we can express a KZ section as $\phi(z_1,z_2,z_3,z_4)=f(z_1,z_2,z_3,z_4)\cdot e+g(z_1,z_2,z_3,z_4)\cdot(12)$, and the KZ equations read

$$\begin{cases}\hbar^{-1}\partial_1 f &= \frac{g}{z_{12}}-\frac{tf+g}{z_{13}}\\ \hbar^{-1}\partial_1 g &= \frac{f}{z_{12}}-\frac{f+tg}{z_{14}}\end{cases}\text{and symm. eqs. for } z_i\mapsto z_{\pi(i)},\ \pi\in\{(12)(34),(13)(24),(14)(23)\} \tag{A.1}$$

where for brevity we denote $z_{ij}:=z_i-z_j$ and $\partial_i:=\partial_{z_i}$. Also, denote $\Delta:=t\hbar$. Then it is straightforward to check that the equations (A.1) are solved by

$$f(z_1,z_2,z_3,z_4)=\frac{A(z_1,z_2,z_3,z_4)}{z_{13}^{\Delta}z_{24}^{\Delta}} \tag{A.2}$$

$$g(z_1,z_2,z_3,z_4)=\frac{B(z_1,z_2,z_3,z_4)}{z_{14}^{\Delta}z_{23}^{\Delta}}. \tag{A.3}$$

where A,B are functions depending on two parameters $c_1,c_2\in\mathbb{C}$:

$$\begin{aligned}A:=&\left(\frac{z_{14}z_{32}}{z_{12}z_{34}}\right)^{1-\Delta}{}_2F_1\left(1-\hbar-\Delta,1+\hbar-\Delta;2-\Delta,\frac{z_{14}z_{32}}{z_{12}z_{34}}\right)c_1\\&+{}_2F_1\left(-\hbar,\hbar;\Delta,\frac{z_{14}z_{32}}{z_{12}z_{34}}\right)c_2\\B:=&\,\hbar^{-1}z_{14}^{\Delta}z_{23}^{\Delta-1}z_{13}^{-\Delta+1}z_{24}^{-\Delta}z_{12}(\partial_1 A).\end{aligned}$$

Note that these solutions involve one integration (in the hypergeometric functions), as opposed to four integrations we would get in our general solutions (see Example 4.1.3).

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