Critical Points for the Gaussian Isoperimetric Problem in $\mathbb{R}^2$

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1 Introduction

Simply stated, we would like to classify all embedded submanifolds $\Sigma^1 \subset \mathbb{R}^2$ without boundary whose Gaussian length is critical with respect to all compactly supported variations that preserve the enclosed Gaussian area. For a fixed enclosed Gaussian area, it is known that the Gaussian length is minimized by lines Σ, as stated more generally in [MR15]. This work also extends the known fact originally due to Abresch and Langer [AL86] that the only closed embedded shrinkers of the mean curvature flow in $\mathbb{R}^2$ are isometric to $S^1_{\sqrt{2}}$. The submanifolds, Σ, of interest, naturally admit descriptions in term of geometric differential equations, and we study their immersed solutions $\mathbb{R} \to \mathbb{R}^2$ more generally (cf. [AL86]'s classification of closed immersed plane shrinkers). In particular, Σ will obey an *inhomogeneous shrinker equation*:

$$H = \frac{1}{2}\langle x, \mathbf{n}\rangle + C_\Sigma$$

where C_Σ is a constant depending on Σ and the choice of an orientation $\mathbf{n}$. Of course the case $C_\Sigma = 0$ will recover the *shrinker equation* governing plane shrinkers. Our two main results are:

Theorem 1. *All such Σ are either lines or strictly convex, in which case they admit an orientation $\mathbf{n}$ so that $H > 0$. Then if $C_\Sigma \geq 0$ it follows that $\Sigma = S^1_R$ for some $0 < R \leq \sqrt{2}$.*

Remark 2. *The condition $C_\Sigma > 0$ when $H > 0$ in the previous theorem is equivalent to the assertion that any compactly supported variation that locally increases the enclosed Gaussian area also locally increases the Gaussian length. See Equation 1 in the next section.*

Theorem 3. *All such Σ are either lines, S^1_R for $R > 0$, or belong in a moduli space Emd_n of such Σ with n-fold rotational symmetry around the origin for $n > 1$. Then the following is known about Emd_n:*

1. *Let $\sim$ denote the equivalence relation associated to rotation about the origin and let (R_Σ, r_Σ) be $(\max_{x\in\Sigma}|x|, \min_{x\in\Sigma}|x|)$. Then $\mathrm{Emd}_n / \sim$ is a smooth connected 1-dimensional family which may be uniquely parameterized by $|R_\Sigma - r_\Sigma| \in (0,\infty)$.*

2. *One boundary limit in the natural Gromov-Hausdorff sense is $S^1_{\sqrt{2(n^2-1)}}$, that is, for the doubly induced Hausdorff distance, d, on sets of subsets of $\mathbb{R}^2$:*
$$\lim_{|R_\Sigma - r_\Sigma|\to 0} d\left(\frac{\mathrm{Emd}_n}{\sim}, \left\{S^1_{\sqrt{2(n^2-1)}}\right\}\right) = 0$$

3. *Let $⬠_{N,R}$ denote the set of regular N-gons inscribed in S^1_R. Then in the other direction we have Gromov-Hausdorff covergence:*
$$\lim_{|R_\Sigma - r_\Sigma|\to \infty} d\left(\frac{\mathrm{Emd}_n}{\sim}, ⬠_{n,|R_\Sigma|}\right) = 0$$

Taken together, these results show that the rigidity theorem for proper embedded shrinkers only extends naturally when a non-negativity condition is enforced. In the absence of such a condition, we can find families of such Σ that vary between circles and regular n-gons centered at the origin for every $n > 1$.

To prove these theorems we adopt what is for the most part a local Euclidean geometric perspective to the issue of analyzing Σ. To that end, the outline of the paper goes as follows:

- In Section 2 we first obtain the differential equations governing such Σ locally.
- In Section 3 we classify the oriented immersed $\Sigma \subset \mathbb{R}^2$ without boundary (and thus often improper) that satisfy these equations everywhere, noting that setting $\{R_\Sigma, r_\Sigma\} = \{\max_{x\in\Sigma}\langle x, \mathbf{n}\rangle, \min_{x\in\Sigma}\langle x, \mathbf{n}\rangle\}$ gives a convenient parameterization.
- In Section 4 we prove Theorem 1 on the backs of the new classification using the Tait-Kneser Theorem. Then the problem reduces to proving a certain 2 real variable inequality which we handle through the use of convexity.
- In Section 5 we continue in the same setting and lay some of the groundwork for Theorem 3 by bounding various Euclidean geometric quantities related to Σ, a key one of which measures the "distance" of Σ from being polygonal.
- In Section 6 we take a decidedly different tack and use a more geometric analytic approach to control the behavior of Σ when it is close to a circle.
- In Section 7 we finally perform the proof of Theorem 3, using the fruits of the previous two sections in addition to some more 2 real variable analysis facilitated by convexity much like in the proof of Theorem 1.

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Contents

2 Defining Equations

Let $\mathbf{n}$ be the outwardly pointing normal for Σ. Any compactly supported tangent variation will preserve Σ, and hence the Gaussian length and enclosed Gaussian area. So we may assume F is a normal variation on Σ. Then we may define $f_t\colon \Sigma \to \mathbb{R}$ so that $F_t|_\Sigma = f_t\mathbf{n}$.

Following the computation in Lemma 3.3.2 of [CM24], by Lemma 3.2.1 of [CM24],

$$\partial_t \mathbf{d}s = f_t H \, \mathbf{d}s,$$

while on the other hand,

$$\partial_t e^{-\frac{x^2}{4}} = -\frac{1}{2} f_t \langle x, \mathbf{n} \rangle e^{-\frac{x^2}{4}},$$

and therefore the rate of change of the Gaussian length under the variation at Σ is given by

$$L_t = \int_\Sigma f_t \left(H - \frac{1}{2}\langle x, \mathbf{n} \rangle \right) e^{-\frac{|x|^2}{4}} \, \mathbf{d}s.$$

Meanwhile, the rate of change of the enclosed Gaussian area under the variation at Σ is just

$$A_t = \int_\Sigma f_t e^{-\frac{|x|^2}{4}} \, \mathbf{d}s.$$

Hence it is an equivalent problem to classify all embedded $\Sigma^1 \subset \mathbb{R}^2$ such that there is a constant C_Σ satisfying

$$L_t = C_\Sigma A_t \iff \tag{1}$$

$$C_\Sigma = H - \frac{1}{2}\langle x, \mathbf{n}\rangle. \tag{2}$$

Alternatively, given a local parameterization in s, we have that equivalently,

$$H_s = \left(\frac{1}{2}\langle x, \mathbf{n}\rangle\right)_s = \frac{1}{2}\langle x_s, \mathbf{n}\rangle + \frac{1}{2}\langle x, \mathbf{n}_s\rangle = \frac{1}{2}\langle x, x_s\rangle\langle x_s, \mathbf{n}_s\rangle = \left(\frac{|x|^2}{4}\right)_s H,$$

and thus there is a constant E_Σ satisfying

$$E_\Sigma = He^{-\frac{|x|^2}{4}}. \tag{3}$$

Remark 4. *Equation 3 seems to have been first noted for plane shrinkers in [CM12] Remark 10.47.*

3 The Oriented Immersed Σ

So from now on suppose Σ is an oriented, immersed, possibly improper, solution of Equation 1 without boundary. We first resolve the case of $E_\Sigma = 0$ separately:

Lemma 5. *The only such Σ with $E_\Sigma = 0$ are exactly the lines.*

Proof. By Equation 3, $H = 0$, and so Σ is a line, and the reverse implication holds as well. □

Thus henceforth we will assume that $E_\Sigma \neq 0$ and therefore Σ is strictly convex by Equation 3. As we shall see, Σ is also then necessarily bounded, offering a means of classification:

Theorem 6. *Suppose Σ is not a line. Then Σ resembles a hypotrochoid; in particular there exists a bijection between equivalence classes of Σ up to rotation, and pairs $\{(r_\Sigma, R_\Sigma)\colon |r_\Sigma| < |R_\Sigma|, r_\Sigma + R_\Sigma \neq 0\}$ so that all of the following hold for $x \in \Sigma$:*

- *$r_\Sigma^2 \leq |x|^2 \leq R_\Sigma^2$, with both bounds obtained and no other critical values.*
- *$\min(r_\Sigma, R_\Sigma) \leq \langle x, \mathbf{n}\rangle \leq \max(r_\Sigma, R_\Sigma)$ with both bounds obtained exactly at critical points for $|x|^2$ and no other critical values.*
- *If $r_\Sigma \neq R_\Sigma$ then*

$$(E_\Sigma, C_\Sigma) = \left(\frac{1}{2}\frac{R_\Sigma - r_\Sigma}{e^{\frac{R_\Sigma^2}{4}} - e^{\frac{r_\Sigma^2}{4}}}, -\frac{1}{2}\frac{R_\Sigma e^{-\frac{R_\Sigma^2}{4}} - r_\Sigma e^{-\frac{r_\Sigma^2}{4}}}{e^{-\frac{R_\Sigma^2}{4}} - e^{-\frac{r_\Sigma^2}{4}}}\right),$$

and otherwise if $r_\Sigma = R_\Sigma = R$, then

$$(E_\Sigma, C_\Sigma) = \left(\frac{1}{R}e^{-\frac{R^2}{4}}, \frac{1}{R} - \frac{R}{2}\right).$$

- *If $r_\Sigma \neq R_\Sigma$ and U is a closed connected subset of Σ containing critical points at the boundaries and nowhere else, then the length of the image of U under the Gauss map $\Sigma \to S^1$ is the half-period*

$$T_\Sigma := \left| \int_{r_\Sigma}^{R_\Sigma} \frac{1}{\sqrt{4\log\left(\frac{\frac{u}{2}+C_\Sigma}{E_\Sigma}\right) - u^2}} \, \boldsymbol{d}u \right| < \pi$$

- *Σ is preserved under reflections connecting critical points to the origin.*
- *Σ is preserved under rotations of $2T_\Sigma$ around the origin.*
- *Σ is smooth.*

and we shall take these properties as definitions for r_Σ and R_Σ.

Proof. Let $\theta\colon \Sigma \to \mathbb{R}$ be a lift of the Gauss map $\Sigma \to S^1$ under an isometric covering $\mathbb{R} \to S^1$. Since H does not vanish, θ is an injection. Therefore there exists a mapping $\theta \mapsto \langle x, \mathbf{n}\rangle$ and moreover this map can be explicitly written out via integration. Indeed, using the definitions of E_Σ and C_Σ, it satisfies the (effectively) separable differential equation (here J is the action of rotation by $\frac{\pi}{2}$ in a direction agreeing with the covering $\mathbb{R} \to S^1$):

$$\begin{aligned}\langle x, \mathbf{n}\rangle_\theta &= \langle x, J(\mathbf{n})\rangle = \operatorname{sgn}(\langle x, x_\theta\rangle)\sqrt{|x|^2 - \langle x, \mathbf{n}\rangle^2} \\ &= \operatorname{sgn}((|x|^2)_\theta)\sqrt{4\log\left(\frac{\frac{1}{2}\langle x, \mathbf{n}\rangle + C_\Sigma}{E_\Sigma}\right) - \langle x, \mathbf{n}\rangle^2}.\end{aligned}$$

In the opposite direction, any solution of this equation satisfies $E_\Sigma e^{\frac{|x|^2}{4}} = \frac{1}{2}\langle x, \mathbf{n}\rangle + C_\Sigma$, and thus upon taking derivatives and using the calculation in Equation 3 we have $E_\Sigma e^{\frac{|x|^2}{4}} = H$ whenever $(|x|^2)_\theta \neq 0$, so any Σ obtained from $\theta \mapsto \langle x, \mathbf{n}\rangle$ is a solution except perhaps at critical points of $|x|^2$, which is sufficient to conclude that Σ is a true solution if also given that these critical points are isolated.

Accordingly, let $f\colon V \subset \mathbb{R}^3 \to \mathbb{R}$ be given by

$$f(u, C, E) = 4\log\left(\frac{\frac{1}{2}u + C}{E}\right) - u^2. \tag{4}$$

so that $f(\langle x, \mathbf{n}\rangle, C_\Sigma, E_\Sigma) = \langle x, \mathbf{n}\rangle_\theta^2$ for all (x, Σ) with $x \in \Sigma$. Then

$$f_{uu} = -\frac{1}{(\frac{1}{2}u + C)^2} - 2 < 0,$$

so it follows that since there in fact exists u so that $f(u, C_\Sigma, E_\Sigma) \geq 0$, there moreover exists a unique pair of possibly non-distinct zeroes, $\{r_\Sigma, R_\Sigma\}$ with $|r_\Sigma| \leq |R_\Sigma|$, of $u \mapsto f(u, C_\Sigma, E_\Sigma)$, and $\langle x, \mathbf{n}\rangle$ always lies between them.

In fact, for every choice of (a, b) such that $a + b \neq 0$ and $|a| \leq |b|$, there exists unique C_Σ and E_Σ so that $(r_\Sigma, R_\Sigma) = (a, b)$. This results from solving the system of equations

$$e^{\frac{r_\Sigma^2}{4}} E_\Sigma - C_\Sigma = \frac{1}{2} r_\Sigma$$
$$e^{\frac{R_\Sigma^2}{4}} E_\Sigma - C_\Sigma = \frac{1}{2} R_\Sigma,$$

which has a unique solution when $|r_\Sigma| \neq |R_\Sigma|$, and when $r_\Sigma = R_\Sigma$, we may take an appropriate limit via replacing one of the equations with its derivative as so,

$$e^{\frac{r_\Sigma^2}{4}} E_\Sigma - C_\Sigma = \frac{1}{2} r_\Sigma$$
$$\frac{r_\Sigma}{2} e^{\frac{r_\Sigma^2}{4}} E_\Sigma = \frac{1}{2},$$

giving solutions as long as $r_\Sigma \neq 0$.

If $|r_\Sigma| = |R_\Sigma| = R$, then from $f(r_\Sigma, C_\Sigma, E_\Sigma) = f(R_\Sigma, C_\Sigma, E_\Sigma)$ we get $r_\Sigma = R_\Sigma$, so evidently $\langle x, \mathbf{n} \rangle = r_\Sigma = R_\Sigma$ always, and therefore $0 = \left(|x|^2\right)_\theta$ always, which is enough to conclude that Σ must be a circle centered at the origin with radius R and orientation chosen appropriately to agree with signs. It can then be checked that this gives a solution with $C_\Sigma = \pm\left(\frac{1}{R} - \frac{R}{2}\right)$ and $E_\Sigma = \pm\left(\frac{1}{R} e^{-\frac{R^2}{4}}\right)$ that corresponds with $r_\Sigma = R_\Sigma = \pm R$.

Otherwise, it follows from Picard–Lindelöf that on any open interval containing no critical points for $|x|^2$ and hence, $f(\langle x, \mathbf{n} \rangle, C_\Sigma, E_\Sigma) > 0$, the mapping $\theta \mapsto \langle x, \mathbf{n} \rangle$ is uniquely determined by an initial value. Indeed, the length of a maximal such interval, U, is also uniquely determined via the formula

$$T_\Sigma := \int_U \mathbf{d}\theta = \operatorname{sgn}(R_\Sigma - r_\Sigma) \int_{r_\Sigma}^{R_\Sigma} \frac{1}{\sqrt{4 \log\left(\frac{\frac{u}{2} + C_\Sigma}{E_\Sigma}\right) - u^2}} \mathbf{d}u$$

obtained by simple integration. In particular, $4 \log\left(\frac{\frac{u}{2} + C_\Sigma}{E_\Sigma}\right)$ as a function of u is concave down and intersects the function u^2 when $u \in \{r_\Sigma, R_\Sigma\}$. Therefore,

$$T_\Sigma < \operatorname{sgn}(R_\Sigma - r_\Sigma) \int_{r_\Sigma}^{R_\Sigma} \frac{1}{\sqrt{(R_\Sigma - u)(u - r_\Sigma)}} \mathbf{d}u = \pi,$$

which shows that U is finite.

On these maximal intervals $\theta \mapsto \langle x, \mathbf{n} \rangle$ is uniquely determined up to a reflection in θ depending on whether $|x|^2$ is decreasing or increasing. In turn it follows that $|x|^2$ attains a global minimum of $|r_\Sigma|^2$ and a global maximum of $|R_\Sigma|^2$. We have that the critical points of $|x|^2$ are also isolated; as otherwise it follows that some circle is a solution which in turn uniquely determines C_Σ and E_Σ that we know must result in $r_\Sigma = R_\Sigma$. Hence, given the location of any critical point, $\theta \mapsto \langle x, \mathbf{n}\rangle$ is uniquely given by stitching the maps obtained on each interval, U, between critical points.

This then determines a unique Σ where $\theta \mapsto x$ can be explictly written in terms of $\theta \mapsto \langle x, \mathbf{n}\rangle$ as $x = \langle x, \mathbf{n}\rangle\mathbf{n} + (\langle x, \mathbf{n}\rangle)_\theta J(\mathbf{n})$. Hence, by leaving the location of any critical point unspecified, Σ will be unique up to rotation around the origin, and moreover from the structure of $\theta \mapsto \langle x, \mathbf{n}\rangle$, Σ is preserved under rotation by $2T_\Sigma$ and reflection over any line through the origin and a critical point. That Σ is smooth is automatic. □

Lemma 7. *A bounded, non-circular immersed solution Σ gives rise to an embedded solution if and only if $T_\Sigma | \pi$ and $r_\Sigma R_\Sigma > 0$.*

Proof. The preceding theorem shows that Σ is the union of rotations and reflections of pieces which we will refer to as half-periods of Σ. The condition that $r_\Sigma R_\Sigma > 0$ is the same as specifying that $\langle x, \mathbf{n}\rangle \neq 0$ for all $x \in \Sigma$, which is equivalet to stating that each half-period lies in the "polar rectangle" with opposite vertices as endpoints. This is a necessary condition for embedded-ness as any intersection of the half-period with the interior of the radial side of the rectangle will also be present in an adjacent half-period via the reflection property. Once this is satisfied, embeddedness corresponds to the perfect overlap of the polar rectangles, which occurs exactly when $T_\Sigma | \pi$. □

4 The Non-Negative Embedded Σ

We are now able to prove to attack Theorem 1. For this section and the one that follows, consider a half-period of Σ, i.e., a connected subset of Σ with no critical points in the interior and bounded by a global minimum and maximum for $|x|$ labeled x_- and x_+ respectively. The following lemma serves as the geometric basis for the proof that follows:

Lemma 8. *Suppose that x_- is a local minimum for curvature on a curve. Then the osculating circle to the curve at x_- contains any neighboring local maximum for curvature, x_+.*

Proof. Note that the curvature is increasing in the direction from x_- to x_+. Therefore, the Tait-Kneser Theorem applies and shows that this segment of the curve lies entirely inside the osculating circle at x_0. □

We now present the main proof.

Proof of Theorem 1. It is clear that Σ may be any line. So suppose that oriented Σ is embedded and not a line, while $E_\Sigma C_\Sigma \geq 0$. The work mainly consists in showing Σ must be a circle.

By the Four Vertex Theorem, $\angle(x_-, x_+) \leq \frac{\pi}{2}$. Lemma 8 also applies to show that x_+ must lie inside the osculating circle at x_-. Equivalently,

$$|x_+ - x_- + H_{x_-}^{-1}\mathbf{n}_{x_-}|^2 \leq H_{x_-}^{-2}.$$

Equation 3 shows that critical points of H correspond to critical points of $|x|$, so we may apply Theorem 6 to obtain that, $x_- = r_\Sigma \mathbf{n}_{x_-}$, and $|x_+| = |R_\Sigma|$. Then it is straightforward to compute:

$$\begin{aligned} H_{x_-}^2 &\geq |x_+ - x_- + H_{x_-}^{-1}\mathbf{n}_{x_-}|^2 \\ &= R_\Sigma^2 + (H_{x_-}^{-1} - r_\Sigma)^2 - 2(H_{x_-}^{-1} - r_\Sigma)\langle x_+, x_- \rangle \\ &\geq R_\Sigma^2 + r_\Sigma^2 - 2r_\Sigma H_{x_-}^{-1} + H_{x_-}^{-2} \end{aligned}$$

and hence $r_\Sigma^2 + R_\Sigma^2 \leq 2r_\Sigma H_{x_-}^{-1}$.

We will now show that in fact $\frac{1}{4}\left(r_\Sigma^2 + R_\Sigma^2\right) \geq \frac{1}{2} r_\Sigma H_{x_-}^{-1}$, so equality must hold, which in fact occurs only when $R_\Sigma = r_\Sigma$.

Using equations 2 & 3, we see that

$$\frac{1}{2} r_\Sigma H_{x_-}^{-1} = (H_{x_-} - C_\Sigma)H_{x_-}^{-1} = 1 - \frac{C_\Sigma}{E_\Sigma} e^{-\frac{r_\Sigma^2}{4}} \leq 1,$$

where we now employ the fact $E_\Sigma C_\Sigma \geq 0$. So let this quantity be a. Further expansion using Theorem 6 shows that at least while $r_\Sigma \neq R_\Sigma$,

$$a = 1 - \frac{C_\Sigma}{E_\Sigma} e^{-\frac{r_\Sigma^2}{4}} = 1 - \left(\frac{R_\Sigma e^{\frac{r_\Sigma^2}{4}} - r_\Sigma e^{\frac{R_\Sigma^2}{4}}}{R_\Sigma - r_\Sigma} \right) e^{-\frac{r_\Sigma^2}{4}} = \frac{e^{\frac{R_\Sigma^2 - r_\Sigma^2}{4}} - 1}{\frac{R_\Sigma}{r_\Sigma} - 1}.$$

As $r_\Sigma^2 + R_\Sigma^2 > 0$, we must have $a > 0$ as well, which in particular forces r_Σ and R_Σ to have the same sign. Then we may treat R_Σ^2 as a variable, u, in the expression to get:

$$\begin{aligned} \frac{R_\Sigma^2 - r_\Sigma^2}{4} &= \ln\left(|R_\Sigma| + \left(\frac{1}{a} - 1\right)|r_\Sigma| \right) + \ln\left(\frac{a}{|r_\Sigma|}\right) \\ \implies \int_{r_\Sigma^2}^{R_\Sigma^2} \frac{1}{4}\, \mathbf{d}u &= \int_{r_\Sigma^2}^{R_\Sigma^2} \frac{1}{2\sqrt{u}\left(\sqrt{u} + \left(\frac{1}{a} - 1\right)|r_\Sigma|\right)}\, \mathbf{d}u \end{aligned}$$

But now since $a < 1$, we can check that the map $f(u) = [2(u + \left(\frac{1}{a} - 1\right)|r_\Sigma|\sqrt{u})]^{-1}$ satisfies both $f' < 0$ and $f'' > 0$. Hence we can bound the RHS as follows:

$$\int_{r_\Sigma^2}^{R_\Sigma^2} f\left(r_\Sigma^2\right) \mathbf{d}u \geq \int_{r_\Sigma^2}^{R_\Sigma^2} f(u)\, \mathbf{d}u = \int_{r_\Sigma^2}^{R_\Sigma^2} \frac{f(u) + f(r_\Sigma^2 + R_\Sigma^2 - u)}{2}\, \mathbf{d}u \geq \int_{r_\Sigma^2}^{R_\Sigma^2} f\left(\frac{r_\Sigma^2 + R_\Sigma^2}{2}\right) \mathbf{d}u.$$

Thus,

$$f\left(r_\Sigma^2\right) \geq \frac{1}{4} \geq f\left(\frac{r_\Sigma^2 + R_\Sigma^2}{2}\right).$$

The first inequality results in $r_\Sigma^2 \leq 2a$, which in turn implies $f(2a) \geq \frac{1}{4}$, and now since $f' < 0$, the second inequality combines to show $r_\Sigma^2 + R_\Sigma^2 \geq 4\text{a}$. Recall that we originally showed the opposite inequality. Hence equality must hold, but this only occurs when $r_\Sigma = R_\Sigma$, so Σ must be a circle as wanted.

Now by 6, any circle centered at the origin with positive radius R is a solution, and we only need to check those that satisfy $E_\Sigma C_\Sigma \geq 0$. But this is equivalent to $\frac{1}{R} - \frac{R}{2} \geq 0$ or $R \leq \sqrt{2}$ as desired. $\square$

5 The Half-Periods Σ

By Theorem 6, we already have that embedded solutions resemble hypotrochoids. Hence embedded solutions can be described as looking like something between a regular polygon and a perfect circle, and we can put a quantity to this by defining

$$g_\Sigma := |r_\Sigma - R_\Sigma \cos(T_\Sigma)|,$$

which we will refer to as the *gap*. It is equivalent to the length of the projection of a half-period of Σ onto the line through the origin with given direction $\mathbf{n}_{x_-}$ as can be seen from the computation

$$\Big|_{x_-}^{x_+} \langle x, \mathbf{n}_{x_-}\rangle = \langle x_+, \mathbf{n}_{x_-}\rangle - \langle x_-, \mathbf{n}_{x_-}\rangle = R_\Sigma \langle \mathbf{n}_{x_+}, \mathbf{n}_{x_-}\rangle - r_\Sigma = R_\Sigma \cos(T_\Sigma) - r_\Sigma.$$

In particular, if Σ is embedded then the *distance* in the Gromov-Hausdorff sense between Σ and the union of line segments joining neighboring points on Σ with maximal distance to the origin is given by g_Σ. As such, g_Σ both gives another descriptive quantity, and another way of accessing the crucial T_Σ for determining embeddedness according to Lemma 7.

Somewhat surprisingly, it is possible to derive an equivalent local condition on Σ in terms of Euclidean geometric quantities, as opposed to say Gaussian ones. For now, let θ be a Gauss map parameterization, s be an arclength parameterization agreeing with the direction of θ, and $\alpha\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ a winding map again with the appropriate direction. Then we can make the following calculation:

Lemma 9. *Working locally we see that the polar area under any connected subset of the curve can be rewritten as:*

$$\int \frac{1}{2}|x|^2\, \boldsymbol{d}\alpha = \int \frac{1}{2}\langle x, \mathbf{n}\rangle\, \boldsymbol{d}s = \int (H - C_\Sigma)\, \boldsymbol{d}s = \int 1\, \boldsymbol{d}\theta - C_\Sigma \int 1\, \boldsymbol{d}s.$$

First we notice that a simple lower bound for T_Σ obtains:

Lemma 10.

$$\cos(T_\Sigma) \leq \frac{r_\Sigma}{R_\Sigma}$$

Proof. This follows from convexity as the half-period lies entirely to one side of the tangent line to Σ at x_-, and the fact that H and R_Σ share the same sign (check the formula for E_Σ in Theorem 6). Thus:

$$\cos(T_\Sigma) = \langle \mathbf{n}_{x_+}, \mathbf{n}_{x_-}\rangle = \frac{\langle x_+, \mathbf{n}_{x_-}\rangle}{R_\Sigma} \leq \frac{\langle x_-, \mathbf{n}_{x_-}\rangle}{R_\Sigma} = \frac{r_\Sigma}{R_\Sigma}$$

□

We will show later on that this bounds doubles as a good estimate for T_Σ. In order to do that, we will need to obtain upper bounds on g_Σ.

The idea here will be to take a half-period of Σ, and split it into two parts. Set the splitting point, x_d, to be distance d away from the origin. As a matter of convenience we will assume that $r_\Sigma R_\Sigma > 0$ so that Σ has a well-defined polar parameterization, though such an assumption should be able to be dropped with some adjustments to the arguments without affecting the order of the bounds subsequently obtained. Then we have that:

$$g_\Sigma = \left| \Big|_{x_-}^{x_+} \langle x, \mathbf{n}_{x_-} \rangle \right| = \left| \Big|_{x_-}^{x_d} \langle x, \mathbf{n}_{x_-} \rangle \right| + \left| \Big|_{x_d}^{x_+} \langle x, \mathbf{n}_{x_-} \rangle \right|$$

and we bound each term separately. In particular, we set ourselves up to be in a position to be able to take advantage of Lemma 9.

We observe two essentially visual facts to directly relate the quantities in Lemma 9.

Lemma 11.

$$\begin{aligned}
\left| \Big|_{x_-}^{x_d} \langle x, \mathbf{n}_{x_-} \rangle \right| &\leq \sin\left(\left| \Big|_{x_-}^{x_d} \theta \right| \right) \left| \Big|_{x_-}^{x_d} s \right| \\
\left| \Big|_{x_d}^{x_+} \langle x, \mathbf{n}_{x_-} \rangle \right| &\leq \sin\left(\left| \Big|_{x_d}^{x_+} \theta \right| \right) \left| \Big|_{x_d}^{x_+} s \right| \\
\left| \Big|_{x_-}^{x_d} s \right| &\leq d - |r_\Sigma| + d \left| \Big|_{x_-}^{x_d} \alpha \right| \\
\left| \Big|_{x_d}^{x_+} s \right| &\leq |R_\Sigma| - d + |R_\Sigma| \left| \Big|_{x_d}^{x_+} \alpha \right|
\end{aligned}$$

Proof. For the first two inequalities, notice that

$$\begin{aligned}
\left| \Big|_I \langle x, \mathbf{n}_{x_-} \rangle \right| = \left| \int_I \sin(\angle(\mathbf{n}, \mathbf{n}_{x_-}))\, \mathbf{d}s \right| &= \left| \int_I \sin(\angle(\mathbf{n}, \mathbf{n}_{x_-}))\, \mathbf{d}s \right| \\
&\leq \max_I |\sin(\angle(\mathbf{n}, \mathbf{n}_{x_-}))| \cdot \left| \Big|_I s \right|
\end{aligned}$$

and the referenced angle is the same as the displacement in θ from x_-. It is therefore maximized at the end of the interval farthest x_- which suffices. For the last two inequalities, note that locally in polar coordinates we have $\mathbf{d}s \leq \mathbf{d}r + r\,\mathbf{d}\alpha$ and since θ and α are monotonic, this inequality can be weakened with $r \leq \max_I r$ and then integrated over I to obtain the desired result. $\square$

So now we derive the following (fairly ugly) upper bound for g_Σ:

Lemma 12. *Assume $r_\Sigma R_\Sigma \geq 0$. Then for any $d \in (\sqrt{r_\Sigma R_\Sigma}, R_\Sigma)$:*

$$g_\Sigma \leq (\pi+1)^2 d^2 \left| \frac{1}{2} R_\Sigma + C_\Sigma \right| e^{\frac{d^2 - R_\Sigma^2}{4}} + \frac{2\pi|R_\Sigma| + d^2(|R_\Sigma| - d)}{|R_\Sigma(R_\Sigma - r_\Sigma)| - 2|R_\Sigma|(|R_\Sigma| - d)}$$

Proof. It is easiest to bound the first term. We only need that since $T_\Sigma < \pi$, the same holds for any displacements of α or θ, in combination with the previous lemma:

$$\left|\Big|_{x_-}^{x_d} s\right| \leq d - |r_\Sigma| + d\left|\Big|_{x_-}^{x_d}\alpha\right| \leq (\pi+1)d$$

$$\left|\Big|_{x_-}^{x_d}\theta\right| = \left|\int_I H\,\mathbf{d}s\right| \leq |H_{x_d}|\cdot\left|\Big|_{x_-}^{x_d} s\right| = |H_{x_+}|\,e^{\frac{d^2-R_\Sigma^2}{4}}\cdot\left|\Big|_{x_-}^{x_d} s\right|$$

$$\implies \left|\Big|_{x_-}^{x_d}\langle x, \mathbf{n}_{x_-}\rangle\right| \leq |H_{x_+}|\,e^{\frac{d^2-R_\Sigma^2}{4}}\cdot\left|\Big|_{x_-}^{x_d} s\right|^2 \leq \left|\frac{1}{2}R_\Sigma + C_\Sigma\right| e^{\frac{d^2-R_\Sigma^2}{4}}\cdot(\pi+1)^2 d^2$$

The last term is the trickier one, and for which we introduced Lemma 9. From the formula for C_Σ in Theorem 6, it is not too hard to check that $C_\Sigma \geq -\frac{r_\Sigma}{2}$. It is also true from our choices of them that s and θ are either both increasing or decreasing so we conclude from Lemma 9 that:

$$\begin{aligned}
\frac{1}{2}d^2\left|\Big|_{x_d}^{x_+}\alpha\right| = \left|\int_I \frac{1}{2}|x|^2\,\mathbf{d}\alpha\right| &= \left|\left|\Big|_{x_d}^{x_+}\theta\right| - C_\Sigma\left|\Big|_{x_d}^{x_+} s\right|\right| \\
&\leq \pi + \frac{|r_\Sigma|}{2}\left(|R_\Sigma| - d + |R_\Sigma|\left|\Big|_{x_d}^{x_+}\alpha\right|\right) \\
\implies \left|\Big|_{x_d}^{x_+}\alpha\right| &\leq \frac{2\pi + |r_\Sigma|(|R_\Sigma| - d)}{d^2 - r_\Sigma R_\Sigma} \\
\implies \left|\Big|_{x_d}^{x_+} s\right| &\leq \frac{2\pi|R_\Sigma| + d^2(|R_\Sigma| - d)}{d^2 - r_\Sigma R_\Sigma} \leq \frac{2\pi|R_\Sigma| + |R_\Sigma|(R_\Sigma^2 - d^2)}{|R_\Sigma(R_\Sigma - r_\Sigma)| - (R_\Sigma^2 - d^2)} \\
\implies \left|\Big|_{x_d}^{x_+}\langle x, \mathbf{n}_{|x|=r_\Sigma}\rangle\right| &\leq \frac{2\pi R_\Sigma + d^2(R_\Sigma - d)}{|R_\Sigma(R_\Sigma - r_\Sigma)| - 2|R_\Sigma|(|R_\Sigma| - d)},
\end{aligned}$$

where we obtain the bound on $\left|\Big|_{x_d}^{x_+} s\right|$ by substituting the just obtained bound on $\left|\Big|_{x_d}^{x_+}\alpha\right|$ into the first equality. $\square$

Now, we do have to be careful about estimating C_Σ, but these estimates are enough to *almost* establish the desired limits. In good times the first and third terms will be negligible while the middle term will act as proxy for $\cos^{-1}(\frac{r_\Sigma}{R_\Sigma})$, enabling us with good upper and lower bounds (indeed, the middle term can be lower bounded using the same techniques though its not necessary for the current argumentation) that let us establish the result. In bad times however, $R_\Sigma - r_\Sigma$ is "small" and we are unable to ignore the first and last terms. For that case we will need a separate set of bounds.

This bound will be more useful in the following form:

Corollary 13. *Suppose $r_\Sigma R_\Sigma > 0$ and $|R_\Sigma - r_\Sigma|$ is not too small in the sense that $\frac{|R_\Sigma - r_\Sigma||R_\Sigma|}{\log|R_\Sigma|}$ is bounded below by some $K_0 \in \mathbb{R}_{>0}$. Then there exists $B, K \in \mathbb{R}_{>0}$ such that for $|R_\Sigma| \geq B$,*

$$g_\Sigma \leq K\frac{\log|R_\Sigma|}{|R_\Sigma - r_\Sigma|}$$

Proof. Notice that according to the formula for C_Σ in Theorem 6, $C_\Sigma = o(R_\Sigma)$. Then pick d so that $|R_\Sigma| - d \sim 12\frac{\log|R_\Sigma|}{|R_\Sigma|}$ in the previous lemma and again perform the asymptotic calculations. $\square$

6 The Round Σ

We will now study the map $\theta \mapsto H$ as the basis for identifying possible Σ. Here we take advantage of H's designation as the *shape operator*; we have a bijection between the equivalence classes of maps $\theta \mapsto H$ under isometries of the domain ($\mathbb{R}^1$) and the equivalence classes of *strictly convex* oriented Σ under isometries of $\mathbb{R}^2$.

Let L be the linear operator defined by $f \overset{L}{\mapsto} f + f_{\theta\theta}$. We see that $\Sigma + v$ is a solution to

$$H - \frac{1}{2}\langle x, \mathbf{n}\rangle = C_\Sigma + \frac{1}{2}\langle v, \mathbf{n}\rangle,$$

and in particular the set of functions $\{\frac{1}{2}\langle v, \mathbf{n}\rangle : C_\Sigma \in \mathbb{R}, v \in \mathbb{R}^2\}$ is precisely the kernel of L. Hence we are looking for those maps $\theta \to H$ that satisfy

$$L(H) = \frac{1}{2}L(\langle x, \mathbf{n}\rangle) + C_\Sigma$$

In particular we have the following important identity:

$$\begin{aligned}
\langle x, \mathbf{n}\rangle + \langle x, \mathbf{n}\rangle_{\theta\theta} &= \langle x, \mathbf{n}\rangle + (\langle x_\theta, \mathbf{n}\rangle)_\theta + (\langle x, \mathbf{n}_\theta\rangle)_\theta = \langle x, \mathbf{n}\rangle + (\langle x, J(\mathbf{n})\rangle)_\theta \\
&= \langle x, \mathbf{n}\rangle + \langle x_\theta, J(\mathbf{n})\rangle - \langle x, \mathbf{n}\rangle = \langle x_\theta, \mathbf{n}_\theta\rangle = \frac{1}{H} \\
\implies L(\langle x, \mathbf{n}\rangle) &= \frac{1}{H}
\end{aligned}$$

Therefore it obtains that an equivalent formulation is

$$H_{\theta\theta} = \frac{1}{2H} - H + C_\Sigma \tag{5}$$

$$\implies H_{\theta\theta\theta} = -\left[1 + \frac{1}{2H^2}\right] H_\theta \tag{6}$$

Thus we have derived a constant free ordinary differential equation which characterizes all and only those immersed oriented solutions up to translation.

In the last section we found a bound for g_Σ when $|R_\Sigma - r_\Sigma|$ isn't very small. We now show that a different method gives decent bounds on T_Σ when the other fails. Indeed, we will use it to show that the limit of T_Σ as $R_\Sigma - r_\Sigma \to 0$ always exists and compute its value. More precisely:

Lemma 14.

$$\frac{\pi}{\sqrt{1 + \frac{1}{2(\frac{1}{2}r_\Sigma + C_\Sigma)^2}}} \leq T_\Sigma \leq \frac{\pi}{\sqrt{1 + \frac{1}{2(\frac{1}{2}R_\Sigma + C_\Sigma)^2}}}$$

Corollary 15.

$$T_{S^1_R} := \lim_{\Sigma \to S^1_R} T_\Sigma = \frac{\pi}{\sqrt{\frac{R^2}{2} + 1}}.$$

The proof is via the following comparison lemma concerning the eigenfunctions produced by L.

Lemma 16. *Suppose that $f\colon I_f \to \mathbb{R}$ and $g\colon I_g \to \mathbb{R}$ are twice-differentiable functions on closed, connected subsets of $\mathbb{R}$, I_f and I_g, that are nonzero in the interior and zero on the boundary of their respective domains. Moreover suppose that*

$$\inf_{I_f} \frac{f''}{f} \geq \sup_{I_g} \frac{g''}{g}.$$

Then the length of I_f, $L(I_f)$ is at least the length of I_g, $L(I_g)$.

Proof. By Mean Value Theorem, there exists $a_f \in I_f$ and $a_g \in I_g$ such that $f'(a_f) = 0$ and $g'(a_g) = 0$. By translating f and g as necessary, we may also assume $a_f = a_g = 0$. We then only need to use that

$$\frac{f''}{f} \geq \frac{g''}{g}$$

holds pointwise on $I_f \cap I_g$ from the hypothesis. Let's look at $h := |fg^{-1}|$, which will be a twice-differentiable positive function on $I_f \cap I_g$. By construction moreover, $h'(0) = 0$. We observe that:

$$\frac{\left[h'g^2\right]'}{hg^2} = \frac{f''}{f} - \frac{g''}{g} \geq 0 \implies h'g^2 \nearrow$$

It then follows that $h(0) > 0$ is a global minimum for h. So we conclude that $I_g \subseteq I_f$, which gives the result, as otherwise the limit of h as one approaches one of the boundary points would be 0, impossible. □

Proof of Lemma 14. Consider the pair $(f,g) = \left(H_\theta, \sin\left(\sqrt{1+\frac{1}{2(\min|H|)^2}}\theta\right)\right)$ as functions of θ. Then Lemma 16 applies since by the derived ODE, we see that

$$\inf_{H_\theta \neq 0} \frac{H_{\theta\theta\theta}}{H_\theta} = \inf_{H_\theta \neq 0} -\left[1 + \frac{1}{2H^2}\right] = -\left[1 + \frac{1}{2(\min|H|)^2}\right] = \sup_{g \neq 0} \frac{g_{\theta\theta}}{g}.$$

Hence for appropriately constructed intervals I_f and I_g, we have that $L(I_f) \geq L(I_g) = \frac{\pi}{\sqrt{1+\frac{1}{2(\min|H|)^2}}}$. But $L(I_f) = T_\Sigma$ so we obtain the lower bound. Similarly the upper bound follows by using the pair $(f,g) = \left(\sin\left(\sqrt{1+\frac{1}{2(\max|H|)^2}}\theta\right), H_\theta\right)$. □

7 The Embedded Σ

We are now ready to describing the space of possible embedded Σ. By Lemma 7, this becomes a problem of determining the behavior of T_Σ. Recall the formula given in Theorem 6:

$$T_\Sigma = \operatorname{sgn}(R_\Sigma - r_\Sigma) \int_{r_\Sigma}^{R_\Sigma} \frac{1}{\sqrt{4\log\left(\frac{\frac{u}{2}+C_\Sigma}{E_\Sigma}\right) - u^2}}\, \mathbf{d}u.$$

Ignoring order we thus obtain a symmetric function $T\colon \mathbb{R}^2 \setminus \{|x| = |y|\} \to \mathbb{R}$ given by

$$T(x,y) \coloneqq \frac{y-x}{|y-x|} \int_x^y \frac{1}{\sqrt{4\log\left(\frac{\frac{u}{2}+C_\Sigma}{E_\Sigma}\right) - u^2}}\, \mathbf{d}u$$

where

$$E_\Sigma = \frac{1}{2} \frac{y-x}{e^{\frac{y^2}{4}} - e^{\frac{x^2}{4}}}$$

$$C_\Sigma = -\frac{1}{2} \frac{ye^{-\frac{y^2}{4}} - xe^{-\frac{x^2}{4}}}{e^{-\frac{y^2}{4}} - e^{-\frac{x^2}{4}}}.$$

It is easy to check that T is smooth on the given domain. Moreover, we suspect that T can be extended to a smooth function defined on all of $\mathbb{R}^2$, as suggested by Corollary 15, though we will not need this full strength.

Our first task is to show that the differential of T is surjective everywhere, therefore implying that the topology of the space of embedded Σ should be fairly simple. We will accomplish this by merely showing that the derivative of T in the direction of $(1,1)$ is always non-zero, which also doubles to show that $|R_\Sigma - r_\Sigma|$ is useful as a possible parameterization. But first we will need some boring calculations.

Lemma 17. *If* $|R_\Sigma| \nearrow$ *and* $|R_\Sigma - r_\Sigma|$ *constant, then* $\max|H| \searrow$.

Proof. A matter of computation. Let t be a dummy variable for the derivative, then we have by the latter condition that $(R_\Sigma)_t = (r_\Sigma)_t$. Hence,

$$(\max|H|)_t = \left(\left|\frac{1}{2}R_\Sigma + C_\Sigma\right|\right)_t = \left(\left|\frac{1}{2}r_\Sigma + C_\Sigma\right|\right)_t \implies$$

$$(\max|H|)_t = \frac{e^{\frac{r_\Sigma^2}{4}}\left(\left|\frac{1}{2}R_\Sigma + C_\Sigma\right|\right)_t - e^{\frac{R_\Sigma^2}{4}}\left(\left|\frac{1}{2}r_\Sigma + C_\Sigma\right|\right)_t}{e^{\frac{r_\Sigma^2}{4}} - e^{\frac{R_\Sigma^2}{4}}}$$

$$= \frac{e^{\frac{r_\Sigma^2}{4}}\left(|E_\Sigma|e^{\frac{R_\Sigma^2}{4}}\right)_t - e^{\frac{R_\Sigma^2}{4}}\left(|E_\Sigma|e^{\frac{r_\Sigma^2}{4}}\right)_t}{e^{\frac{r_\Sigma^2}{4}} - e^{\frac{R_\Sigma^2}{4}}}$$

$$= e^{\frac{R_\Sigma^2 + r_\Sigma^2}{4}}|E_\Sigma|\left(\frac{\frac{R_\Sigma}{2}(R_\Sigma)_t - \frac{r_\Sigma}{2}(r_\Sigma)_t}{e^{\frac{r_\Sigma^2}{4}} - e^{\frac{R_\Sigma^2}{4}}}\right) = -e^{\frac{R_\Sigma^2 + r_\Sigma^2}{4}} E_\Sigma^2(|R_\Sigma|)_t,$$

where we use that $\operatorname{sgn}(E_\Sigma) = \operatorname{sgn}(R_\Sigma)$ to obtained the desired. $\square$

Lemma 18. *If $r_\Sigma R_\Sigma > 0$, $|R_\Sigma|$ constant and $|R_\Sigma - r_\Sigma| \nearrow$, then* $\max|H| \nearrow$.

Proof. For once assume $r_\Sigma, R_\Sigma > 0$. From $\max|H| = \left|\frac{1}{2}R_\Sigma + C_\Sigma\right|$ it suffices to show that $C_\Sigma \nearrow$. On the other hand, from Theorem 6, we have that $-2C_\Sigma$ is the slope of the line connecting the points $(x, f(x))$ for $f(x) = 2x\sqrt{-\ln x}$ and $x = e^{-\frac{R_\Sigma^2}{4}}, e^{-\frac{r_\Sigma^2}{4}}$. On the other hand $f''(x) = \frac{2\ln x - 1}{2x(\sqrt{-\ln x})^3} < 0$ for all $x < 1$, so the graph is concave down on this interval. Hence if $|R_\Sigma - r_\Sigma| \nearrow$ then $e^{-\frac{r_\Sigma^2}{4}} \nearrow$, and we know $e^{-\frac{R_\Sigma^2}{4}} < e^{-\frac{r_\Sigma^2}{4}} < 1$, so it follows that $-2C_\Sigma \searrow$ which suffices. □

Lemma 19. *If $|R_\Sigma| \nearrow$ and $|R_\Sigma - r_\Sigma|$ constant, then $T_\Sigma \searrow$.*

Proof. Again let t be a dummy variable for the derivative with $(R_\Sigma)_t = (r_\Sigma)_t$. Now recall our favorite function $f\colon V \subset \mathbb{R}^3 \to \mathbb{R}$ from the proof of Theorem 6 (Equation 4) defined by $(\langle x, \mathbf{n}\rangle, C_\Sigma, E_\Sigma) \overset{f}{\mapsto} \langle x, J(\mathbf{n})\rangle^2$. If the first input is made to vary in step with R_Σ and r_Σ, that is, $(\langle x, \mathbf{n}\rangle)_t = (R_\Sigma)_t$, then we compute that

$$f_t = 4\left[\frac{\frac{1}{2}(R_\Sigma)_t + (C_\Sigma)_t}{\frac{1}{2}\langle x, \mathbf{n}\rangle + C_\Sigma} - \frac{(E_\Sigma)_t}{E_\Sigma}\right] - 2\langle x, \mathbf{n}\rangle.$$

In particular, when treated as a function of $\langle x, \mathbf{n}\rangle$, we see that by Lemma 17,

$$\operatorname{sgn}(f_t'') = \operatorname{sgn}\left(\frac{1}{2}(R_\Sigma)_t + (C_\Sigma)_t\right) = \operatorname{sgn}((\max H)_t) = -\operatorname{sgn}(E_\Sigma) = -\operatorname{sgn}(R_\Sigma),$$

so we have convexity. Moreover, since $f(r_\Sigma, C_\Sigma, E_\Sigma) = f(R_\Sigma, C_\Sigma, E_\Sigma) = 0$ is constant, we have that $f_t(r_\Sigma) = f_t(R_\Sigma) = 0$, and hence for $\langle x, \mathbf{n}\rangle \in (r_\Sigma, R_\Sigma)$, $\operatorname{sgn}(f_t) = -\operatorname{sgn}(f_t'') = \operatorname{sgn}(R_\Sigma)$. So since

$$\begin{aligned}(T_\Sigma)_t &= \operatorname{sgn}(R_\Sigma - r_\Sigma)\left(\int_{r_\Sigma}^{R_\Sigma} f(u, C_\Sigma, E_\Sigma)^{-\frac{1}{2}}\,\mathbf{d}u\right)_t \\ &= \operatorname{sgn}(R_\Sigma - r_\Sigma)\int_{r_\Sigma}^{R_\Sigma} -\frac{1}{2}f(u, C_\Sigma, E_\Sigma)^{-\frac{3}{2}} f_t(u)\,\mathbf{d}u,\end{aligned}$$

it follows that $\operatorname{sgn}((T_\Sigma)_t) = -\operatorname{sgn}(R_\Sigma - r_\Sigma)\operatorname{sgn}(R_\Sigma) = -1$ as desired. □

Next, we combine the results of the previous two sections to obtain bounds for all $r_\Sigma R_\Sigma > 0$

Lemma 20. *Assuming $r_\sigma R_\Sigma > 0$, there exists a constant $K \in \mathbb{R}_{>0}$ such that*

$$g_\Sigma < K|R_\Sigma|^{\frac{1}{3}}(\log|R_\Sigma|)^{\frac{2}{3}}$$

Proof. Note that g_Σ is bounded when $|R_\Sigma|$ is, so we need only worry about the asymptotic behavior as $|R_\Sigma| \to \infty$ and that $|R_\Sigma|$ is sufficiently large. We split into two possible cases:

If $|R_\Sigma - r_\Sigma| \leq K_0 |R_\Sigma|^{-\frac{1}{3}} (\log|R_\Sigma|)^{\frac{1}{3}}$ for some constant $K_0 \in \mathbb{R}_{>0}$, then using Lemma 14 and a standard Taylor series bound for $\cos x$, we have that:

$$g_\Sigma = |r_\Sigma - R_\Sigma \cos(T_\Sigma)| \leq |R_\Sigma|(1 - \cos T_\Sigma) \leq \frac{1}{2}|R_\Sigma| T_\Sigma^2 \leq |R_\Sigma| \frac{\pi^2}{2 + \frac{1}{(\max|H|)^2}}.$$

By Lemma 18, we may assume that $|R_\Sigma - r_\Sigma| = K_0 |R_\Sigma|^{-\frac{1}{3}} (\log|R_\Sigma|)^{\frac{1}{3}}$. Further unpacking in terms of R_Σ and r_Σ gives:

$$\begin{aligned} g_\Sigma &\leq |R_\Sigma| \frac{\pi^2}{2 + \frac{1}{\left(\frac{R_\Sigma}{2} + C_\Sigma\right)^2}} = \frac{\pi^2}{4} |R_\Sigma| |R_\Sigma - r_\Sigma|^2 \left(1 + \frac{|R_\Sigma - r_\Sigma|^2}{2} - \frac{1}{e^{\frac{R_\Sigma^2 - r_\Sigma^2}{4}}}\right)^{-1} \\ &\leq \frac{\pi^2}{4} |R_\Sigma| |R_\Sigma - r_\Sigma|^2 = K |R_\Sigma|^{\frac{1}{3}} (\log|R_\Sigma|)^{\frac{2}{3}} \end{aligned}$$

for $|R_\Sigma|$ sufficiently large and some constant $K \in \mathbb{R}_{>0}$ as wanted.

Otherwise if $|R_\Sigma - r_\Sigma| > K_0 |R_\Sigma|^{-\frac{1}{3}} (\log|R_\Sigma|)^{\frac{1}{3}}$, then we get the result automatically by Corollary 13. □

Now as promised we can show that the lower bound on T_Σ in Lemma 10 doubles as an asymptotic estimate for T_Σ:

Corollary 21.

$$\lim_{|R_\Sigma| \to \infty} \cos(T_\Sigma) = \frac{r_\Sigma}{R_\Sigma}$$

Proof of Corollary 21. We have by the previous lemma that

$$\left| \frac{r_\Sigma}{R_\Sigma} - \cos(T_\Sigma) \right| = \frac{g_\Sigma}{|R_\Sigma|} < K \left(\frac{\log|R_\Sigma|}{|R_\Sigma|} \right)^{\frac{2}{3}},$$

for some $K \in \mathbb{R}_{>0}$, the latter of which goes to 0 as $|R_\Sigma| \to \infty$. □

We now prove Theorem 3 in full.

Proof of Theorem 3. Any equivalence class of $\frac{\mathrm{Emd}_n}{\sim}$ corresponds to a unique pair $(r_\Sigma, R_\Sigma) \in \{(x, y) \in \mathbb{R}^2 \colon x + y, y - x > 0\}$. This is a simply connected domain on which T is a smooth function, and furthermore Lemma 19 gives that T also has no critical points, so it follows that the level set $\{T = t\}$ is a smooth 1-dimensional subset of the domain. As defined in Theorem 6, $0 < T_\Sigma < \pi$, and we now show that any non-empty level set is parameterized by $y - x \in \mathbb{R}_{>0}$.

Fix $y - x = c$. According to Lemma 10 and Lemma 21 respectively,

$$\lim_{x \to 0^+} T\left(x - \frac{c}{2}, x + \frac{c}{2}\right) = \pi, \lim_{x \to \infty} T\left(x - \frac{c}{2}, x + \frac{c}{2}\right) = 0.$$

Therefore Lemma 19 provides for the existence of (x, y) with $y - x = c$ satisfying $T(x, y) = t$ any $t \in (0, \pi)$. This concludes the proof of the first property by choosing $t = \frac{\pi}{n}$.

For the second property observe that Corollary 15 provides an explicit continuous extension of T to $\{(x,x) \in \mathbb{R}^2_{>0}\}$. Hence the level set $\{T = t\}$ will include as a limit point $(2(\frac{\pi^2}{t^2} - 1), 2(\frac{\pi^2}{t^2} - 1))$. For $t = \frac{\pi}{n}$, this gives that as $|R_\Sigma - r_\Sigma| \to 0$, the members of Emd_n are bounded by the shrinking annulus $\{|r_\Sigma| < |x| < |R_\Sigma|\}$ with limit $S^1_{2(n^2-1)}$, which is sufficient to conclude the limit in the Gromov-Hausdorff sense.

For the third property we have that the Hausdorff distance between Σ and the regular polygon with vertices being those points of Σ farthest away from the origin is g_Σ. So now with a little bit of work once it is recognized that $|R_\Sigma| \geq \frac{1}{2}(|R_\Sigma| + |r_\Sigma|) \geq \frac{1}{2}|R_\Sigma - r_\Sigma|$, we have that by Corollary 21, since T_Σ is fixed, there exists constants M and K_1 so that for any $\Sigma \in \mathrm{Emd}_n$ with $|R_\Sigma - r_\Sigma| > M$, we have $|R_\Sigma - r_\Sigma| > K_1|R_\Sigma|$, and in particular Corollary 13 then provides that there exists a constant K_2 so that $g_\Sigma < K_2(\log|R_\Sigma|)|R_\Sigma|^{-1}$, which is sufficient to show $g_\Sigma \to 0$ as $|R_\Sigma - r_\Sigma| \to \infty$ as desired. □

In particular, the proof of Theorem 3 used very little of the embeddedness condition. Empirical evidence suggests that in fact the third property holds in a sense for any family of immersed solutions, but for that we do not have sufficiently strong bounds to prove with.

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