Fine Mixed Subdivisions of a Dilated Triangle

Fedir Yudin
Mentor: Yuan Yao

September 4, 2023

Abstract

An upward equilateral triangle of side n can be partitioned into n unit upward equilateral triangles and $\frac{n(n-1)}{2}$ unit 60-120 rhombi. In this paper, we focus on understanding such partitions with a fixed arrangement of unit triangles. We formulate a criterion for such a partition being unique, identify a set of operations that connects all such partitions, and determine which parts are common to all such partitions. Additionally, we discuss an operation that connects all possible partitions with different arrangements of triangles.

1 Introduction

Consider an upward equilateral triangle of side n, divided into n^2 unit triangles. Suppose that we cut out n upward unit triangles. Note that the obtained region has an equal number of upward and downward triangles, so it's natural to ask whether it's tileable with unit 60-120 rhombi.

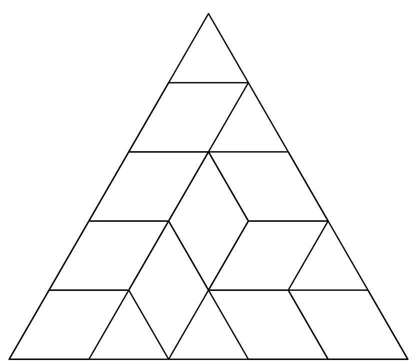


Figure 1: A fine mixed subdivision.

Sections 3-6 explore the structure of the set of tilings for a fixed arrangement of triangular holes, and Section 7 discusses operations that moves the holes and connect all possible tilings.

2 Preliminaries

Definition 2.1 ([3]). A *fine mixed subdivision* of an upward equilateral triangle of side length n is a partition into n upward unit triangles (holes) and $\binom{n}{2}$ unit 60-120 rhombi.

Suppose we're given a set of triangular holes and what to find out whether there is a fine mixed subdivision with this set of holes. Note that every upward equilateral triangle T of side k consists of k more upward triangles than downward triangles. Also, note that A rhombus covers an equal number of upward and downward triangles in T if it does not intersect the boundary of T, and covers one upward triangle if it intersects the boundary of T. Hence, there are at most k triangular holes that lie in T. Turns out, if this condition is satisfied for every T, then the desired fine mixed subdivision exists.

Definition 2.2. We will say that an arrangement of n upward unit triangular holes in an upward triangle of side n is *spread-out* if every upward grid triangle of size k contains at most k holes. We will also refer to the side length of such a triangle as its *size*.

Theorem 2.3 ([2])**.** *An arrangement of holes is a part of some fine mixed subdivision if and only if it's spread-out.*

Definition 2.4. For an arrangement of triangular holes, we say an upward triangle of side length k is *saturated* if it contains exactly k triangular holes.

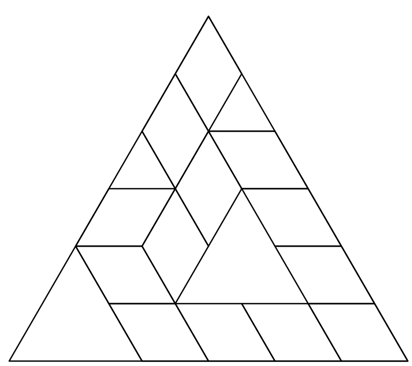
Lemma 2.5. *If a triangle is saturated, then no rhombus can intersect its boundary in any tiling.*

Proof. Consider a saturated triangle T. A rhombus covers an equal number of upward and downward triangles in T if it does not intersect the boundary of T, and covers one upward triangle if it intersects the boundary of T. Since the number of upward triangles in T covered by a rhombus is equal to the number of downward triangles in T covered by a rhombus, no rhombus can intersect the boundary of T. □

3 Big Triangles

Suppose we are given an arrangement of unit triangular holes and would like to tile the rest with rhombi. If T is a saturated triangle for the given arrangement, by Lemma 2.5 this problem can be separated into two independent subproblems: tiling T and tiling the region outside T. Thus, it makes sense to consider a more general problem, where the triangular holes do not necessarily have unit side length.

More precisely, consider an upward equilateral triangle Δ of side length n divided into n^2 unit triangles, and suppose that we cut out several disjoint upward triangular holes whose sides sum to n.



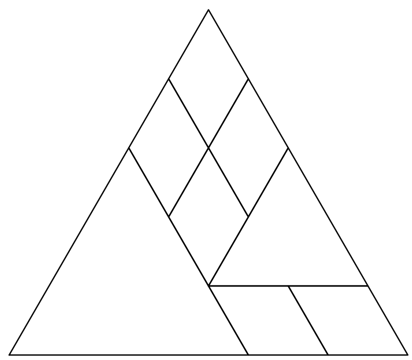
We say a set of triangles S is *spread-out* if for any upward triangle T inside Δ,

$$\sum_{s\in S}\text{size}(s\cap T)\le \text{size}(T).$$

Call T *saturated* (by S) if equality holds. Note that all elements of S are saturated triangles.

Similarly to the case of unit triangular holes, one may prove that if $\Delta\setminus\cup_{s\in S}s$ is tileable with rhombi, then S is spread-out. If we replace all triangles in S with upward unit triangles in their bottom rows, we get a spread-out arrangement of unit triangles, so S being spread-out is sufficient for $\Delta\setminus\cup_{s\in S}s$ to be tileable tileable with rhombi.

For upward grid triangles s_1, s_2, denote by $s_1\vee s_2$ the minimal upward grid triangle that contains both s_1 and s_2. Suppose that saturated triangles $s_1, s_2\in S$ touch at a point. Note that then $s_1\vee s_2$ is saturated, and there is only one way to tile $(s_1\vee s_2)\setminus(s_1\cap s_2)$. Also, no other triangles in S can intersect $s_1\vee s_2$. Therefore, there is an obvious bijection between tilings of Δ with holes S and tilings of Δ with holes $S\setminus\{s_1,s_2\}\cup\{s_1\vee s_2\}$.



Consider any set S of triangular holes and a saturated triangle T. Let A be the set of triangles in S completely outside T, B be the set of triangles in S completely inside T, and $\{s_1, s_2, \dots, s_k\}$ be the rest of triangles in S.

For each i, let $p_i = T\cap s_i$, and pick q_i such that $s_i = p_i\vee q_i$. Note that a tiling of Δ with holes S corresponds to a tiling of Δ with holes holes $A\cup\{T\}\cup\{q_1, q_2, \dots q_k\}$ and a tiling of T with holes $B\cup\{p_1, p_2, \dots p_k\}$.

Suppose that we start with some arrangement of holes and begin performing changes of the following type: two triangles s_1, s_2 touching at a point can be replaced with $s_1\vee s_2$. Eventually, we will get an arrangement of triangular holes that are pairwise not touching. If this arrangement consists of a single hole coinciding with the big triangle, then the original arrangement of holes has a unique tiling. We claim that the converse is also true.

Theorem 3.1. *Let S be a spread-out set of triangles. If no two triangles in S touch, then $\Delta\setminus\cup_{s\in S}s$ is tileable in at least two ways.*

We will delay the proof until the end of the next section.

4 A Graph Representation of a Fine Mixed Subdivision

Consider some fine mixed subdivison. Note that it divides the plane into regions: triangular holes, rhombi, and the outside region. Consider the graph G whose vertices are these regions, and where there is an edge $a \to b$ if and only if a contains an upward triangle adjacent to a downward triangle contained by b. We say that G is the graph corresponding to the original fine mixed subdivison.

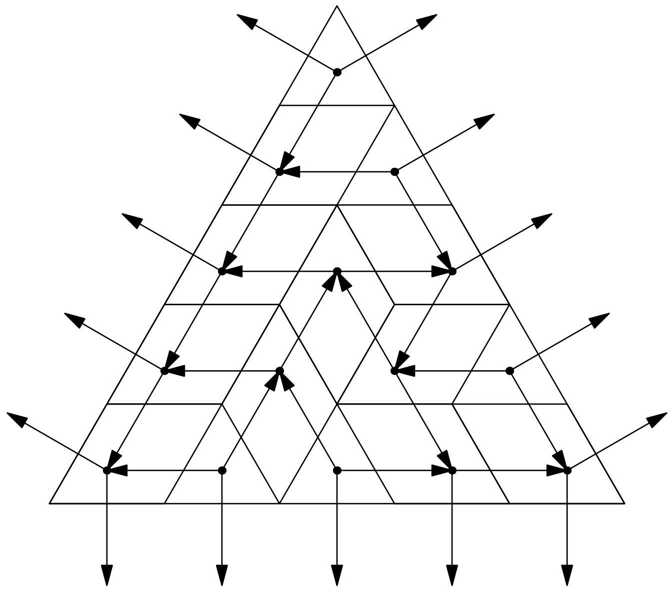


Figure 2: A fine mixed subdivision and the corresponding graph.

Note that that all vertices corresponding to rhombi have indegree and outdegree equal to 2; each vertex corresponding to a triangular hole of side k has outdegree $3k$ and indegree 0; and the vertex corresponding to the outside region has outdegree 0 and indegree $3n$.

Consider any cycle of the graph. This cycle must consist of rhombi that occupy a periodic sequence of triangles where each next one touches the previous. Then this sequence of triangles can be tiled in another way. We will call replacing one of these tilings with the other one a "cycle flip".

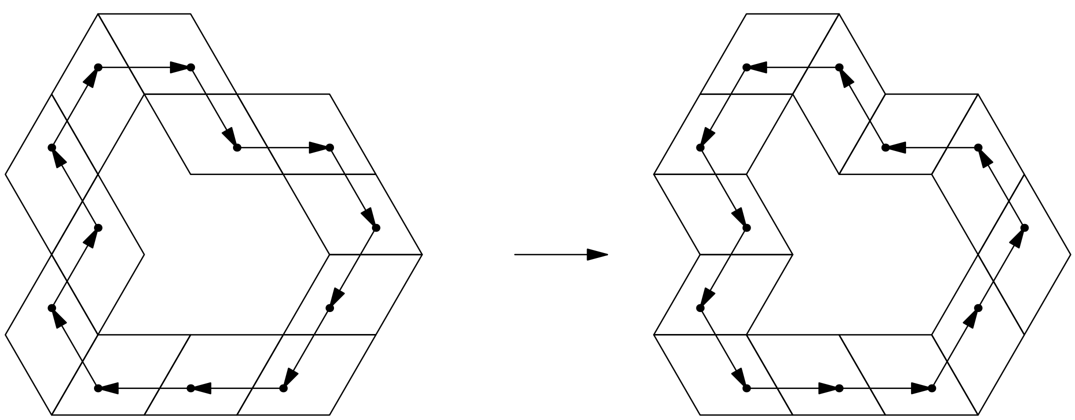


Figure 3: A cycle flip.

Lemma 4.1. *Consider any fine mixed subdivision $\mathcal{F}$, and let a and b be two rhombi such that $a \to b$ is an edge of its corresponding graph. Let c be the rhombus formed by the upward triangle of a and* *the downward triangle of b. Then there exists a fine mixed subdivision with the same arrangement of triangular holes as $\mathcal{F}$ and using c if and only if a is reachable from b in the graph corresponding to $\mathcal{F}$.*

Proof. If a is reachable from b, then there is a simple cycling containing the edge $a \to b$. After flipping this cycle, we get the desired fine mixed subdivision.

Suppose that there is a fine mixed subdivision $\mathcal{G}$ using c. Build a sequence of rhombi as follows: let $f_1 = a$. Let g_1 be the rhombus in $\mathcal{G}$ containing the downward triangle of f_1. Let f_2 be the rhombus in $\mathcal{F}$ containing the upward triangle of g_1. Let g_2 be the rhombus in $\mathcal{G}$ containing the downward triangle of f_2, and so on. Since f_i uniquely determines f_{i-1} and f_{i+1}, there is an index $k > 1$ such that $f_k = f_1$. Then vertices $a = f_1, f_2, f_3, \dots f_{k-1} = b$ form an cycle in the graph corresponding to $\mathcal{F}$, as desired. ☐

This gives us another criterion for the uniqueness of a tiling: any tiling is unique for its arrangement of triangles if and only if the corresponding graph has no oriented cycles.

The notion of a cycle flip also gives us a simple way to prove Theorem 3.1.

Proof of Theorem 3.1. Consider a spread-out arrangement where no two trianglular holes touch, and any rhombus a of some tiling. We claim that there is at least one rhombus b such that the graph contains the edge $a \leftarrow b$. Indeed, each of the two edges going into a is coming from either a triangle or a rhombus; if both come from a triangle then the two triangles are touching. Thus, we can construct an infinite sequence or edges $a_1 \leftarrow a_2 \leftarrow a_3 \leftarrow \dots$, so there must be a cycle in the graph, and we can perform a cycle flip, giving us a second tiling. ☐

5 "Google Drive Flips"

Using the the idea from the proof of Lemma 4.1, we see that any two fine mixed subdivisions with the same arrangement of triangular holes differ by several cycle flips. Additionally, if we place 3 unit triangles in the corners and one triangle of side $n-3$ in the middle we get an arrangement of triangular holes that admits only two fine mixed subdivisions.

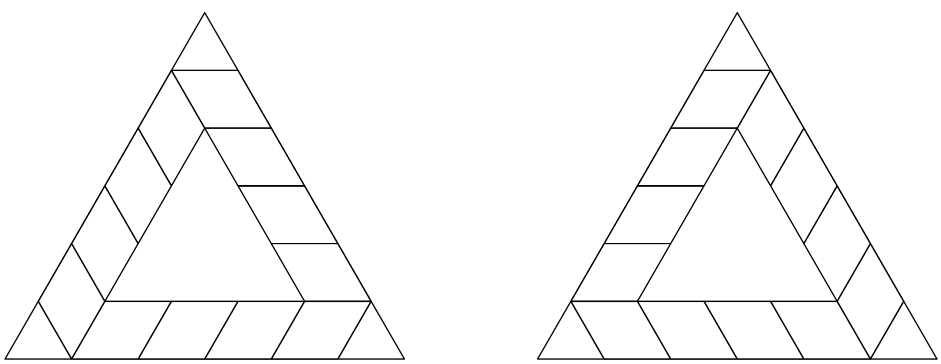


Figure 4: Arrangement of triangles admitting exactly two fine mixed subdivisions for $n = 6$.

Definition 5.1. For $n \geq 3$, consider the region obtained by removing three corner unit triangles from a triangle of side n, as well as removing a triangle of side $n-3$ in the middle. There are two configurations of rhombi that tile this region. Let *Clockwise Google Drive* (CW-GD) be the configuration where vertical rhombi go along the left side of the triangle, and let *Counter-clockwise Google Drive* (CCW-GD) be the other one. A *GD flip* consists of replacing a CW-GD with a CCW-GD. In particular, a GD flip is a cycle flip.

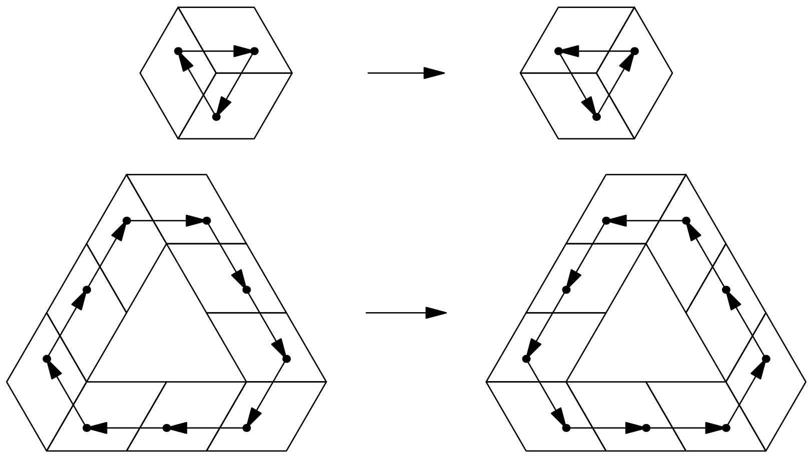


Figure 5: GD flips.

Theorem 5.2. *Any two tilings with the same arrangement of triangles are connected by a sequence of GD flips and their inverses.*

Lemma 5.3. *A tiling has a clockwise cycle if and only if it has a CW-GD.*

Proof. Take a minimal clockwise cycle C (in terms of enclosed area).

First, we claim that there are no edges going from a rhombus of C to a rhombus inside C. Suppose there is such an edge $a \to b$. Without loss of generality, we may assume that this the boundary between a and b is horizontal, that is, a is directly below b. Then we can construct a sequence of rhombi starting with b, a where each next rhombus is directly below the previous one. Eventually, this sequence must hit C again. Thus, we've constructed a path surrounded by C connecting two vertices of C. This path creates a clockwise cycle smaller than C, which is a contradiction.

Now consider the polygon surrounded by C. Note that if we traverse it clockwise, its edges can only go in three directions, corresponding to traversing an upward triangle in clockwise direction. This means that all internal angles of this polygon are equal to 60° or 300°. If some is equal to 300°, then only one of the triangles surrounding can belong to C, which is impossible. Since all angles of the polygon are equal to 60°, it has to be an equilateral triangle, so C is a CW-GD, as desired. □

Proof of Theorem 5.2. Suppose that we begin with some tiling and start performing GD flips while possible. Since the number of vertical rhombi does not change after a GD flip, and the sum of x-coordinates of their centers decreases with every GD flip, we must eventually stop. Note that there is at most one tiling with no clockwise cycles: if there were two, they would differ by several cycle flips, so one of them would have a clockwise cycle. This shows that starting from any tiling,

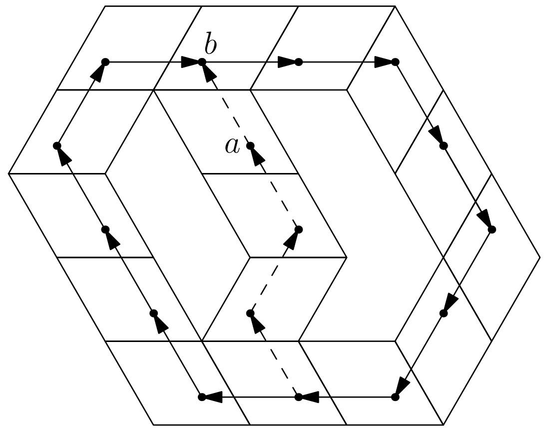


Figure 6: Proof of Lemma 5.3.

we can get to the unique tiling with no clockwise cycles by performing GD flips, and then back to any other tiling by performing their inverses. □

We can construct the unique tiling with no clockwise cycles more explicitly:

Proposition 5.4. *If in the process of sliding triangles (see [2], Section 3) we slide the triangle left whenever possible, we end up with a tiling that has no CW-GD.*

Proof. Suppose there is a CW-GD. Then its topmost rhombus arose from us sliding a triangle to the right when we could slide it to the left. □

6 Forced Segments

Given a fixed arrangement of triangular holes, it is natural to ask if there is a tiling of the remaining region including a certain rhombus. This is equivalent to determining if a certain segment of the triangular grid is forced to be a part of the tiling. Note that Lemma 2.5 shows that all segments belonging to a boundary of a saturated triangle are forced, and Lemma 4.1 shows that a segment between two rhombi is forced if and only if they belong to different connected components.

Given a fined mixed subdivision with corresponding graph G, consider the subgraph of G induced by vertices corresponding to rhombi. In this graph, pick a strongly connected component that has no edges going in from other strongly components. Let A be the set of rhombi in this component, as well as all triangular holes adjacent to them. Note that in G, there are no edges going from a vertex outside A into a vertex in A.

Consider the union of regions corresponding to vertices of A. Clearly, it's connected. Consider the polygon P that is the outside boundary of this region. This polygon, when traversed clockwise, can only have edges going in the directions, corresponding to traversing an upward triangle in clockwise direction.

Let T be the smallest upward triangle containing P. Note that if X and Y are the topmost vertices of P on left and right sides of T, respectively, then the distance from X to Y along the boundary of P is not less than the distance from X to Y along the boundary of T (in both cases, we are traversing polygons clockwise). Indeed, every unit segment on the path from X to Y

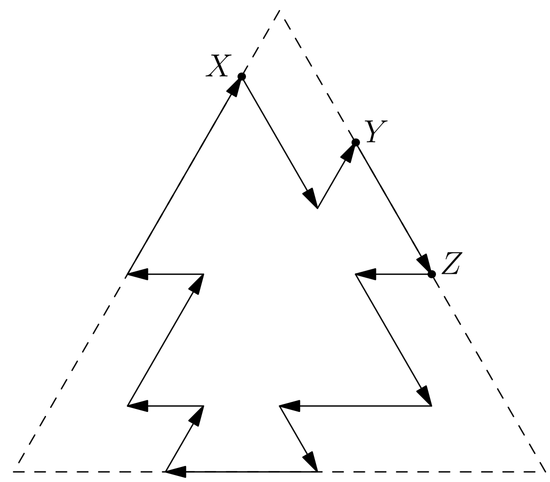


Figure 7: A polygon P traversed clockwise and the corresponding triangle T (dashed).

along the boundary of P increases the x-coordinate by at most $\frac{1}{2}$, and each unit segment on the path from X to Y along the boundary of T increases the x-coordinate by exactly $\frac{1}{2}$. Similarly, if Z is the bottommost vertex of P on the right side of T, the distance from Y to Z along the boundary of P is not less that the distance from Y to Z along the boundary of T. This shows that $\text{perimeter}(P) \geq \text{perimeter}(T)$.

However, note that every unit segment on the boundary of P corresponds to an edge going from a vertex in A to a vertex not in A, and there are no edges going from a vertex not in A into a vertex in A, so the total outdegree of vertices of A larger than the total indegree of vertices in A by at least $\text{perimeter}(P)$. This means that the sum of sizes of triangular holes in A is at least $\text{perimeter}(P)/3 \geq \text{size}(T)$. Since all these triangular holes lie entirely inside T, all inequalities must turn into equalities, and T is saturated.

In particular, all segments on the boundary of P between X and Y must go down-right or up-right, and all segments on the boundary of P between Y and Z must go down-right. This implies that $T \setminus P$ consists of three regions containing three vertices of T, and there is only one way to tile these regions.

This also implies that A is simply connected – if A had "holes", then there would be edges going from vertices of A to vertices in the holes, so the difference of total out- and in-degrees of vertices of A would be strictly greater than $\text{perimeter}(P)$.

Now, we may replace T with a triangular hole and repeat the process. This shows that regions occupied by strongly connected components of G have the following form: interior of some saturated triangle, minus interiors of saturated triangles inside it, and minus some rhombi in its three corners.

7 Trapezoid Flips

Definition 7.1. A *trapezoid flip* consists of replacing a triangle and an adjacent rhombus with a different triangle and an adjacent rhombus occupying the same trapezoid.

It is known (see [3], Corollary 4.5) that any two fine mixed subdivisions are connected by at most $\frac{2n(n-1)}{3}$ trapezoid flips.

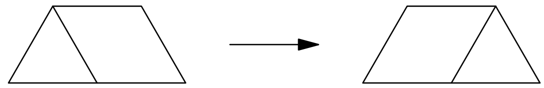


Figure 8: A trapezoid flip.

Proof. For any triangle hole s, let $h_1(s)$, $h_2(s)$, and $h_3(s)$ denote the distances from sides s to the parallel sides of the big triangle. Note that for any triangle s with $h_i(s) > 0$, there is a trapezoid flip that decreases $h_i(s)$ by 1. Then the distance from a fine mixed subdivision with triangle set S to the unique fine mixed subdivision with all triangles at the bottom is at most $\sum_{s\in S} h_i(s)$. Then the distance between two fine mixed subdivisions with triangle sets S_1 and S_2 is at most $\sum_{s\in S_1} h_i(s) + \sum_{s\in S_2} h_i(s)$. Since for any s we have $h_1(s)+h_2(s)+h_3(s) = n-1$, the sum of this expression over $i = 1, 2, 3$ is equal to $2n(n-1)$, so for some choice of i its at most $\frac{2n(n-1)}{3}$, as desired.

□

We show that this bound is tight up to terms of order $O(n)$.

Proposition 7.2. *Place an equilateral triangle of side n over a grid of unit hexagons. Consider two fine mixed subdivisions as shown below: one obtained by tiling each hexagon in clockwise manner and one obtained by tiling each hexagon in counterclockwise manner. At least $\frac{2n^2}{3} - n - O(1)$ trapezoid flips are needed to change from one tiling to the other.*

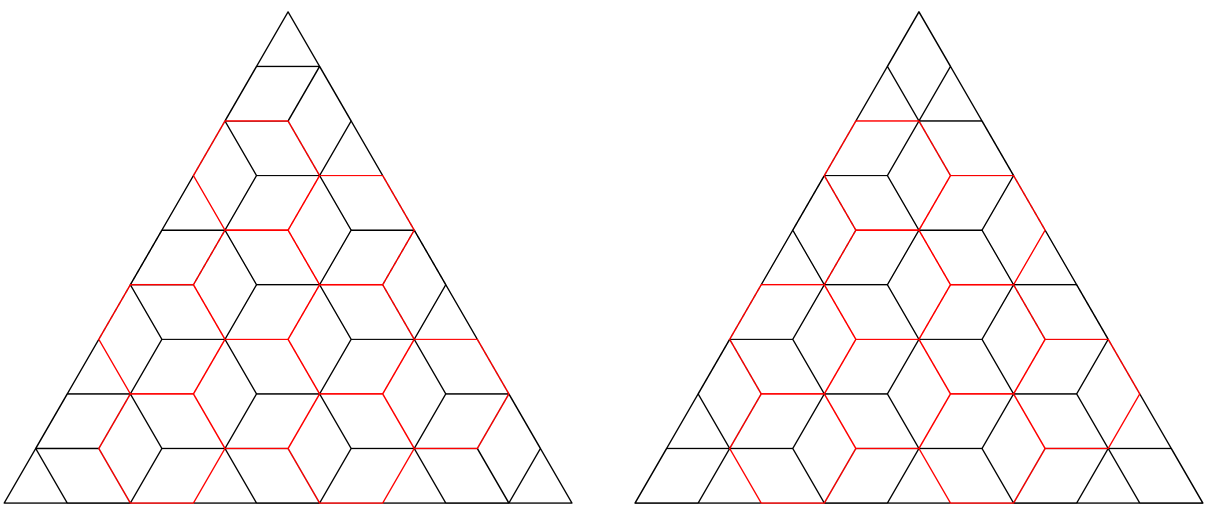


Figure 9: Two tilings that require many trapezoid flips to connect, with common hexagon boundaries highlighted in red.

Proof. For each rhombus in the final tiling, consider the last trapezoid flip involving this triangle. Since these flips are all distinct and the two tilings have no rhombi in common, this gives us $\frac{n^2-n}{2}$ distinct trapezoid flips.

For each hexagon, consider the first trapezoid flip involving some rhombus inside this hexagon in the initial tiling. These flips are clearly pairwise distinct. Additionally, they result in a rhombus that does not belong to any hexagon, so these moves are distinct from the previously considered $\frac{n^2-n}{2}$. Note that there are $\frac{n^2}{6} - \frac{n}{2} - O(1)$ hexagons lying inside the triangle. Therefore, we have $\frac{n^2-n}{2} + \frac{n^2}{6} - \frac{n}{2} - O(1) = \frac{2n^2}{3} - n - O(1)$ distinct trapezoid flips, as desired. $\square$

8 Further Questions

The rhombus tilings of a hexagon with sides a, b, c, a, b, c are in bijection with 3-dimensional Yound tableuax that fit in a $a \times b \times c$ box. A GD flip of size 0 corresponds to removing a cube:

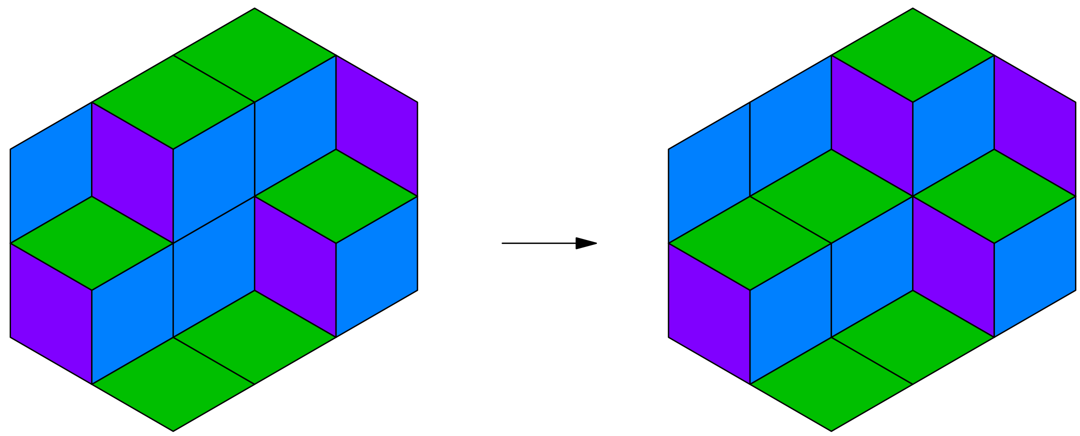


Figure 10: A GD flip of size 0 (note that this is rotated by 60° counterclockwise relative to conventions in the rest of the paper).

Partitions of regions that may contain triangular holes of some fixed orientation correspond to similar structures in certain 3-dimensional manifolds:

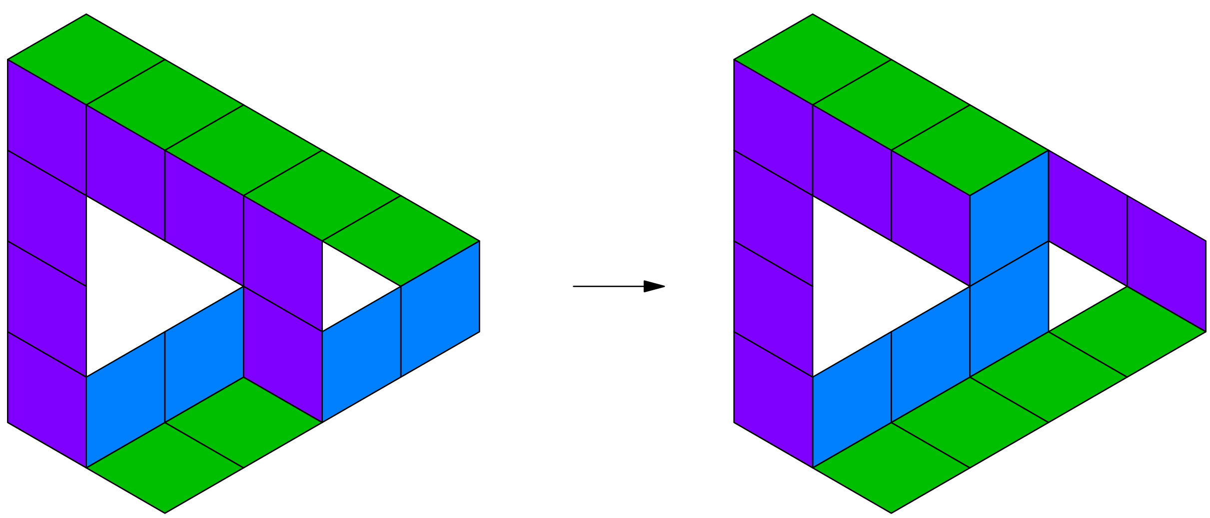


Figure 11: A GD flip of nonzero size appears to correspond to removing some non-Euclidean polyhedron (note that this is rotated by 60° counterclockwise relative to conventions in the rest of the paper).

One attempt of studying this correspondence is by looking at what we called the *depth function*. Fix a direction parallel to one of the lines of the triangle grid. For any node p of the triangular grid, consider the ray starting at p and going in the chosen direction, and let $d(p)$ denote the number of rhombi whose short diagonals belong to that ray. Note that the values of d at neighboring nodes differ by at most one, and that a GD flip changes the values of d of nodes in the triangle surrounded by the flip by one, and does not change all other values of d.

Note that the unique tiling with no clockwise cycles corresponds to a structure where no polyhedron can be removed, i.e., the "empty" one. Similarly, the unique tiling with no counterclockwise cycles corresponds to a structure where no polyhedron can be added, i.e., the "full" one. For each node, the difference of values of d for full and empty configurations corresponds to how "thick" the region that we're filling with polyhedra is at that point.

References

[1] Frederico Ardila and Sara Billey, Flag arrangements and triangulations of products of simplices, *Advances in Mathematics*. **214** (2007), no. 2, 495–524.

[2] Derek Liu and Yuan Yao, Arrangement of Simplices in Fine Mixed Subdivisions, https://math.mit.edu/research/highschool/primes/materials/2022/Liu-Yao.pdf.

[3] Francisco Santos, The Cayley trick and triangulations of products of simplices. *Contemporary Mathematics* **374** (2005): 151-178.