Combinatorial Results in Partition Theory

UROP+ Final Report

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Abstract

We prove a number of combinatorial results in partition theory, relating to the double Glaisher map introduced by Keith, and also expanding on work by Herden et al. on partitions with designated summands and the Andrews-Merca identity.

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1 Introduction

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n, denoted $\lambda \vdash n$, is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ with

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = n.$$

We can also express partitions in terms of their multiplicities: the partition

$$(1^{m_1}, 2^{m_2}, \ldots)$$

is the one in which each $i \geq 1$ appears m_i times. We use both notations in this report.

Many results in partition theory may be proved both analytically with the use of generating functions and with combinatorial methods. A famous and fundamental identity by Euler states that the number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.

Glaisher's generalization of this identity states that the number of partitions of n into parts not divisible by k is equal to the number of partitions of n where each part appears less than k times. The first class of partitions are called k-regular, and the second class are k-distinct.

We provide here a sketch of the bijective proof of this theorem. Consider a k-distinct partition λ. For each part $m = jk^\ell$, with $j, \ell \in \mathbb{Z}_{\geq 0}$, and $k \nmid j$, replace m with k^ℓ copies of j in the image. This maps λ to a k-regular partition. One may check that this map is invertible, and hence gives a bijection.

The body of this report is divided into three sections. In Section 2, we discuss the double Glaisher map introduced by Keith.

In [4], Keith generalizes the above bijection to a bijection between s-regular, t-distinct partitions and s-distinct, t-regular partitions. We prove a number of results regarding the generalized map, and relate the results to those proven by O'Hara [5].

In Section 3, we prove several congruences conjectured by Herden et al. in [2], and make some generalizations. These congruences relate to partitions with designated summands, in which one part of each size is designated by '. For instance, the partitions with designated summands of 4 are

$$(4'), (3', 1'), (2', 2), (2, 2'), (2', 1', 1), (2', 1, 1'), (1', 1, 1, 1), (1, 1', 1, 1), (1, 1, 1', 1), (1, 1, 1, 1').$$

In Section 4, we generalize another result of Herden et al. in [1] related to the Andrews-Merca identity. We then demonstrate a connection between our generalization and both the Andrews-Merca identity and the Beck identity of the first kind.

2 The Double Glaisher Map

2.1 Definitions and Overview

We first repeat the definition of the generalized Glaisher map given by Keith in [4]. Given a partition λ, the generalized Glaisher map ϕ_k is defined by the following procedure:

1. For each integer $m \geq 1$ with $k \nmid m$, define a matrix A_m so that the ℓth column of A_m is the base-k representation of the number of occurrences of mk^ℓ in λ (where the first row is 1, the second is k, and so on).
2. For each m, let A'_m be the transpose of A_m.
3. Undo the process in step 1 to get the partition $\phi_k(\lambda)$.

Note that by definition, ϕ_k is an involution. The matrices corresponding to each m are called *part-frequency matrices* by Keith.

Example. Consider the case where $k = 3$ and $\lambda = (18^4, 9^1, 6^2, 1^7)$. Then, we have two part-frequency matrices, for $m = 1$ and $m = 2$, shown in the first row below. Applying the procedure described above gives $\phi_3(\lambda) = (6^9, 3^2, 2^{15}, 1^{10})$.

$$\lambda : \quad \begin{array}{c|ccc} \mathbf{1} & & & \\ \hline & 1 & 0 & 1 \\ & 2 & 0 & 0 \\ & 0 & 0 & 0 \end{array} \qquad \begin{array}{c|ccc} \mathbf{2} & & & \\ \hline & 0 & 2 & 1 \\ & 0 & 0 & 1 \\ & 0 & 0 & 0 \end{array}$$

$$\phi_2(\lambda) : \quad \begin{array}{c|ccc} \mathbf{1} & & & \\ \hline & 1 & 2 & 0 \\ & 0 & 0 & 0 \\ & 1 & 0 & 0 \end{array} \qquad \begin{array}{c|ccc} \mathbf{2} & & & \\ \hline & 0 & 0 & 0 \\ & 2 & 0 & 0 \\ & 1 & 1 & 0 \end{array}$$

Figure 1: The top two matrices represent $\lambda = (18^4, 9^1, 6^2, 1^7)$. The bottom two matrices represent $\phi_3(\lambda) = (6^9, 3^2, 2^{15}, 1^{10})$. The number in the top-left corner of each matrix is m, and the matrices A_i

Recall that an s-regular partition is one in which no part is divisible by s, and a t-distinct partition is one in which every part appears less than t times. Throughout this section, we always take s and t to be integers greater than 1.

To motivate the definition of ϕ_k, consider the case where we begin with a k-regular partition λ. Then, only the first column of each A_m can contain non-zero values, and thus the transposes have only non-zero values in the first rows, corresponding exactly to k-distinct partitions. It is not hard to check that the generalized map ϕ_k in fact restricts to the original Glaisher bijection on the subset of k-regular partitions.

We also define the following set of partitions, for notational convenience.

Definition. Let $\mathcal{RD}_{s,t}(n)$ denote the set of all s-regular, t-distinct partitions of n.

2.1.1 Overview

As Keith shows in [3], when s and t are coprime, the composed map $\phi_s\phi_t$ immediately gives a bijection from s-regular, t-distinct partitions to s-distinct, t-regular partitions. If s and t are not coprime, the bijection is more involved, and Keith does this working individually with the primes dividing s and t.

In this section, we prove a number of properties resulting from iterating the double Glaisher map $\phi_s\phi_t$ on partitions $\lambda \in \mathcal{RD}_{s,t}(n)$, addressing in part the questions posed by Keith. We also make connections to work by O'Hara [5] regarding a general class of bijections between sets of partitions.

2.2 One Iteration of the Double Glaisher Map

Definition. For integers $s, t > 1$ and $k \in \mathbb{N}$, define $f_{s,t}(k)$ to be the smallest non-negative integer so that $s \mid t^{f_{s,t}(k)}k$. If no such integer exists, set $f_{s,t}(k) = +\infty$. Then, let $g_{s,t}(k) = t^{f_{s,t}(k)}$.

For example, we have $f_{6,10}(3) = 1$, and $f_{10,6}(3) = +\infty$.

Theorem 2.1. *Let $\lambda \vdash n$ be a partition that is both s-regular and t-regular. Then, $\phi_t(\lambda) \in \mathcal{RD}_{s,t}(n)$ if and only if for all $j \in \mathbb{N}$, j appears in λ less than $g_{s,t}(j)$ times.*

Proof. Since λ is t-regular, $\phi_t(\lambda)$ is t-distinct. Thus, $\phi_t(\lambda) \in \mathcal{RD}_{s,t}(n)$ if and only if it is s-regular. Note that if $s \mid j$, $g_{s,t}(j) = 1$, and by definition, j does not appear in λ.

Now, suppose some part k in $\phi_t(\lambda)$ is divisible by s. Then, write $k = t^\ell j$, with $t \nmid j$ and $\ell \in \mathbb{Z}$. Since $s \mid k$, we have $\ell \geq f_{s,t}(j)$. Since k appears in $\phi_t(\lambda)$, j must appear at least $t^\ell \geq g_{s,t}(j)$ times in λ.

The reverse argument holds as well. If j appears at least $g_{s,t}(j)$ times in λ, $t^m j$ must appear in $\phi_t(\lambda)$ for some $m \geq f_{s,t}(j)$, which implies $s \mid t^m j$ and $\phi_t(\lambda) \notin \mathcal{RD}_{s,t}(n)$. The result follows. ∎

Corollary 2.2. *Take any partition $\lambda \in \mathcal{RD}_{s,t}(n)$. Then, $\mu = \phi_s\phi_t(\lambda) \in \mathcal{RD}_{t,s}(n)$ if and only if for all $j \in \mathbb{N}$, j appears in $\phi_t(\lambda) = \phi_s(\mu)$ less than $\min(g_{s,t}(j), g_{t,s}(j))$ times.*

Proof. Note that for $\lambda \in \mathcal{RD}_{s,t}(n)$, $\phi_t(\lambda)$ is t-regular. Furthermore, every part of $\phi_t(\lambda)$ divides some part of λ, so since λ is s-regular, $\phi_t(\lambda)$ is as well.

So, for any $\lambda \in \mathcal{RD}_{s,t}(n)$, $\phi_t(\lambda)$ is s-regular, t-regular. Let $\delta = \phi_t(\lambda)$. Then, $\mu \in \mathcal{RD}_{t,s}(n)$ if and only if $\phi_s(\delta) \in \mathcal{RD}_{t,s}(n)$. By Theorem 2.1, this occurs if and only if any j appears less than $g_{t,s}(j)$ times in δ. Likewise, $\phi_t(\delta) \in \mathcal{RD}_{s,t}(n)$, so any j appears less than $g_{s,t}(j)$ times in δ. ■

Corollary 2.3. *The generating function for the number of $\lambda \in \mathcal{RD}_{s,t}(n)$ such that $\phi_s\phi_t(\lambda) \in \mathcal{RD}_{t,s}(n)$ is*

$$\prod_{j=1}^{\infty} \frac{1 - q^{j \min(f_{s,t}(j), f_{t,s}(j))}}{1 - q^j}.$$

Note that we take $q^{+\infty} = 0$ in the case where $f_{s,t}(j) = f_{t,s}(j) = +\infty$.

Corollary 2.4. *Suppose that $s = t^k$ or $t = s^k$ for some integer k. Then, $\phi_s\phi_t(\lambda) \in \mathcal{RD}_{t,s}(n)$ for all $\lambda \in \mathcal{RD}_{s,t}(n)$.*

Proof. It suffices to show that for any s-regular, t-regular partition μ, $\phi_t(\mu) \in \mathcal{RD}_{s,t}(n)$ if and only if $\phi_s(\mu) \in \mathcal{RD}_{t,s}(n)$.

Let $a = \min(s,t)$ and $b = \max(s,t)$. No multiples of a appear in μ, since μ is a-regular. For j with $a \nmid j$, we have $g_{s,t}(j) = g_{t,s}(j) = b$. By Theorem 2.1, this implies $\phi_t(\mu) \in \mathcal{RD}_{s,t}(n)$ if and only if $\phi_s(\mu) \in \mathcal{RD}_{t,s}(n)$, as desired. ■

In fact, as we show in Theorem 2.19, for any other s, t with $\gcd(s,t) > 1$, there exists some s-regular, t-distinct partition λ such that $\phi_s\phi_t(\lambda)$ is not both s-distinct and t-regular.

2.3 Cycles of the Double Glaisher Map

We first make a few observations about the behavior of the Glaisher map.

Lemma 2.5. *The Glaisher map ϕ_k is the composition of moves swapping k copies of an integer m for one copy of km, or vice versa.*

Proof. Consider the part-frequency matrices as used in the definition of the Glaisher map. Note that conjugation of the matrix moves the entry corresponding to k^a copies of jk^b to an entry corresponding to k^b copies of jk^a. It is not hard to see that this can be accomplished by repeatedly merging k equal parts or splitting multiples of k.

Since the Glaisher map consists of several of these moves, it can be performed only by merging and splitting. ∎

Note that $\phi_s\phi_t$ is invertible (its inverse is $\phi_t\phi_s$), and thus induces a permutation on partitions of n. Hence, $\phi_s\phi_t$ splits the partitions of n into several cycles. In particular, for any partition λ, we have $(\phi_s\phi_t)^k\lambda = \lambda$ for some $k > 0$.

Theorem 2.6. *Take $\lambda \in \mathcal{RD}_{s,t}(n)$. Suppose we make a series of moves as defined in Lemma 2.5 for ϕ_s and ϕ_t, starting with λ and ending at a partition $\mu \in \mathcal{RD}_{s,t}(n)$. Then, $\lambda = \mu$.*

Proof. For each $k \in \mathbb{N}$, define the set $S_k = \{ks^at^b \mid a, b \in \mathbb{Z}\} \cap \mathbb{Z}$. Note that the operations defined in Lemma 2.5 for ϕ_s and ϕ_t preserve the sets S_k, i.e. if we perform an operation on parts of size ks^at^b, the parts created will have sizes $ks^{a\pm1}t^b$ or $ks^at^{b\pm1}$, all of which are also in S_k.

Thus, it suffices to restrict our attention to the parts inside any fixed set S_k. If we show that λ and μ coincide on S_k for each k, this implies $\lambda = \mu$, as the S_k cover $\mathbb{N}$.

We split into two cases.

Case 1: $\log_s(t) \notin \mathbb{Q}$.

In this case, note that $s^at^b = 1$ for $a, b \in \mathbb{Z}$ if and only if $a = b = 0$. As described above, we need only consider parts in S_k for some fixed k.

Consider a coordinate grid, where the cell at (a, b) represents ks^at^b. The moves described Lemma 2.5 correspond to taking s from a cell and putting 1 in the cell to the right, or taking t from a cell and putting 1 in the cell above, or the inverse operations.

Now, note that if $(a, b) \in S_k$, all cells above and to the right of (a, b) are also in S_k. Furthermore, the elements of S_k not divisible by either s or t are exactly those cells which do not have an element of S_k either below or to the left. Call these cells *corners*.

First, $\phi_t(\lambda)$ is s-regular, t-regular, so all non-zero entries of $\phi_t(\lambda)$ are at corners. Since ϕ_t can be performed using the moves in Lemma 2.5, it suffices to consider going from $\phi_t(\lambda)$ to $\phi_t(\mu)$, both of which are s-regular and t-regular. For convenience, define $\lambda' = \phi_t(\lambda)$, $\mu' = \phi_t(\mu)$.

Consider the lowest element in the grid that appears in either λ' or μ'. Call this corner (x_1, y_1). Moving up the grid, call the next corner (x_2, y_2), and so on. Note that there may be infinitely many corners. If there are finitely many corners, we add a final "corner" with $y = +\infty$. These corners satisfy $x_1 > x_2 > \ldots$ and $y_1 < y_2 < \ldots$.

Example. We give an example of the grid in the diagram below, where we take $k = 48$, $s = 6$, and $t = 24$. The left diagram shows the partition $\lambda = (192^1, 12^1, 2^4, 3^{24})$ and the right diagram shows the partition $\mu = (192^1, 72^1, 12^1, 2^4)$, obtained from λ by making a move of the form described in Lemma 2.5.

The cells (a, b) in gray are the ones for which $ks^a t^b \notin \mathbb{Z}$. The corners in the region $-2 \leq x, y \leq 2$ are $(-2, 1)$, $(-1, 0)$, $(0, -1)$ and $(2, -2)$.

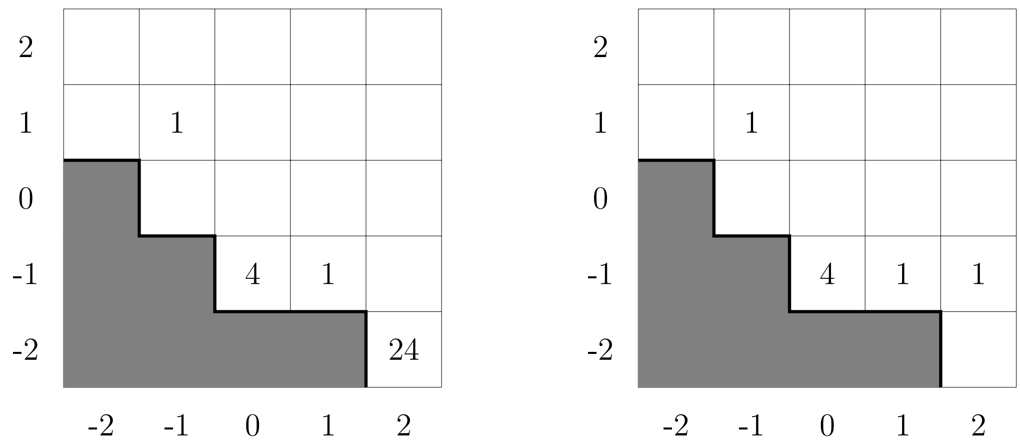

Figure 2: When $k = 48$, $s = 6$, $t = 24$, the left diagram is the partition $(192^1, 12^1, 2^4, 3^{24})$, and the right diagram is the partition $(192^1, 72^1, 12^1, 2^4)$.

Consider a corner (x_i, y_i). Note that the smallest $\ell > y_i$ for which $s \mid ks^{x_i} t^{\ell}$ is exactly y_{i+1}, if any such ℓ exists. So, in the notation from Theorem 2.1, we have

$$f_{s,t}(ks^{x_i} t^{y_i}) = y_{i+1} - y_i.$$

Note that this holds even in the case where $y_{i+1} = +\infty$. In all cases, by Theorem 2.1, the total value of the parts at (x_i, y_i) in both λ' and μ' is less than $ks^{x_i} t^{y_{i+1}}$.

Claim. *Suppose we make a series of moves of the form described in Lemma 2.5 for ϕ_s and ϕ_t from λ' to get another partition δ. Then, for each $i \geq 1$, the sum of all parts with $y < y_i$ does not decrease from λ' to δ.*

Proof. We induct on i. When $i = 1$, this holds, as the sum of parts below y_1 is 0 in λ'. Also, if $y_i = +\infty$, this is clear.

Now, suppose the claim holds for $i \leq \ell$, for some $\ell \geq 1$. Consider $i = \ell + 1$, and assume $y_{\ell+1} < +\infty$. Suppose the total of all parts below y_ℓ in λ' is equal to A, and the total of the parts at (x_ℓ, y_ℓ) in λ is equal to B. As shown above, $B < ks^{x_\ell} t^{y_{\ell+1}}$.

Consider the sum of all parts with $y < y_{\ell+1}$ throughout the process as we go from λ' to δ. Note that this sum changes if and only if we make a swap of the form

$$t \times (ks^x t^{y_{\ell+1}-1}) \leftrightarrow ks^x t^{y_{\ell+1}},$$

where $x \geq x_\ell$. In particular, the sum of these parts is constant modulo $ks^{x_\ell} t^{y_{\ell+1}}$.

Suppose the sum of parts with $y < y_{\ell+1}$ in δ is C. By the inductive hypothesis, we have $C \geq A$. The above argument shows

$$C \equiv A + B \pmod{ks^{x_\ell} t^{y_{\ell+1}}}.$$

Recall that $B < ks^{x_\ell} t^{y_{\ell+1}}$. Thus, we must have $C \geq A + B$. This completes the inductive step. So, for each y_i, the sum of all parts with $y < y_i$ in λ' is at most the sum in δ. $\square$

Now, since we can go from λ' to μ', the sum of parts with $y < y_i$ in λ' is at most the total of such parts in μ'. Since all moves are reversible, and μ' is also s-regular and t-regular, the reverse inequality holds as well.

So, for each y_i, the sum of all parts with $y < y_i$ in λ' is equal to the sum in μ'. The only parts in λ' and μ' with $y_i \leq y < y_{i+1}$ are at (x_i, y_i). Thus, for each i, the multiplicities of (x_i, y_i) in λ' and μ' are equal, so $\lambda' = \mu'$. Applying ϕ_t, we have $\lambda = \mu$.

Case 2: $\log_s(t) \in \mathbb{Q}$.

Suppose $s = m^a$, $t = m^b$, where $m, a, b \in \mathbb{N}$. Without loss of generality, suppose $\gcd(a, b) = 1$. (If $\gcd(a, b) = d$, we may simply replace m with m^d.)

Note that for any $k \in \mathbb{N}$, S_k is of the form $\{\ell, m\ell, m^2\ell, \dots\}$, where $\ell \in \mathbb{N}$. (Since we assume $\gcd(a, b) = 1$, some integer linear combination of a and b is 1, so $m\ell \in S_k$.)

Recall from the previous case that we may restrict ourselves to only one S_k. Since all the S_k have the same structure, without loss of generality, consider only S_1.

We use the part-frequency matrix in base m. Let (x, y) denote the entry in the column labelled m^x and row labelled m^y, i.e. the coefficient of m^y in the base m representation of the multiplicity of m^x.

Since λ and μ are s-regular and t-distinct, all non-zero entries for both λ and μ lie in the region $0 \leq x < a$, $0 \leq y < b$. Let $\mathcal{R}_{a,b}$ denote this region.

We define a labelling for cells (x, y), where $x + y \leq a + b - 2$, of the part-frequency matrix with labels from 0 to $a-1$ in the following way. Label the cell (x, y) with x for all $0 \leq x < a$, $0 \leq y < b$. Furthermore, impose the condition that the cell (x, y) has the same label as $(x - a, y + a)$ and $(x - b, y + b)$ for all (x, y) (whenever these are valid entries in the matrix).

In the diagram below, we exhibit the labelling for $a = 5$, $b = 8$.

	0	1	2	3	4	5	6	7	8	9	10	11
0	0	1	2	3	4	0	1	2	3	4	3	4
1	0	1	2	3	4	0	1	2	3	4	4	
2	0	1	2	3	4	0	1	2	3	4		
3	0	1	2	3	4	3	4	3	4			
4	0	1	2	3	4	3	4	4				
5	0	1	2	3	4	3	4					
6	0	1	2	3	4	4						
7	0	1	2	3	4							
8	3	4	3	4								
9	3	4	4									
10	3	4										
11	4											

Figure 3: Example labelling for $a = 5$, $b = 8$. The regions $\mathcal{R}_{a,b}$ and $\mathcal{R}_{b,a}$ are outlined.

Claim. *This labelling is consistent, i.e. it is not possible to go from any cell in the region $\mathcal{R}_{a,b}$ to any other cell using the equivalence relations described above.*

Proof. Note that the equivalence relations both fix $x+y$. Thus, we may restrict our attention to a fixed diagonal of the form $x + y = k$, with $0 \leq k \leq a + b - 2$. The x-values for the portion of the diagonal that lies inside the region $\mathcal{R}_{a,b}$ are

$$\max(0, k - b + 1) \leq x \leq \min(a - 1, k).$$

In the below argument, a "move" simply indicates moving by a or b along the diagonal, while keeping $x \geq 0$ and $y \geq 0$.

For contradiction, suppose some sequence of moves starts at $x = u$ and ends at $x = v$, where $u \neq v$ and

$$\max(0, k - b + 1) \leq u, v \leq \min(a - 1, k).$$

Note that for both u and v, the only legal move, if any, is to:

- increase by a if $a < b$,
- decrease by b if $a > b$.

When $a = b$, there are never any possible moves.

Suppose we have a sequence of moves

$$u = x_0 \to x_1 \to x_2 \to \cdots \to x_\ell = v.$$

Assume that this path is of minimal length. Then, $x_i \neq x_{i+2}$ for any i. Further, since $k \leq a + b - 2$, it is not possible to make consecutive moves by a and b in the same direction.

Thus, the possible moves after increasing by a are to increase by a again, or decrease by b. Likewise, possible moves after decreasing by b are to decrease by b again or increase by a. So, since the first move is either an increase by a or decrease by b, every move in the sequence is either an increase by a or decrease by b.

However, as shown above, the final move must either be a decrease by a or increase by b, contradiction. So, of the cells of the diagonal in the region $\mathcal{R}_{a,b}$, no cell can reach another. $\square$

Next, we show that all cells in the region $0 \leq x + y \leq a + b - 2$ have a uniquely defined label using the equivalence relations defined above.

Consider the graph with vertices labelled $0, 1, \ldots, k$, where two vertices u and v are connected by an edge if and only if $|u - v| \in \{a, b\}$. Since $k < a + b$, any vertex v has at most one neighbor in each of the sets $\{v - a, v + b\}$ and $\{v + a, v - b\}$. In particular, each vertex has degree at most 2.

Claim. *This graph is acyclic.*

Proof. Suppose we had some cycle with vertices $v_1, v_2, \ldots, v_\ell$ in that order, where v_1 and v_ℓ are adjacent, and all v_i are distinct. The argument in the previous claim shows that we may assume only one of the moves $+a$ or $-a$ appears as we move along the cycle. Without loss of generality, let this be $+a$ (by reversing the cycle if needed).

Then, again by the above argument, the only other possible moves are $-b$. Since $\gcd(a,b) = 1$, in order to make a series of such moves starting at v_1 and ending at v_1, we must have at least a moves of the form $-b$ and b moves of the form $+a$.

So, the cycle must have at least $a+b$ vertices. However, since we are on the diagonal $x+y=k$, there are only $k+1 < a+b$ vertices in the graph, contradiction. $\square$

Since every vertex has degree at most two, the connected components of the graph are paths, possibly of one isolated vertex.

Also, note that there are $\max(k-a+1,0)$ edges of the form $(v, v+a)$, and there are $\max(k-b+1,0)$ edges of the form $(v,v+b)$. Thus, the graph has $k+1$ vertices and $\max(k-a+1,0)+\max(k-b+1,0)$ edges. Each connected component with ℓ vertices has $\ell-1$ edges. So, the number of connected components in the graph is

$$k+1-\max(k-a+1,0)-\max(k-b+1,0).$$

Recall from the proof of the previous claim that x-values of the cells on the diagonal inside the region $\mathcal{R}_{a,b}$ is

$$\max(0,k-b+1) \leq x \leq \min(a-1,k).$$

In particular, there are

$$\min(a-1,k)-\max(0,k-b+1)+1$$

such cells. We have $k-\max(k-a+1,0)=\min(a-1,k)$, so the number of connected components is equal to the number of cells on the diagonal in $\mathcal{R}_{a,b}$.

By the previous claim, each cell in $\mathcal{R}_{a,b}$ is in a different connected component. So, every vertex is reachable from exactly one of the cells in $\mathcal{R}_{a,b}$, so every cell has a well-defined, unique label.

Claim. *For a cell (x,y) with label k, if $x+y<k+b-1$, $(x,y+1)$ also has label k.*

Proof. Since (x,y) has label k, there is a series of moves from $(k,x+y-k)$ to (x,y). By assumption, $x+y-k<b-1$, so $(k,x+y-k+1)$ also has label k.

By making the same series of moves (shifted by one), we can reach the cell $(x,y+1)$ from $(k,x+y-k+1)$, so $(x,y+1)$ also has label k. $\square$

Note that for any k, the cells (x,y) with label k all satisfy $x+y \leq k+b-1$.

Claim. *Fix $k<a-1$. Consider the cells (x,y) with label k such that $x+y=k+b-1$. For all such cells, the cell $(x,y+1)$ has the same label as (k,b).*

Proof. Consider two cells (x_1, y_1) and (x_2, y_2) such that $x_1 + y_1 = k + b - 1 = x_2 + y_2$ and both cells have label k. Then, as shown above, some series of moves connects (x_1, y_1) and (x_2, y_2). Using this same series of moves (shifted by one), we can go from $(x_1, y_1 + 1)$ to $(x_2, y_2 + 1)$. So, these two cells have the same label.

In particular, since $(k, b-1)$ has label k, this is the label of the cell (k, b). ☐

Now, define a directed graph on vertices labelled $0, 1, \ldots, a-1$. For each $v < a-1$, there is exactly one outgoing edge from v, to the vertex that is the label of (v, b). Call this label $f(v)$. Note that $f(v) > v$, since all cells (x, y) with labels at most v satisfy $x + y < v + b$.

For instance, referring to Fig. 3, in the case where $a = 5$ and $b = 8$, we have

$$f(0) = 3, f(1) = 4, f(2) = 3, f(3) = 4.$$

Since all cells with $x + y = a + b - 2$ have label $a - 1$, there is a directed path from v to $a - 1$ for all v. So, since the graph has $a - 1$ edges, it must be a tree, with all edges directed towards $a - 1$. Furthermore, note that for any edge $u \to v$, we have $u < v$.

Claim. *Take $\lambda \in \mathcal{RD}_{s,t}$. Suppose a series of moves as described in Lemma 2.5 sends λ to a partition λ'. Then, for all $0 \leq i \leq a-1$, the total value of all cells with label i or that are above cells with label i does not decrease from λ to λ'.*

We define the value of the cell (x, y) to simply be m^{x+y}. To compute the total value of cells with label i, sum over all cells of label i, multiplying the value of each cell by the number in that cell. Note that we do not even require the entries in the part-frequency matrix to remain less than m.

Example. If we take $m = 2$, both of the below matrices would be possible intermediate steps, corresponding to the partition $\lambda = (4^7, 2^4, 1^6)$

$$\begin{array}{c|ccc} & & & \\ \hline & 2 & 4 & 3 \\ & 2 & 0 & 0 \\ & 0 & 0 & 1 \end{array} \qquad \begin{array}{c|ccc} & & & \\ \hline & 6 & 0 & 1 \\ & 0 & 0 & 3 \\ & 0 & 1 & 0 \end{array}$$

Figure 4: Two different matrices with $m = 2$, both representing the partition $(4^7, 2^4, 1^6)$.

Proof. We induct on i. First, consider $i = 0$. The set of cells with label 0 is closed under moves along a diagonal by a or b, by construction.

Since no other labels lie above a cell with label 0, the only way for the sum of entries in cells of label 0 to change is to move within a column. Specifically, some cell directly below a cell with label 0 must decrease by 1, and the cell with label 0 above increases by m, or the cell below increases by 1 and the cell above decreases by m.

Each of these operations changes the sum of parts with label 0 by m^b. So, the total value of all parts with label 0 is constant modulo m^b. Since it is initially less than m^b, it does not decrease.

We now prove the inductive step. The argument is similar. Suppose the claim holds for all $i < k$. Consider $i = k$.

Again, by construction, the set of cells with label k or above cells with label k is closed under moves along diagonals. Since we include all cells above cells with label k, the only way for this sum to change is the same as in the base case – a cell below a cell with label k must decrease by 1, and the cell with label k increases by m, or the cell below increases by 1 and the cell above decreases by m.

So, the sum in these cells is constant modulo m^{k+b}. Now, consider the sum in cells not labelled k which lie above cells labelled k. Recall that all these labels must be less than k, since the edges in the directed graph constructed above all go from smaller to larger labels.

Let A be the sum in cells not labelled k which are above cells labelled k in λ, and let B be the sum in cells labelled k in λ. Also, let C be the sum over all the cells covered by either A or B in λ'. The inductive hypothesis implies $C \geq A$, and the above argument shows

$$C \equiv A + B \pmod{m^{k+b}}.$$

Since λ is t-distinct, $B < m^{k+b}$. So, $C \geq A + B$, which completes the induction. □

Thus, from λ to μ, the total value in cells with label i or lying above cells with label i does not decrease. The same argument shows that the sum does not decrease from μ to λ. So, for each i, the total value in these cells is equal in μ and λ.

It is easy to see that this implies the total value of cells with label i in λ and μ are equal, since no cell lies above a cell with strictly smaller label.

Since all non-zero entries initially are in $\mathcal{R}_{a,b}$, and all entries are less than m, this implies $\lambda = \mu$. ■

Corollary 2.7. *Take $\lambda \in \mathcal{RD}_{s,t}(n)$. Then, $(\phi_s\phi_t)^k\lambda \in \mathcal{RD}_{s,t}$ if and only if $\lambda = (\phi_s\phi_t)^k\lambda$.*

2.3.1 Special Choices of s and t

In this section, we describe the behavior of the map $\phi_s\phi_t$ in the case where $s \mid t$, and especially focus on the case $\log_s(t) \in \mathbb{Q}$.

We first describe how ϕ_{m^k} acts on the part-frequency matrix with base m.

Lemma 2.8. *The map ϕ_{m^k} swaps the entries at (x_1k+y_1, x_2k+y_2) and (x_2k+y_1, x_1k+y_2), where x_1, x_2, y_1, y_2 are non-negative integers, and $y_1, y_2 < k$.*

Proof. Consider the part-frequency matrix for m^k that contains the part m^{y_1}. The coefficient of $(m^k)^{x_2}$ in the number of occurrences $m^{x_1k+y_1}$ is encoded in the part-frequency matrix for m in the cells $(x_1k + y_1, x_2k)$ to $(x_1k + y_1, x_2k + k - 1)$. Thus, when we transpose the base m^k matrix, we move the cells $(x_1k + y_1, x_2k), \ldots, (x_1k + y_1, x_2k + k - 1)$ to the cells $(x_2k + y_1, x_1k), \ldots, (x_2k + y_1, x_1k + k - 1)$, in that order.

In particular, we move $(x_1k + y_1, x_2k + y_2)$ to $(x_2k + y_1, x_1k + y_2)$. □

Corollary 2.9. *Suppose that $s = t^k$ or $t = s^k$ for some integer k. Then, for any partition λ,*

$$(\phi_s\phi_t)^2\lambda = \lambda.$$

Proof. It suffices to consider when $s = t^k$, since $(\phi_s\phi_t)^{-1} = \phi_t\phi_s$.

Consider a part-frequency matrix in base t, and consider the entry starting at the cell $(x_1k + y_1, x_2k + y_2)$, where $0 \leq y_1, y_2 < k$. By Lemma 2.8, this entry moves as follows:

$$\begin{aligned}(x_1k + y_1, x_2k + y_2) &\xrightarrow{\phi_t} (x_2k + y_2, x_1k + y_1), \xrightarrow{\phi_s} (x_1k + y_2, x_2k + y_1)\\ &\xrightarrow{\phi_t} (x_2k + y_1, x_1k + y_2) \xrightarrow{\phi_s} (x_1k + y_1, x_2k + y_2).\end{aligned}$$

Thus, $(\phi_s\phi_t)^2$ moves each entry back to its original position. So, it sends λ to itself. ■

We now turn our attention to the more general situation where $s \mid t$ or $t \mid s$.

Theorem 2.10. *Suppose $s \mid t$ or $t \mid s$, and for some k,*

$$(\phi_s\phi_t)^k\lambda = \mu,$$

with $\lambda \in \mathcal{RD}_{s,t}(n)$ and $\mu \in \mathcal{RD}_{t,s}(n)$. Then,

$$(\phi_s\phi_t)^{2k}\lambda = \lambda.$$

Proof. Consider the case where $s \mid t$. The other case follows by swapping s and t and switching the roles of λ and μ.

Since $s \mid t$, λ is t-regular, so $\phi_t(\lambda) = \lambda$. Also, $s < t$, so μ is t-distinct. Thus, $\phi_t(\mu) = \mu$. So, we have

$$(\phi_t\phi_s)^k\lambda = \phi_t(\phi_s\phi_t)^k\lambda = \phi_t\mu = \mu.$$

Thus, $\lambda = (\phi_s\phi_t)^k\mu$, so

$$(\phi_s\phi_t)^{2k}\lambda = \lambda. \qquad \blacksquare$$

Using both Lemma 2.8 and Theorem 2.10, we can in fact describe the set of partitions $\lambda \in \mathcal{RD}_{s,t}(n)$ which are mapped to a partition in $\mathcal{RD}_{t,s}(n)$ by repeatedly applying $\phi_s\phi_t$ in the case where $\log_s(t) \in \mathbb{Q}$.

In what follows, take $s = m^a$, $t = m^b$, for integers $m \geq 2$, and a, b, with $\gcd(a, b) = 1$. As in the proof of Theorem 2.6, we restrict ourselves to parts in S_1.

We first prove some preliminary lemmas.

Lemma 2.11. *Suppose we have a partition $\lambda = ((m^k)^{m^\ell}) \in \mathcal{RD}_{s,t}(n)$ where $k + \ell < a + b$. Then, any number of applications of ϕ_s or ϕ_t send λ to a partition of the form*

$$((m^u)^{m^{k+\ell-u}}),$$

where $\min(u, k - \ell - u) < \min(a, b)$.

Proof. We show this by induction. It suffices to show that for any partition $\mu = ((m^u)^{m^v})$ with $\min(u, v) < \min(a, b)$ and $u + v < a + b$, both $\phi_s(\mu)$ and $\phi_t(\mu)$ are also of this form.

By Lemma 2.8, the map ϕ_{m^k} sends the cell $(x_1k + y_1, x_2k + y_2)$ in the part-frequency matrix with base m to the cell $(x_2k + y_1, x_1k + y_2)$, where $0 \leq y_1, y_2 < k$.

In particular, when we apply $\phi_{\min(s,t)}$, the induction hypothesis means either $x_1 = 0$ or $x_2 = 0$, so in the image, one of the coordinates is less than $\min(a, b)$.

When we apply $\phi_{\max(s,t)}$, we either have $x_1 = x_2 = 0$, in which case nothing happens, or exactly one of the x_i is 1 and the other is 0. In the latter case, one of the coordinates will be at least $\max(a, b)$ in the image, and hence the other coordinate is less than $\min(a, b)$, as we assume $u + v < a + b$.

This completes the induction. $\blacksquare$

Now, consider the labelling used in the proof of Theorem 2.6.

Lemma 2.12. *For any diagonal $x + y = k$, where $0 \leq k \leq a + b - 2$, the labels of cells in $\mathcal{R}_{b,a}$ on this diagonal are all distinct, and is the same as the set of labels of cells on this diagonal in $\mathcal{R}_{a,b}$.*

Proof. By symmetry over the main diagonal, the diagonal $x + y = k$ has an equal number of cells in both $\mathcal{R}_{a,b}$ and $\mathcal{R}_{b,a}$. The argument from the proof of Theorem 2.6 that shows all cells in $\mathcal{R}_{a,b}$ have different labels also shows that all cells in $\mathcal{R}_{b,a}$ have different labels.

Thus, since the only possible labels are those that occur in the cells in $\mathcal{R}_{a,b}$ (from the way we defined the labelling), the two sets of labels are the same. ■

Lemma 2.13. *For any label ℓ and diagonal $x + y = k$, where $0 \le k \le a + b - 2$, the number of cells on the diagonal with label ℓ and $x < \min(a, b)$ is equal to the number of cells on the diagonal with label ℓ and $y < \min(a, b)$.*

Proof. If ℓ does not appear on the diagonal at all, this is clear.

Now, suppose ℓ appears on the diagonal. Then, some cell $(x_0, y_0) \in \mathcal{R}_{a,b}$ has label ℓ, where $x_0 + y_0 = k$. If $(x_0, y_0) \in \mathcal{R}_{b,a}$, then (x_0, y_0) is the only cell on this diagonal with label ℓ, and the result follows.

If not, start from (x_0, y_0), and alternate applying $\phi_{\min(s,t)}$ and $\phi_{\max(s,t)}$, starting with $\phi_{\min(s,t)}$. By considering the adjacency graph used in the proof of Theorem 2.6, it is not hard to see that we do not visit any cell twice in this process unless we reach a cell in $\mathcal{R}_{b,a}$, which is fixed by $\phi_{\max(s,t)}$. Thus, the process stops when we reach the cell on the diagonal $x + y = k$ in the region $\mathcal{R}_{b,a}$ that has label k.

In fact, appealing again to the graph, this process will visit every cell on the diagonal with label ℓ where either coordinate is less than $\min(s, t)$, since each connected component of the graph is a path.

Also, note that the final (non-identity) step is $\phi_{\min(s,t)}$, since $\phi_{\max(s,t)}$ fixes all cells in $\mathcal{R}_{b,a}$. Hence, there are an odd number of moves, so there are an even number of cells visited.

It is not hard to see that the only cells with $x < \min(a, b)$ or $y < \min(a, b)$ that are fixed by either ϕ_s or ϕ_t are those in $\mathcal{R}_{a,b}$ or $\mathcal{R}_{b,a}$. Thus, each application of ϕ_s or ϕ_t is a non-trivial move, and the cells visited are all distinct.

Furthermore, the coordinate which is less than $\min(a, b)$ alternates between the x-coordinate and the y-coordinate. Thus, since this sequence of cells has an even number of cells, and alternates between $x < \min(a, b)$ and $y < \min(a, b)$, there are an equal number of cells on this diagonal with label ℓ in the regions $x < \min(a, b)$ and $y < \min(a, b)$. ■

Let $c(\ell, k)$ be the number of cells on the diagonal $x+y = k$ with label ℓ such that $x < \min(a, b)$ and $y \ge \min(a, b)$. Then, let $d(u, v) = c(u, u + v)$.

Example. Referring back to the labelling from Fig. 3, if we again take the example $a = 5$, $b = 8$, we have

$$d(3, 4) = c(3, 7) = 0,$$

and

$$d(3, 5) = c(3, 8) = 2.$$

The following corollary then follows immediately from the above argument.

Corollary 2.14. *Suppose $0 \le \ell < a$, and $\ell \le k < \ell + b$. Then, if $\lambda = ((m^u)^{m^v})$, with $\lambda \in \mathcal{RD}_{s,t}(m^{u+v})$ (or equivalently $(u,v) \in \mathcal{R}_{a,b}$), we have*

$$(\phi_s\phi_t)^{d(u,v)}\lambda \in \mathcal{RD}_{t,s}(m^{u+v}).$$

This corollary allows us to completely characterize the $\lambda \in \mathcal{RD}_{s,t}(n)$ that are mapped to $\mathcal{RD}_{t,s}(n)$ by repeatedly applying $\phi_s\phi_t$.

Theorem 2.15. *Consider a partition $\lambda \in \mathcal{RD}_{s,t}(n)$. Over all part-frequency matrices with base m, let $\mathcal{R}(\lambda)$ be the set of cells with non-zero entries in at least one matrix. Let $d(\lambda)$ be the set of values $d(u,v)$ over all cells $(u,v) \in \mathcal{R}(\lambda)$.*

Then, λ maps to $\mathcal{RD}_{t,s}(n)$ under some number of iterations of $\phi_s\phi_t$, i.e. $(\phi_s\phi_t)^k\lambda \in \mathcal{RD}_{t,s}(n)$ for some k, if and only if all non-zero elements of $d(\lambda)$ have equal ν_2, where ν_2 is the 2-adic valuation.

Proof. By Corollary 2.14, each individual cell $(u,v) \in \mathcal{R}_{a,b}$ is mapped to $\mathcal{R}_{b,a}$ in $d(u,v)$ steps. Thus, by Theorem 2.10, the integers k for which $(\phi_s\phi_t)^k((m^u)^{m^v})$ is s-distinct and t-regular are exactly those that are $d(u,v)$ modulo $2d(u,v)$, when $d(u,v) \neq 0$.

Thus, $(\phi_s\phi_t)^k\lambda \in \mathcal{RD}_{t,s}(n)$ if and only if for each $0 \neq d \in d(\lambda)$, we have $k \equiv d \pmod{2d}$. Such a solution exists if and only if $\nu_2(d)$ is constant over all $0 \neq d \in d(\lambda)$. ∎

2.3.2 Connection to O'Hara's Algorithm

As Keith notes in [4], O'Hara [5] gives an algorithm that implies a bijection between $\mathcal{RD}_{s,t}$ and $\mathcal{RD}_{t,s}$. We first explicitly describe this bijection.

With the notation used by O'Hara, we define two sieve-equivalent families of pairwise-disjoint multisets. For each $n > 0$:

- if $st \mid n$, define $A_n = B_n = \{n\}$,
- if $s \mid n$ and $st \nmid n$, define $A_n = \{n\}$, and $B_n = \left\{\frac{n}{s}, \ldots, \frac{n}{s}\right\}$, where there are s copies of $\frac{n}{s}$,
- if $s \nmid n$, define $A_n = \{n, \ldots, n\}$, where there are t copies of n, and $B_n = \{tn\}$.

By definition, both the A_i and B_i are pairwise disjoint families of multisets. Also, for each i, A_i and B_i have the same sum. Thus, the two families are sieve-equivalent.

Now, A_0 consists of all the s-regular, t-distinct partitions, and B_0 consists of all the s-distinct, t-regular partitions. So, applying O'Hara's algorithm, which repeatedly replaces multisets A_i appearing in the partition with the corresponding B_i, we get a bijection from $\mathcal{RD}_{s,t}(n)$ to $\mathcal{RD}_{t,s}(n)$.

Theorem 2.16. *If $(\phi_s\phi_t)^k(\lambda) = \mu$ for $\lambda \in \mathcal{RD}_{s,t}(n)$ and $\mu \in \mathcal{RD}_{t,s}(n)$, then μ is also the image of λ in O'Hara's bijection.*

Proof. Let μ' be the image of λ from O'Hara's bijection. By Lemma 2.5, we can go from μ to λ using only moves that merge s or t equal parts, or do the reverse.

The swaps in O'Hara's algorithm are also of this form, so we can then go from λ to μ' using moves of this form. So, we can go from μ to μ' using only moves that merge s or t equal parts, or the reverse. By Theorem 2.6, this implies $\mu = \mu'$. ■

Consider now Lemma 2 from [5], which, in this case, implies that given any partition $\lambda \vdash n$, there is at most one partition in $\mathcal{RD}_{s,t}(n)$ which can be reached by swapping multisets A_i for B_i or vice-versa.

This is similar to Theorem 2.6, if we take $\lambda \in \mathcal{RD}_{s,t}(n)$. In fact, in the second case from the proof of our theorem, where $\log_s(t) \in \mathbb{Q}$, it is exactly equivalent.

Since we start with an s-regular, t-distinct partition, the sum of parts in S_1 of λ is less than

$$\sum_{i=0}^{a-1}(t-1)(m^i) < tm^a = st.$$

So, we will never reach any multiple of st. Thus, any reachable partitions using our moves are also reachable making swaps of the form defined by O'Hara.

In particular, we only ever merge t copies of parts that are not divisible by s, or s copies of parts that are not divisible by t, or perform the inverse operations. To complete the argument, take $\lambda \in \mathcal{RD}_{s,t}(n)$. Since λ is reachable from itself (by doing nothing), no other $\mu \in \mathcal{RD}_{s,t}(n)$ is reachable.

However, in the first case from the proof, our result is more general. For instance, if we take $s = 6$ and $t = 4$, we may start from the partition $\lambda = (16^1, 4^2) \in \mathcal{RD}_{s,t}(24)$. Then, in our proof, we allow the sequence of operations

$$(16^1, 4^2) \to (4^6) \to (24^1).$$

However, $24 = st$ is never reachable using swaps in O'Hara's setup, since multiples of st are always fixed.

In fact, for some $\lambda \in \mathcal{RD}_{s,t}(n)$, it is possible to reach a partition in $\mathcal{RD}_{t,s}(n)$ with iterations of $\phi_s\phi_t$ even when reaching intermediate partitions unreachable from O'Hara's algorithm. As an explicit example, we may take the case of $s = 4$, $t = 6$, and $\lambda = (6^4, 3^4, 1^2) \in \mathcal{RD}_{4,6}(38)$, for which

$$(\phi_s\phi_t)^3(\lambda) = (24^1, 2^6, 1^2),$$

which contains $24 = st$ as a part, and

$$(\phi_s\phi_t)^{10}(\lambda) = (16^1, 8^1, 4^2, 2^2, 1^2) \in \mathcal{RD}_{6,4}(38).$$

2.4 Small Values of n

We now characterize the behavior of $\phi_s\phi_t$ on $\mathcal{RD}_{s,t}(n)$ for $n \leq \operatorname{lcm}(s,t) + \min(s,t)$.

Definition. Let $\mathcal{P}(n)$ be the set of all partitions of n, and let $p(n) = |\mathcal{P}(n)|$.

In each of the three theorems below, let $s, t > 1$ be integers such that $s \neq t^k$ and $t \neq s^k$ for any integer k and $\gcd(s,t) > 1$.

Theorem 2.17. *If $n < \operatorname{lcm}(s,t)$, $\phi_s\phi_t(\lambda) \in \mathcal{RD}_{t,s}(n)$ for all $\lambda \in \mathcal{RD}_{s,t}(n)$.*

Proof of Theorem 2.17. For $\lambda \in \mathcal{RD}_{s,t}(n)$, $\phi_t(\lambda)$ is both s-regular and t-regular. So, $\phi_s\phi_t(\lambda)$ is s-distinct.

In particular, $\mu = \phi_s\phi_t(\lambda) \notin \mathcal{RD}_{t,s}(n)$ if and only if μ is not t-regular. Since $\phi_t(\lambda)$ is s-regular, ϕ_s repeatedly merges s equal parts. So, $\phi_s\phi_t(\lambda)$ is not t-regular if and only if this merging creates a part that is a multiple of t. Note that any part created by merging must also be a multiple of s, and thus any multiple of t is at least $\operatorname{lcm}(s,t)$.

So, when $n < \operatorname{lcm}(s,t)$, all parts of $\phi_s\phi_t(\lambda)$ are less than $\operatorname{lcm}(s,t)$, and thus $\phi_s\phi_t(\lambda)$ is t-regular. ■

Theorem 2.18. *If $\operatorname{lcm}(s,t) \leq n < \operatorname{lcm}(s,t) + \min(s,t)$, all but $p(n - \operatorname{lcm}(s,t))$ partitions $\lambda \in \mathcal{RD}_{s,t}(n)$ satisfy $\phi_s\phi_t(\lambda) \in \mathcal{RD}_{t,s}(n)$.*

Furthermore, the $\lambda \in \mathcal{RD}_{s,t}(n)$ that fail are exactly the partitions such that $\phi_s\phi_t(\lambda) = (\operatorname{lcm}(s,t)^1) \cup \mu$, where $\mu \in \mathcal{P}(n - \operatorname{lcm}(s,t))$. For these λ, we have $(\phi_s\phi_t)^2\lambda \in \mathcal{RD}_{s,t}(n)$.

Proof of Theorem 2.18. Suppose $\operatorname{lcm}(s,t) \leq n < \operatorname{lcm}(s,t) + \min(s,t)$. As shown above, $\operatorname{lcm}(s,t)$ (or some multiple) must appear in $\phi_s\phi_t(\lambda)$ if $\phi_s\phi_t(\lambda) \notin \mathcal{RD}_{t,s}(n)$. Since $\min(s,t) < \operatorname{lcm}(s,t)$, $\operatorname{lcm}(s,t)$ must appear in $\phi_s\phi_t(\lambda)$ with multiplicity exactly 1.

Let s^ℓ and t^k be the largest powers of s and t, respectively, that divide $\operatorname{lcm}(s,t)$.

Claim. *For any $\mu \vdash n - \operatorname{lcm}(s,t)$, $\phi_s\phi_t(\operatorname{lcm}(s,t)^1 \cup \mu) \in \mathcal{RD}_{t,s}(n)$.*

Proof. Since $n - \operatorname{lcm}(s,t) < t$, all parts of μ are less than t, and μ is t-distinct. So,

$$\phi_t(\operatorname{lcm}(s,t)^1 \cup \mu) = \left(\frac{\operatorname{lcm}(s,t)}{t^k}\right)^{t^k} \cup \mu.$$

This partition is t-regular, since t^k is the largest power of t dividing $\operatorname{lcm}(s,t)$. It is also s-regular, since $\operatorname{lcm}(s,t) < st$, so $\frac{\operatorname{lcm}(s,t)}{t^k} < s$. Thus, $\phi_s\phi_t(\operatorname{lcm}(s,t)^1 \cup \mu)$ is s-distinct.

We now show it is also t-regular. Note that the only part of $\phi_t(\operatorname{lcm}(s,t)^1 \cup \mu)$ that can appear s or more times is $\frac{\operatorname{lcm}(s,t)}{t^k}$. Also, $\phi_t(\operatorname{lcm}(s,t)^1 \cup \mu)$ is t-regular, since $\operatorname{lcm}(s,t)^1 \cup \mu$ is t-distinct.

By the same argument in the proof of Theorem 2.17, $\phi_s\phi_t(\operatorname{lcm}(s,t)^1 \cup \mu)$ is not t-regular if and only if it contains $\operatorname{lcm}(s,t)$ as a part. Since $\frac{\operatorname{lcm}(s,t)}{t^k}$ is the only part that is merged when we apply ϕ_s, this is possible only if

$$s^m \cdot \frac{\operatorname{lcm}(s,t)}{t^k} = \operatorname{lcm}(s,t)$$

for some m. So, $\log_s(t) \in \mathbb{Q}$. In particular, either $s \mid t$ or $t \mid s$. Rearranging the above equality, we have

$$\frac{\operatorname{lcm}(s,t)}{t^k} = \frac{\operatorname{lcm}(s,t)}{s^m}.$$

If $s \mid t$, then the LHS is equal to 1, so $t = s^m$. Similarly, if $t \mid s$, $s = t^k$. Both these cases are impossible, by assumption. So, $\phi_s\phi_t(\operatorname{lcm}(s,t) \cup \mu) \in \mathcal{RD}_{t,s}(n)$. □

Note that the partitions $\lambda \in \mathcal{RD}_{s,t}(n)$ such that $\phi_s\phi_t(\lambda) \notin \mathcal{RD}_{t,s}(n)$ are exactly those with $\phi_s\phi_t(\lambda) = \operatorname{lcm}(s,t)^1 \cup \mu$, for some $\mu \vdash n - \operatorname{lcm}(s,t)$.

By the above claim (swapping s and t), every such μ gives such a λ. Furthermore, for each such λ, we have

$$(\phi_s\phi_t)^2\lambda = \phi_s\phi_t(\operatorname{lcm}(s,t)^1 \cup \mu) \in \mathcal{RD}_{t,s}.$$

Finally, since there are exactly $p(n - \operatorname{lcm}(s,t))$ choices of μ, there are exactly $p(n - \operatorname{lcm}(s,t))$ choices of λ where $\phi_s\phi_t(\lambda) \notin \mathcal{RD}_{t,s}(n)$. ■

Theorem 2.19. *When $n = \operatorname{lcm}(s,t) + \min(s,t)$, all but $p(\min(s,t)) - 1$ partitions $\lambda \in \mathcal{RD}_{s,t}(n)$ satisfy $\phi_s\phi_t(\lambda) \in \mathcal{RD}_{t,s}(n)$.*

The partitions λ such that $\phi_s\phi_t(\lambda) = (\operatorname{lcm}(s,t)^1) \cup \mu$, for $\mu \in \mathcal{P}(\min(s,t))$ and $\mu \neq (\min(s,t)^1), (1^{\min(s,t)})$ satisfy $(\phi_s\phi_t)^2\lambda \in \mathcal{RD}_{s,t}(n)$.

Define $\mu_1 = (\operatorname{lcm}(s,t)^1) \cup (1^{\min(s,t)})$ and $\mu_2 = (\operatorname{lcm}(s,t)^1) \cup (\min(s,t)^1)$. Then,

- *If $s < t$, $\phi_s\phi_t(\lambda) = \mu_1$ is impossible, and when $\lambda = \phi_t\phi_s(\mu_2)$, $(\phi_s\phi_t)^k\lambda \notin \mathcal{RD}_{t,s}(n)$ for any k.*

- *If $s > t$, $\phi_s\phi_t(\lambda) = \mu_2$ is impossible, and when $\lambda = \phi_t\phi_s(\mu_1)$, $(\phi_s\phi_t)^k\lambda \notin \mathcal{RD}_{t,s}(n)$ for any k.*

In particular, there is exactly one $\lambda \in \mathcal{RD}_{s,t}(n)$ for which iterating $\phi_s\phi_t$ fails to send λ to a partition in $\mathcal{RD}_{t,s}(n)$. For this λ, we have $(\phi_s\phi_t)^4\lambda = \lambda$.

Remark. We can alternatively describe the partition $\lambda \in \mathcal{RD}_{s,t}(\mathrm{lcm}(s,t) + \min(s,t))$ that does not map to $\mathcal{RD}_{t,s}(\mathrm{lcm}(s,t) + \min(s,t))$ as

$$\phi_t\left(\left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup 1^{\min(s,t)}\right),$$

where s^ℓ is the largest power of s dividing $\mathrm{lcm}(s,t)$.

Proof of Theorem 2.19. We use the same notation as in the above proofs.

In this case, $n = \mathrm{lcm}(s,t) + \min(s,t)$. As in the proofs of the previous parts, if $\mathrm{lcm}(s,t)$ does not appear in $\phi_s\phi_t(\lambda)$, then $\phi_s\phi_t(\lambda) \in \mathcal{RD}_{t,s}(n)$.

Furthermore, if $\phi_s\phi_t(\lambda) = \mathrm{lcm}(s,t)^1 \cup \mu$, where μ is not equal to either $\min(s,t)^1$ or $1^{\min(s,t)}$, then μ is s-regular, t-regular, and $\min(s,t)$-distinct, and the argument from the proof of Theorem 2.18 still works. In particular, in these cases, $(\phi_s\phi_t)^2\lambda \in \mathcal{RD}_{t,s}(n)$. There are $p(\min(s,t)) - 2$ such choices of μ, and hence $p(\min(s,t)) - 2$ such λ.

Now, recall that we defined above the partitions $\mu_1 = (\mathrm{lcm}(s,t)^1) \cup (1^{\min(s,t)})$ and $\mu_2 = (\mathrm{lcm}(s,t)^1) \cup (\min(s,t)^1)$. If $s > t$, we have

$$\phi_s(\mu_1) = \left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup 1^t,$$
$$\phi_s(\mu_2) = \left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup t^1.$$

Recall that $\gcd(s,t) > 1$, so $\frac{\mathrm{lcm}(s,t)}{s^\ell} < t$. Thus,

$$\phi_t\phi_s(\mu_2) = \phi_t\left(\left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup t^1\right) = \phi_t\left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup \phi_t(t^1) = \phi_t\left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup 1^t,$$

where the second equality follows since $\left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell}$ and t^1 do not have any common parts. Note that $\phi_t\phi_s(\mu_2)$ is not t-distinct, and thus $\phi_s\phi_t(\lambda) = \mu_2$ is not possible for $\lambda \in \mathcal{RD}_{s,t}(n)$.

We also have

$$\phi_t\phi_s(\mu_1) = \phi_t\left(\left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup 1^t\right).$$

Since $\phi_s(\mu_1)$ is t-regular, $\phi_t\phi_s(\mu_1)$ is t-distinct. Using a similar argument as in the proof of Theorem 2.18, $\phi_t\phi_s(\mu_1)$ cannot contain $\mathrm{lcm}(s,t)$ as a part, and is thus s-regular. So, $\phi_t\phi_s(\mu_1) \in \mathcal{RD}_{s,t}(n)$.

The case when $s < t$ is similar. In this case, we have

$$\phi_s(\mu_1) = \left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup s^1,$$
$$\phi_s(\mu_2) = \left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup 1^s,$$

so

$$\phi_t\phi_s(\mu_1) = \phi_t\left(\left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup s^1\right) = \phi_t\left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup \phi_t(s^1) = \phi_t\left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup s^1$$

and

$$\phi_t\phi_s(\mu_2) = \phi_t\left(\left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup 1^s\right).$$

Thus, $\phi_t\phi_s(\mu_1)$ is not s-regular, so $\phi_s\phi_t(\lambda) = \mu_1$ is impossible. Since $\frac{\mathrm{lcm}(s,t)}{s^\ell} < t$, $\phi_t\left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell}$ is t-distinct, so $\phi_t\phi_s(\mu_2)$ is also t-distinct, as $\frac{\mathrm{lcm}(s,t)}{s^\ell} \neq 1$. Note that $\phi_s(\mu_2)$ is s-regular, t-regular. Again, by the same argument as in Theorem 2.18, $\mathrm{lcm}(s,t)$ cannot appear in $\phi_t\phi_s(\mu_2)$, so $\phi_t\phi_s(\mu_2)$ is also s-regular.

Hence, $\phi_t\phi_s(\mu_2) \in \mathcal{RD}_{s,t}(n)$.

As mentioned in the remark, the λ mapping to the valid μ_i in both cases is equal to

$$\phi_t\left(\left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup 1^{\min(s,t)}\right) \in \mathcal{RD}_{s,t}(n).$$

We show that for this choice of λ, we have $(\phi_s\phi_t)^4\lambda = \lambda$, and $(\phi_s\phi_t)^m\lambda \notin \mathcal{RD}_{t,s}(n)$ for any m. First, if $\lambda \in \mathcal{RD}_{t,s}(n)$, then λ is fixed by both ϕ_s and ϕ_t, which is clearly not the case, since $\mu_1, \mu_2 \notin \mathcal{RD}_{s,t}(n)$.

Next, we have

$$\begin{aligned}\phi_s\phi_t(\lambda) &= \phi_s\left(\left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup 1^{\min(s,t)}\right)\\ &= \phi_s\left(\left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell}\right) \cup \phi_s(1^{\min(s,t)})\\ &= \mathrm{lcm}(s,t)^1 \cup \phi_s(1^{\min(s,t)}).\end{aligned}$$

This is not t-regular, and hence is not in $\mathcal{RD}_{t,s}(n)$.

Continuing,

$$\begin{aligned}\phi_t(\mathrm{lcm}(s,t)^1 \cup \phi_s(1^{\min(s,t)})) &= \phi_t(\mathrm{lcm}(s,t)^1) \cup \phi_t\phi_s(1^{\min(s,t)})\\ &= \left(\frac{\mathrm{lcm}(s,t)}{t^k}\right)^{t^k} \cup \min(s,t)^1.\end{aligned}$$

So, we have

$$(\phi_s\phi_t)^2\lambda = \phi_s\left(\left(\frac{\mathrm{lcm}(s,t)}{t^k}\right)^{t^k} \cup \min(s,t)^1\right).$$

Note that the argument from above with μ_1 and μ_2 shows that $(\phi_s\phi_t)^2\lambda \notin \mathcal{RD}_{t,s}(n)$ (where we need to swap the roles of s and t in the proof).

Claim. *The partitions λ and $(\phi_s\phi_t)^2\lambda$ are both $\max(s,t)$-distinct and $\max(s,t)$-regular.*

Proof. First, consider when $s < t$. Recall that

$$\lambda = \phi_t\left(\left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup 1^{\min(s,t)}\right).$$

We show that

$$\left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup (1^{\min(s,t)}) = \left(\frac{\mathrm{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup 1^s$$

is t-regular and t-distinct. This then implies that λ is t-regular and t-distinct.

It suffices to show that $s^\ell < t$, since $t \nmid \frac{\mathrm{lcm}(s,t)}{s^\ell}$. If $s \mid t$, then $\mathrm{lcm}(s,t) = t$. Since t is not a power of s, $s^\ell < t$. If $s \nmid t$, then

$$\frac{\mathrm{lcm}(s,t)}{s} = \frac{t}{\gcd(s,t)},$$

which is coprime with $\frac{s}{\gcd(s,t)} > 1$. So, $s^2 \nmid \mathrm{lcm}(s,t)$ in this case, so $s^\ell = s < t$. Thus, λ is t-distinct and t-regular.

Note that $\mathrm{lcm}(s,t) < st < t^2$, so $k = 1$. Recall that

$$\begin{aligned}
(\phi_s\phi_t)^2\lambda &= \phi_s\left(\left(\frac{\mathrm{lcm}(s,t)}{t^k}\right)^{t^k} \cup \min(s,t)^1\right) \\
&= \phi_s\left(\left(\frac{\mathrm{lcm}(s,t)}{t}\right)^{t} \cup s^1\right) \\
&= \phi_s\left(\frac{\mathrm{lcm}(s,t)}{t}\right)^{t} \cup \phi_s(s^1) \\
&= \phi_s\left(\frac{\mathrm{lcm}(s,t)}{t}\right)^{t} \cup 1^s.
\end{aligned}$$

As shown above, this is t-regular, since it does not contain $\mathrm{lcm}(s,t)$ as a part. Thus, it suffices to show that this is t-distinct. If $s \mid t$, this is equal to $\phi_s(1^t) \cup 1^s$. Since $s \mid t$, the multiplicity of 1 in this partition is equal to s, and the multiplicity of any other part is at most $\frac{t}{s}$.

If $s \nmid t$, $\frac{\mathrm{lcm}(s,t)}{t} \neq 1$. Thus, since $s < t$, ϕ_s merges some number of copies of $\frac{\mathrm{lcm}(s,t)}{t}$, and the partition is again t-distinct. So, in both cases, $(\phi_s\phi_t)^2\lambda$ is t-regular and t-distinct.

We now consider the case where $s > t$. The arguments are similar.

We have

$$\lambda = \phi_t\left(\left(\frac{\mathrm{lcm}(s,t)}{s}\right)^{s} \cup 1^t\right).$$

If $t \mid s$, this equals

$$\phi_t(1^{s+t}).$$

As above, $s = \mathrm{lcm}(s,t)$ cannot appear in this partition, so this is s-regular. Since $t \mid s$ and $1 < t < s$, all parts have multiplicity at most $\frac{s+t}{t} < s$, so this partition is s-distinct as well.

If $t \nmid s$, $\frac{\mathrm{lcm}(s,t)}{s} \neq 1$, so

$$\lambda = \phi_t\left(\frac{\mathrm{lcm}(s,t)}{s}\right)^{s} \cup \phi_t(1^t) = \phi_t\left(\frac{\mathrm{lcm}(s,t)}{s}\right)^{s} \cup t^1.$$

Since $t < s$, ϕ_t merges some number of copies of $\frac{\mathrm{lcm}(s,t)}{s}$, and the partition is again s-distinct. So, λ is s-regular and s-distinct.

For $(\phi_s\phi_t)^2\lambda$, we have

$$(\phi_s\phi_t)^2\lambda = \phi_s\left(\left(\frac{\operatorname{lcm}(s,t)}{t^k}\right)^{t^k} \cup t^1\right),$$

so it suffices to show that

$$\left(\frac{\operatorname{lcm}(s,t)}{t^k}\right)^{t^k} \cup t^1$$

is s-regular and s-distinct.

By a similar argument as in the other case, we have $t^k < s$, so this partition is s-distinct. Since $\frac{\operatorname{lcm}(s,t)}{t^k}, t < s$, it is also s-regular, as desired. □

We now show that $(\phi_s\phi_t)^4\lambda = \lambda$. The argument is similar to the one used in the proof of Theorem 2.10. First, if $s < t$, then

$$(\phi_s\phi_t)^2\lambda = \phi_t\phi_s\phi_t\lambda = (\phi_t\phi_s)^2\lambda,$$

so $\lambda = (\phi_s\phi_t)^4\lambda$. Similarly, if $s > t$,

$$(\phi_s\phi_t)^2\lambda = \phi_t(\phi_s\phi_t)^2\lambda = (\phi_t\phi_s)^2\lambda,$$

and we again have $\lambda = (\phi_s\phi_t)^4\lambda$.

It now only remains to check that $(\phi_s\phi_t)^3\lambda \notin \mathcal{RD}_{t,s}(n)$. First, if $s > t$, then

$$(\phi_s\phi_t)^3\lambda = \phi_t\phi_s\lambda = \phi_t\lambda = \left(\frac{\operatorname{lcm}(s,t)}{s^\ell}\right)^{s^\ell} \cup 1^t,$$

which is not s-distinct.

If $s < t$,

$$(\phi_s\phi_t)^3\lambda = \phi_s(\phi_t\phi_s)^2\lambda = \phi_s(\phi_s\phi_t)^2\lambda = \left(\frac{\operatorname{lcm}(s,t)}{t^k}\right)^{t^k} \cup s^1,$$

which is again not s-distinct, since $t > s$.

So, $(\phi_s\phi_t)^m\lambda \notin \mathcal{RD}_{t,s}(n)$ for any m, as desired. ■

3 Partitions with Designated Summands

We prove the congruences conjectured by Herden et al. at the end of [2], and present some more general statements.

3.1 Notation

We use the same notation as in [2].

Definition. Let $\mathrm{PD}_k(n)$ be the number of partitions with designated summands of n where no part is divisible by k.

Recall that a partition with designated summands is one in which exactly one part of each size is marked with '. For example, $(4', 2, 2', 1', 1, 1)$ is counted by $\mathrm{PD}_3(11)$, but $(3', 3, 2', 1, 1', 1)$ is not.

Using this notation, we can now state the congruences listed in Conjecture 1 of [2].

Theorem 3.1 (Conjecture 1, [2])**.** *For $n \geq 0$, the following hold:*

$$
\begin{aligned}
\mathrm{PD}_2(16n+12) &\equiv 0 \pmod 4,\\
\mathrm{PD}_2(24n+20) &\equiv 0 \pmod 4,\\
\mathrm{PD}_2(25n+5) &\equiv 0 \pmod 4,\\
\mathrm{PD}_2(32n+24) &\equiv 0 \pmod 4,\\
\mathrm{PD}_2(48n+26) &\equiv 0 \pmod 4,\\
\mathrm{PD}_9(54n+3r) &\equiv 0 \pmod 3, \ \textit{for } r \in \{5, 11, 15, 17\}.
\end{aligned}
$$

The results presented below are more or less direct consequences of Theorems 10, 11, and 19 given by Herden et al.

We first prove a generalization encompassing the first and fourth congruences, then a generalization of both the second and fifth congruences. Finally, we will prove the third and sixth congruences separately, and give a slight generalization of the third congruence.

3.2 Proofs of Congruences

3.2.1 First and Fourth Congruences

Theorem 3.2. *Let n be a positive integer that is not the sum of two perfect squares. Equivalently, $\nu_p(n)$ is odd for some $p \equiv 3 \pmod 4$. Then,*

$$\mathrm{PD}_2(n) \equiv 0 \pmod 4.$$

Proof. We first consider the case where n is even. Write $n = 2m$. By Theorem 11 of [2],

$$\sum_{n\geq 0} \mathrm{PD}_2(2m)q^m \equiv 1 + 2\sum_{k\geq 1,\ 3\nmid k} q^{k^2} + \sum_{k,\ell\geq 1,\ 3\nmid k,\ell} q^{k^2+\ell^2} \pmod 4.$$

Note that since $m = \frac{n}{2}$, $\nu_p(m)$ is odd for some $p \equiv 3 \pmod 4$, so m is also not the sum of two squares. Thus, the coefficient of q^m in the above series is 0, as desired.

Now, consider the case when n is odd. Write $n = 2m+1$. By Theorem 10 of [2], $\mathrm{PD}_2(n)$ is congruent modulo 4 to the number of solutions to

$$m = 3j(3j-1) + 3k(3k-1).$$

Rearranging, this equation becomes

$$2n = 4m + 2 = 36j^2 - 12j + 1 + 36k^2 - 12k + 1 = (6j-1)^2 + (6k-1)^2.$$

Since n is not the sum of two squares, $2n$ is not either, so $\mathrm{PD}_2(n) \equiv 0 \pmod 4$ in this case as well. ■

As a corollary, we obtain Theorem 1.2 of Sellers from [6].

Corollary 3.3 (Theorem 1.2, [6])**.** *For all $\alpha \geq 0$ and all $n \geq 0$,*

$$\mathrm{PD}_2(2^\alpha(4n+3)) \equiv 0 \pmod 4.$$

Taking the special cases $\alpha = 2$ and $\alpha = 3$ give the first and fourth congruences conjectured by Herden et al., respectively.

3.2.2 Second and Fifth Congruences

Theorem 3.4. *Let n be a positive integer such that n is not a perfect square, and $n \not\equiv 2 \pmod 3$. Then,*

$$\mathrm{PD}_2(2n) \equiv 0 \pmod 4.$$

Proof. We again use Theorem 11 from [2], which states that

$$\sum_{n\geq 0} \mathrm{PD}_2(2n)q^n \equiv 1 + 2\sum_{k\geq 1,\ 3\nmid k} q^{k^2} + \sum_{k,\ell\geq 1,\ 3\nmid k,\ell} q^{k^2+\ell^2} \pmod 4.$$

Since n is not a perfect square, the first summation does not contain the term q^n. Also, if $3 \nmid k, \ell$, we must have $k^2 + \ell^2 \equiv 2 \pmod 3$, so the second summation also does not contain a term of the form q^n.

Thus, the coefficient of q^n is 0 mod 4, as desired. ∎

This theorem implies the second and fifth congruences conjectured in [2].

Corollary 3.5. *We have*

$$\mathrm{PD}_2(24n + 20) \equiv 0 \pmod 4,$$
$$\mathrm{PD}_2(48n + 26) \equiv 0 \pmod 4.$$

Proof. Note that $12n + 10, 24n + 13 \not\equiv 2 \pmod 3$. Also, 2 and 5 are not quadratic residues modulo 4 and 8, respectively, so $12n + 10$ and $24n + 13$ are not squares. Both results then follow from Theorem 3.4. ∎

3.2.3 Third Congruence

We now prove the third congruence:

$$\mathrm{PD}_2(25n + 5) \equiv 0 \pmod 4.$$

We first prove a preliminary lemma.

Lemma 3.6. *Let n be a positive integer. The number of solutions to $n = k^2+\ell^2$ with $k, \ell \in \mathbb{Z}$ is*

- *0 if $\nu_p(n)$ is odd for any $p \equiv 3 \pmod 4$,*

- $4\prod_{p\equiv 1 \pmod 4}(\nu_p(n)+1)$ *otherwise, where the product is over primes* $p \equiv 1 \pmod 4$*.*

Proof. Since -1 is not a quadratic residue modulo any prime $p \equiv 3 \pmod 4$, we have $p \mid k^2+\ell^2$ if and only if $p \mid k, \ell$ when $p \equiv 3 \pmod 4$. Thus, $\nu_p(n)$ must be even for all $p \equiv 3 \pmod 4$.

Now, consider the factorization of n in $\mathbb{Z}[i]$, up to units. Recall that $\mathbb{Z}[i]$ is a UFD, and the irreducible elements (up to units) are exactly the primes $p \in \mathbb{Z}$ with $p \equiv 3 \pmod 4$, $\pi_2 = 1+i$, and pairs of conjugates π_p, $\overline{\pi_p}$, where $p \equiv 1 \pmod 4$ and $|\pi_p| = |\overline{\pi_p}| = \sqrt{p}$.

Note that there is a bijection between solutions (k, ℓ) and elements $k+\ell i \in \mathbb{Z}[i]$ with $|k+\ell i| = \sqrt{n}$. For each such α with $|\alpha| = \sqrt{n}$, we have $\alpha\overline{\alpha} = n$. Thus, each irreducible factor appears equally often on both sides.

Thus, for each $p \equiv 3 \pmod 4$, p must appear in the factorization of α exactly $\frac{1}{2}\nu_p(n)$ times. Likewise, π_2 must appear $\nu_2(n)$ times. For $p \equiv 1 \pmod 4$, there are $\nu_p(n)+1$ possibilities for the factors π_p, $\overline{\pi_p}$, namely

$$\pi_p^{\nu_p(n)}, \pi_p^{\nu_p(n)-1}\overline{\pi_p}, \ldots, \overline{\pi_p}^{\nu_p(n)}.$$

Since factorizations are unique only up to units, and the units in $\mathbb{Z}[i]$ are $\pm 1, \pm i$, we multiply by another factor of 4. This gives the desired formula. ■

Theorem 3.7. *For any prime* $p \equiv 5 \pmod{24}$ *and integer* $n \geq 0$*, we have*

$$\mathrm{PD}_2(p^2 n + p) \equiv 0 \pmod 4.$$

Note that taking $p = 5$ gives the third congruence.

Proof. We split into cases based on the parity of n.

Case 1: n is even.

Then, p^2n+p is odd. As shown above, by Theorem 10 of [2], $\mathrm{PD}_2(p^2n+p)$ is equivalent modulo 4 to the number of integer solutions to

$$2p(pn+1) = 2p^2n + 2p = (6k-1)^2 + (6\ell-1)^2.$$

If there are no solutions, we are immediately done. Otherwise, the LHS must be 2 mod 3. Since $p \equiv 2 \pmod 3$, this implies $pn+1 \equiv 2 \pmod 3$. Since n is even, $pn+1$ is odd. So, for some odd prime $q \neq p$, $\nu_q(pn+1)$ is odd. If $q \equiv 3 \pmod 4$, there are no solutions, so assume $q \equiv 1 \pmod 4$.

Then, since $2p(pn+1) \equiv 2 \pmod 4$, all solutions to $2p(pn+1) = a^2 + b^2$ have a, b both odd. Furthermore, for a fixed solution (a, b) to $a^2 + b^2 = 2p(pn+1)$, all of

$$(a, b), (a, -b), (-a, b), (-a, -b)$$

are also solutions, and exactly one pair has both elements congruent to -1 mod 6.

So, by Lemma 3.6, the number of solutions to

$$2p(pn+1) = (6k-1)^2 + (6\ell-1)^2,$$

is equal to $\prod_{r \equiv 1 \pmod 4} (\nu_r(2p^2n + 2p) + 1)$.

Note that taking $r = p$ and $r = q$ gives two even terms in the product, so the product is divisible by 4. Thus, $\mathrm{PD}_2(p^2n + p) \equiv 0 \pmod 4$.

Case 2: n is odd.

Then, $p^2n + p$ is even. Note that $\frac{p^2n+p}{2}$ is not a perfect square, since $\nu_p\left(\frac{p^2n+p}{2}\right) = 1$. Thus, again by Theorem 11 of [2], $\mathrm{PD}_2(p^2n + p)$ is equivalent modulo 4 to the number of solutions to $\frac{p^2n+p}{2} = k^2 + \ell^2$, where $3 \nmid k, \ell$ and k, ℓ are positive integers.

If there are no solutions, we are done. Otherwise, by Lemma 3.6, the number of solutions is

$$\prod_{r \equiv 1 \pmod 4} \left(\nu_r\left(\frac{p^2n+p}{2}\right) + 1\right) = \prod_{r \equiv 1 \pmod 4} (\nu_r(p^2n + p) + 1),$$

since we require k and ℓ to be positive and $k, \ell \neq 0$ for any solution (k, ℓ).

Since $p^2n + p = p(pn+1)$, $p \nmid pn + 1$, and $p \equiv 1 \pmod 4$, this product is even. If it is not divisible by 4, we must have $pn + 1 = a^2$ or $pn + 1 = 2a^2$ for some integer a, since $\nu_r(pn+1)$ is even for all $r \equiv 3 \pmod 4$.

However, recall that $3 \nmid k, \ell$, so $p^2n + p = 2(k^2 + \ell^2) \equiv 1 \pmod 3$. Thus, $pn + 1 \equiv 2 \pmod 3$, so $pn + 1$ is not a square.

It remains to show that $\frac{pn+1}{2}$ is not a square. Since $p \equiv 5 \pmod 8$,

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} = -1,$$

where $\left(\frac{\bullet}{p}\right)$ is the Legendre symbol. So, $\frac{1}{2}$ is not a quadratic residue modulo p. Hence, $\frac{pn+1}{2}$ is not a perfect square. Thus, for some odd prime $q \equiv 1 \pmod 4$ with $q \neq p$, we must have $\nu_q(p^2n + p)$ odd, which finishes. ■

3.2.4 Sixth Congruence

Finally, we prove the last conjectured congruence.

Theorem 3.8. *Let n be a non-negative integer. For $r \in \{5, 11, 15, 17\}$,*

$$\mathrm{PD}_9(54n + 3r) \equiv 0 \pmod 3.$$

Proof. By Theorem 19 of [2], $\mathrm{PD}_9(3n)$ is equivalent modulo 3 to the number of solutions to

$$n = k_0^2 + {k_0'}^2 + 3k_1^2 + 3{k_1'}^2,$$

where $k_m, k_m' \in \mathbb{Z}, \mathbb{N}$, when k_m, k_m' are even or odd, respectively.

Note that the number of odd k_m, k_m' has the same parity as n. In particular, for fixed n, the parity is fixed. Thus, allowing the k_m, k_m' to be any integers multiplies the number of solutions (when considered modulo 3)by 2^n.

So, it suffices to show that the number of solutions to

$$m = a^2 + b^2 + 3c^2 + 3d^2$$

is a multiple of 3, when $m \equiv 5, 11, 15, 17 \pmod{18}$ and $a, b, c, d \in \mathbb{Z}$.

In the case $r = 15$, we have $3 \mid a, b$. Writing $a = 3e$, $b = 3f$, this reduces to the equation

$$6n + 5 = c^2 + d^2 + 3e + 3f,$$

for some n. The cases $r = 5, 11, 17$ also give $m \equiv 5 \pmod 6$. Thus, it suffices to show that the number of solutions to the equation

$$m = a^2 + b^2 + 3c^2 + 3d^2$$

is divisible by 3 when $m \equiv 5 \pmod 6$.

We claim the number of solutions to the above equation is congruent modulo 3 to the number of ways to write $6n + 5$ as the sum of eight squares.

This follows by considering the generating functions and applying the Frobenius endomorphism. We instead present a combinatorial argument. Consider all solutions to

$$n = a_1^2 + \cdots + a_8^2,$$

and take the multisets $S_1 = \{a_1, a_2, a_3\}$, $S_2 = \{a_4, a_5, a_6\}$. Then, define the map

$$(a_1, \ldots, a_8) \mapsto (S_1, S_2, a_7, a_8).$$

Note that for any fixed choice of (S_1, S_2, a_7, a_8), the number of solutions $(a_1, \ldots, a_8)$ in the preimage is a multiple of 3 unless each of S_1 and S_2 contain three equal elements. Furthermore, in the case where S_1 and S_2 both contain three equal elements, there is exactly one solution in the preimage. This proves the congruence.

Now, a result of Jacobi, stated in [7], states that the number of ways to express a positive integer m as the sum of eight squares is

$$16 \sum_{d|m} (-1)^{m+d} d^3.$$

Setting $m = 6n + 5$, all factors of m are odd, so $(-1)^{m+d} = 1$ for all $d \mid m$.

Thus, we have

$$16 \sum_{d|m} (-1)^{m+d} d^3 \equiv \sum_{d|m} d^3 \equiv \sum_{d|m} d \pmod 3.$$

Since $m \equiv 2 \pmod 3$, there exists some $p \equiv 2 \pmod 3$ with $\nu_p(m)$ odd. So, $3 \mid \sigma(p^{\nu_p(m)})$. We also have $\sigma(p^{\nu_p(m)}) \mid \sigma(m)$, which finishes. ■

4 A Connection Between the Andrews-Merca Identity and the Beck Identity of the First Kind

4.1 Definitions and Statement of Result

We generalize some definitions by Herden et al. from [1].

Definition. Let $\mathcal{D}_k(n)$ denote the set of all k-distinct partitions of n and $\mathcal{O}_k(n)$ denote the set of all k-regular partitions of n.

Definition. Given two partitions $\lambda \vdash n$ and $\mu \vdash m$, let $\lambda \cup \mu \vdash m+n$ be the partition such that the multiplicity of any integer k in $\lambda \cup \mu$ is the sum of the multiplicities of k in λ and μ.

Definition. Let $a_{s,t}(n)$ be the total number of parts (with multiplicity) that are divisible by s, among all the t-distinct partitions of n.

For example, 3-distinct partitions of 5 are

$$(5), (4,1), (3,2), (3,1,1), (2,2,1),$$

so we have

$$a_{2,3}(5) = 0+1+1+0+2 = 4.$$

Definition. We define the following sets:

- $\mathcal{B}_{s,t,0}(n) = \{(\lambda, a, b) \mid a, b \in \mathbb{N},\ \lambda \in \mathcal{O}_t(n-ab),\ s \mid a,\ t \mid b\}$,
- $\mathcal{B}_{s,t,1}(n) = \{(\lambda, a, b) \mid a, b \in \mathbb{N},\ \lambda \in \mathcal{O}_t(n-ab),\ s \mid a,\ t \nmid b\}$,
- $\mathcal{C}_{s,t,0}(n) = \{(\lambda, a, b) \mid a, b \in \mathbb{N},\ \lambda \in \mathcal{D}_t(n-ab),\ t \mid a,\ s \mid b\}$,
- $\mathcal{C}_{s,t,1}(n) = \{(\lambda, a, b) \mid a, b \in \mathbb{N},\ \lambda \in \mathcal{D}_t(n-ab),\ t \nmid a,\ s \mid b\}$.

Then, let $b_{s,t,i}(n) = |\mathcal{B}_{s,t,i}(n)|$ and $c_{s,t,i}(n) = |\mathcal{C}_{s,t,i}(n)|$ for $i \in \{0,1\}$. Also, define $\mathcal{B}_{s,t}(n) = \mathcal{B}_{s,t,0}(n) \cup \mathcal{B}_{s,t,1}(n)$, and $\mathcal{C}_{s,t}(n) = \mathcal{C}_{s,t,0}(n) \cup \mathcal{C}_{s,t,1}(n)$

With these definitions in mind, we now state the main result of this section, generalizing Theorem 2 of [1].

Theorem 4.1. *We have* $b_{s,t,0}(n) = c_{s,t,0}(n)$, $b_{s,t,1}(n) = c_{s,t,1}(n)$, *and*

$$a_{s,t}(n) = b_{s,t,1}(n) - (t-1)b_{s,t,0}(n) = c_{s,t,1}(n) - (t-1)c_{s,t,0}(n).$$

We show below that this theorem specializes not only to the Andrews-Merca identity generalized by Herden et al., but also to a general form of the Beck identity of the first kind.

Example. Continuing with $s = 2$, $t = 3$, $n = 5$ from above, we have the sets

$$\begin{aligned}
&\mathcal{B}_{2,3,0}(5) = \emptyset,\\
&\mathcal{B}_{2,3,1}(5) = \{((2,1),2,1),((1,1,1),2,1),((1),2,2),((1),4,1)\},\\
&\mathcal{C}_{2,3,0}(5) = \emptyset,\\
&\mathcal{C}_{2,3,1}(5) = \{((2,1),1,2),((3),1,2),((1),2,2),((1),1,4)\},
\end{aligned}$$

for which $b_{s,t,1} = 4 = c_{s,t,1}$ and $b_{s,t,0} = 0 = c_{s,t,0}$.

4.2 Generating Function Proof

We first compute the generating function for $a_{s,t}$.

Theorem 4.2. *The generating function for $a_{s,t}(n)$ is*

$$\sum_{n=1}^{\infty} a_{s,t}(n)q^n = \frac{(q^t;q^t)_\infty}{(q;q)_\infty} \sum_{n=1}^{\infty} \left(\frac{q^{sn}}{1-q^{sn}} - \frac{tq^{stn}}{1-q^{stn}} \right),$$

where we use the q-Pochhammer symbol

$$(a;q)_\infty = \prod_{i=0}^{\infty} (1-aq^i).$$

Proof. We have

$$\begin{aligned}
\sum_{n=1}^{\infty} a_{s,t}(n)q^n &= \left(\prod_{s\nmid n, n\in\mathbb{N}} \frac{1-q^{tn}}{1-q^n} \right) \frac{\partial}{\partial z} \left(\prod_{m=1}^{\infty} (1 + zq^{sm} + \cdots + z^{t-1}q^{(t-1)sm}) \right) \Bigg|_{z=1}\\
&= \prod_{n=1}^{\infty} \frac{1-q^{tn}}{1-q^n} \cdot \sum_{m=1}^{\infty} \frac{\partial}{\partial z} \log(1 + zq^{sm} + \cdots + z^{t-1}q^{(t-1)sm}) \Big|_{z=1}\\
&= \frac{(q^t;q^t)_\infty}{(q;q)_\infty} \sum_{m=1}^{\infty} \frac{q^{sm} + 2q^{2sm} + \cdots + (t-1)q^{(t-1)sm}}{1 + q^{sm} + \cdots + q^{(t-1)sm}}\\
&= \frac{(q^t;q^t)_\infty}{(q;q)_\infty} \sum_{m=1}^{\infty} \frac{(1-q^m)(q^{sm} + 2q^{2sm} + \cdots + (t-1)q^{(t-1)sm})}{1-q^{stm}}\\
&= \frac{(q^t;q^t)_\infty}{(q;q)_\infty} \sum_{m=1}^{\infty} \frac{q^{sm} + q^{2sm} + \cdots + q^{(t-1)sm} - (t-1)q^{stm}}{1-q^{stm}}
\end{aligned}$$

$$= \frac{(q^t;q^t)_\infty}{(q;q)_\infty} \sum_{m=1}^{\infty} \left(\frac{q^{sm} + q^{2sm} + \cdots + q^{stm}}{1 - q^{stm}} - \frac{tq^{stm}}{1 - q^{stm}} \right)$$
$$= \frac{(q^t;q^t)_\infty}{(q;q)_\infty} \sum_{m=1}^{\infty} \left(\frac{q^{sm}}{1 - q^{sm}} - \frac{tq^{stm}}{1 - q^{stm}} \right). \qquad \blacksquare$$

Analytic Proof of Theorem 4.1. We have the generating functions

$$\sum_{n=1}^{\infty} (b_{s,t,0}(n) + b_{s,t,1}(n))q^n = \frac{(q^t;q^t)_\infty}{(q;q)_\infty} \sum_{m=1}^{\infty} \frac{q^{sm}}{1 - q^{sm}} = \sum_{n=1}^{\infty} (c_{s,t,0}(n) + c_{s,t,1}(n))q^n.$$

The first equality follows by taking $a = sm$ and choosing a t-regular partition. The second equality follows by taking $a = m$, and choosing a t-distinct partition.

By a similar argument, we have

$$\sum_{n=1}^{\infty} b_{s,t,0}(n)q^n = \frac{(q^t;q^t)_\infty}{(q;q)_\infty} \sum_{m=1}^{\infty} \frac{q^{stm}}{1 - q^{stm}} = \sum_{n=1}^{\infty} c_{s,t,0}(n)q^n.$$

The result follows. $\blacksquare$

4.3 Combinatorial Proof

We now give a combinatorial argument for Theorem 4.1. The argument is quite similar to the argument in section 3.3 of [1], but somewhat more direct.

Combinatorial proof of Theorem 4.1. We first show that $b_{s,t,i}(n) = c_{s,t,i}(n)$ for $i \in \{0, 1\}$. Fix a and b. Then, by Glaisher's theorem, $|\mathcal{O}_t(m)| = |\mathcal{D}_t(m)|$ for all m, so the number of elements counted by $\mathcal{B}_{s,t,i}$ whose last two entries are (a, b) is equal to the number of elements counted by $\mathcal{C}_{s,t,i}$ whose last two elements are (b, a). Summing over all (a, b) finishes.

We now show that $a_{s,t}(n) = c_{s,t,1}(n) - (t-1)c_{s,t,0}(n)$. For each tuple $(\lambda, a, b) \in \mathcal{C}_{s,t}(n)$ consider $f(\lambda, a, b) = \lambda \cup (b^a) \vdash n$.

Fix $\mu \vdash n$ and b. Consider all choices of λ and a such that $(\lambda, a, b) \in \mathcal{C}_{s,t}(n)$ and $f(\lambda, a, b) = \mu$. Such (λ, a, b) exist only if b appears in μ at least once. Furthermore, any $k \neq b$ must occur in μ less than t times.

Let ℓ be the multiplicity of b in μ, and write $\mu = \mu' \cup (b^\ell)$. Then, the set of $(\lambda, a, b) \in \mathcal{C}_{s,t}(n)$ with $f(\lambda, a, b) = \mu$ are exactly those such that $\lambda = \mu' \cup (b^i)$ and $a = \ell - i$, for each $0 \leq i \leq \max(\ell, t-1)$.

Now, if $\ell \geq t$, exactly one of these will be in $\mathcal{C}_{s,t,0}(n)$, since t divides exactly one element of the set $\{\ell - t + 1, \ldots, \ell\}$, and the others will be in $\mathcal{C}_{s,t,1}(n)$. Thus, the contribution from this pair (μ, b) to $c_{s,t,1}(n) - (t-1)c_{s,t,0}(n)$ is zero.

If $\ell < t$, all of these tuples are in $\mathcal{C}_{s,t,1}(n)$, and the contribution of this (μ, b) is exactly ℓ, which is the multiplicity of b in μ.

So, if (μ, b) has nonzero contribution, we have $\mu \in \mathcal{D}_t(n)$. Furthermore, summing over all (μ, b), the total contribution is exactly the number of multiples of s in partitions in $\mathcal{D}_t(n)$. Thus, we have

$$c_{s,t,1}(n) - (t-1)c_{s,t,0}(n) = a_{s,t}(n). \qquad \blacksquare$$

4.4 Special Cases

We now show that for certain choices of s and t, Theorem 4.1 implies general forms of both the Andrews-Merca identity and the Beck identity of the first kind.

4.4.1 Andrews-Merca Identity

Consider the case $s = t = k$. Then, given (λ, a, b) in any of the four classes $\mathcal{B}_{s,t,0}$, $\mathcal{B}_{s,t,1}$, $\mathcal{C}_{s,t,0}$, or $\mathcal{C}_{s,t,1}$, we can uniquely recover all three of λ, a, b from the partition $\lambda \cup (a^b)$, since there will either be exactly one part appearing more than k times or exactly one part that is divisible by k.

Furthermore, for (λ, a, b) in $\mathcal{B}_{k,k,i}$ or $\mathcal{C}_{k,k,i}$, we have $\lambda \cup (a^b)$ in $B_{k,i}$ or $C_{k,i}$, respectively, where we define $B_{k,i}$ and $C_{k,i}$ as in [1].

Thus, Theorem 4.1 specializes to Theorem 2 of [1] when $s = t$.

4.4.2 Beck Identity of the First Kind

Consider when $s = 1$. Then, $a_{1,t}(n)$ is the total number of parts in the t-distinct partitions of n. For each $(\lambda, a, b) \in \mathcal{B}_{1,t}(n)$, consider the partition $f(\lambda, a, b) = \lambda \cup (b^a)$.

For $(\lambda, a, b) \in \mathcal{B}_{1,t,0}(n)$, $f(\lambda, a, b)$ is a partition with exactly one multiple of t (possibly repeated). Further, given $f(\lambda, a, b)$, we can uniquely determine all three of λ, a, b, since b is the only part of $f(\lambda, a, b)$ divisible by t.

Furthermore, for any $\mu \vdash n$ with exactly one multiple of t, possibly repeated, there is exactly one choice of $(\lambda, a, b) \in \mathcal{B}_{1,t,0}(n)$ such that $f(\lambda, a, b) = \mu$. So, $b_{1,t,0}(n)$ is the number of partitions of n with exactly one multiple of t.

For $(\lambda, a, b) \in \mathcal{B}_{s,t,1}(n)$, we have $f(\lambda, a, b) \in \mathcal{O}_t(n)$.

Lemma 4.3. *For any $\mu \in \mathcal{O}_t(n)$, the number of $(\lambda, a, b) \in \mathcal{B}_{1,t,1}(n)$ with $f(\lambda, a, b) = \mu$ is equal to the number of parts of μ.*

Proof. Let $\mu = (\mu_1^{e_1}, \ldots, \mu_m^{e_m})$. Then, b must appear in μ. Say $b = \mu_i$ for some i. Since $\mu = \lambda \cup (b^a)$, $1 \le a \le e_i$.

For each such a, we can write $\mu = \mu' \cup (b^a)$ for a unique partition $\mu' \in \mathcal{O}_t(n - ab)$. Hence, each choice of a gives exactly one tuple $(\lambda, a, b) \in \mathcal{B}_{s,t,1}(n)$. Summing over all b, the number of possible (λ, a, b) is $e_1 + \cdots + e_m$, as desired. ■

Corollary 4.4. *The total number of parts in all t-regular partitions of n is equal to $b_{s,t,1}(n)$.*

We now make correspondences to the notation defined by Yang in [8]. Yang defines $\mathcal{O}_{1,k}(n)$ as the set of partitions of n with exactly one multiple of k and $\ell(\lambda)$ as the number of parts in a partition λ.

As shown above, we have $|\mathcal{B}_{1,t,0}(n)| = |\mathcal{O}_{1,t}(n)|$, and

$$b_{1,t,1}(n) = \sum_{\lambda \in \mathcal{O}_t(n)} \ell(\lambda).$$

Also, by definition,

$$a_{1,t}(n) = \sum_{\lambda \in \mathcal{D}_t(n)} \ell(\lambda).$$

By Theorem 4.1, we have $a_{1,t}(n) = b_{1,t,0} - (t-1)b_{1,t,1}$. Changing the notation, this is exactly

$$\sum_{\lambda \in \mathcal{D}_t(n)} \ell(\lambda) = \sum_{\lambda \in \mathcal{O}_t(n)} \ell(\lambda) - (t-1)\left|\mathcal{O}_{1,t}(n)\right|,$$

which rearranges to Theorem 1.5 of Yang.

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References

[1] Daniel Herden, et al. Counting the Parts Divisible by k in All the Partitions of n Whose Parts Have Multiplicity Less than k. *Integers*, 2022.

[2] Daniel Herden, et al. Partitions with Designated Summands Not Divisible by 2^ℓ, 2 and 3^ℓ Modulo 2, 4, and 3. *Integers*, volume 23, 2023. ISSN 1553-1732.

[3] William J. Keith. Partitions into parts simultaneously regular, distinct and/or flat. In *Combinatorial and Additive Number Theory II: CANT, New York, NY, USA, 2015 and 2016*, Springer Proceedings in Mathematics & Statistics. 2019.

[4] William J. Keith. A bijection for partitions simultaneously s-regular and t-distinct. *Integers*, volume 23, 2023. ISSN 1553-1732.

[5] Kathleen M O'Hara. Bijections for partition identities. *Journal of Combinatorial Theory, Series A*, volume 49(1):pages 13–25, September 1988. ISSN 0097-3165.

[6] James A. Sellers. New Infinite Families of Congruences Modulo Powers of 2 for 2–Regular Partitions with Designated Summands. 2023.

[7] Kenneth S. Williams. An Arithmetic Proof of Jacobi's Eight Squares Theorem. *Far East Journal of Mathematical Sciences*, volume 3:pages 1001–1005, 2001.

[8] Jane Y.X. Yang. Combinatorial proofs and generalizations of conjectures related to Euler's partition theorem. *European Journal of Combinatorics*, volume 76:pages 62–72, February 2019. ISSN 0195-6698.