CORKS IN KNOT-SURGERED ELLIPTIC SURFACES

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Abstract. We find explicit embeddings of corks in knot traces whose smooth structures have been modified by Fintushel-Stern knot surgery. These traces occur naturally in many closed 4-manifolds, so this setting generalizes the original case of performing Fintushel-Stern surgery on a regular fiber of an elliptic fibration. In particular, we find corks in $E(1)_J$ which undo the FS-surgery. Along the way, we discuss the "one is enough" conjecture for Fintushel-Stern surgery on simply-connected manifolds.

1. Introduction

Introduced in [FS98], Fintushel and Stern's technique for changing the smooth structure of certain 4-manifolds has been a central source of exotic pairs of smooth, simply-connected 4-manifolds. Their technique, called *Fintushel-Stern surgery*, *FS-surgery*, or *knot-surgery* takes a smooth 4-manifold X satisfying some mild conditions, a knot $J \subset S^3$, called the companion of the surgery, and produces a smooth manifold X_J which is homeomorphic to X. The technique was originally introduced to study the $K3$-surface, and it can be used to produce countably many distinct exotic smooth structures on this manifold. The $K3$ surface fits into the family of elliptic fibrations, $E(n)$, which are singular Lefschetz fibrations of T^2 over S^2. The first elliptic fibration arises from the fact that CP^2 is foliated by a 1-parameter family of elliptic curves which intersect each other in the same nine points. Blowing up these points, we obtain a singular fibration $(T^2 \to E(1) \xrightarrow{\pi} S^2) \cong CP^2 \# 9\overline{CP}^2$. $E(n)$ is then defined to be the fiber sum of n copies of $E(1)$ along generic regular fibers. The $K3$-surface is diffeomorphic to $E(2)$, and any $E(n)$ can be modified using FS-surgery with different companion knots to produce countably many exotic copies of itself.

These examples have proven testing grounds for many famous conjectures about smooth 4-manifolds: see [Ste05] for a good, if slightly dated, summary. In particular, the first examples of corks in 4-manifolds were discovered in blown-up elliptic surfaces by Akbulut [Akb91]. He found an effective embedding of the Mazur cork in $E(2) \# \overline{CP}^2$, which is an exotic smooth structure on $3CP^2 \# 20\overline{CP}^2$. Later work of Akbulut and Yasui found countably many distinct embeddings of this cork in the same manifold, each of which produces a different exotic structure when twisted [AY09]. In this case, the exotic manifolds are obtained from $E(n) \# \overline{CP}^2$ by performing FS-surgery. Akbulut and Yasui use the extra $\overline{CP}^2$-summand in an essential way to find a single cork whose different embeddings undo the the different FS-surgeries, which further motivates the following problem:

Problem 1.1. Find embeddings of corks in $E(1)$ (and then in $E(n)$), modified by FS-surgery, whose cork twists undo the FS-surgery.

So far, this problem has resisted the community's efforts since it was first posed in the 1990's. We present a partial solution: a procedure to find corks whose twists undo FS-surgery on any *odd* elliptic fibration with any companion knot. We provide a concrete example of an explicit cork produced by our method:

Corollary 1.1. *The cork in Figure 1 embeds into $E(1)_{4_1}$ in such a way that its twist (the dot-zero swap in the diagram) is diffeomorphic to $E(1)$. The method by which it was found can find a similar cork in any manifold of the form $E(2n-1)_J$.*

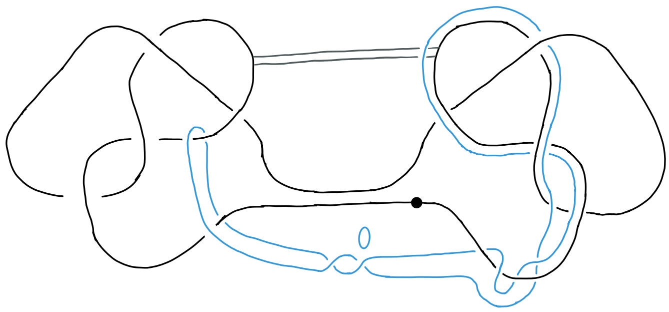

FIGURE 1. A cork whose twist turns $E(1)_{4_1}$ into $E(1)$. The ribbon 1-handle is carved along the spun disk for $4_1 \# \overline{4_1}$, while the blue 2-handle is attached along an unknot in S^3.

The two component link in Figure 1 is not symmetric, so the dot-zero swap we describe in the theorem does not correspond to cutting out and re-gluing this contractible manifold by an involution. Rather, it corresponds to cutting out this manifold and gluing in the complement of a natural embedding of this manifold in S^4. We describe an elementary procedure in Proposition 2.12 which shows how to convert this type of cork to the standard type. We give a concrete example for a similar cork in Figure 17 and the paragraph which follows it. It should be understood that we claim a solution to Problem 1.1 in the standard sense of the words 'cork twist' for odd elliptic fibrations and arbitrary companion knots.

We approached this problem by looking for corks inside of knot traces modified by FS-surgery rather than FS-surgered elliptic surfaces directly. While being much simpler to analyze *a priori*, we can produce infinite families of exotic 4-manifolds from knot traces using standard computations of the Seiberg-Witten invariant (Theorem 2.4). We find corks that undo FS-surgery on knot traces by analyzing explicit Kirby-calculus representatives of the 1-stable diffeomorphisms from the surgered traces to their unsurgered counterparts. We based our analysis on the diffeomorphisms described in the four excellent papers: [Auc03, Akb02, CPY19, Bay18]. The full statement of our theorem concerning knot traces is the following:

Theorem 1.2. *Let K, J be knots in S^3 and assume $g_3(K) = 1$ and $\mathrm{Arf}(K) = 0$. Let $X_0(K)_J$ denote Fintushel-Stern surgery on $X_0(K)$ with companion knot J, as described in Definition 2.1. The procedure described in Section 4.1 produces a Kirby diagram of a cork $C(K, J) \hookrightarrow X_0(K)_J$ whose twist recovers $X_0(K)$.*

Moreover, the Kirby diagram of $C(K, J)$ found by this procedure consists of two slice knots C_1 and C_2 with prescribed ribbon disks, such that their linking number is one, C_1 is dotted, and C_2 is zero-framed. The twist is given by swapping the framing data of C_1 and C_2 in the given diagram of the embedding.

Theorem 1.2 is significant because we can apply it to many closed 4-manifolds modified by FS-surgery, since the torus along which we perform the surgery often lives in an embedded knot trace. Even better, we can drop the $\mathrm{Arf}(K) = 0$ hypothesis from the statement of Theorem 1.2 when the ambient manifold is non-spin. This is the case for the odd elliptic surfaces $E(2n-1)$: they are non-spin manifolds, and any regular fiber along which we may perform FS-surgery can be chosen to live in a cusp neighborhood, which is diffeomorphic to the 0-trace of the right-handed trefoil. The method of the proof of Theorem 1.2 can then be applied to find a cork for any companion knot J in any odd-index elliptic surface, though the crossing numbers of the diagrams increase very rapidly with that of the companion. On the other hand, even-index elliptic surfaces are spin manifolds, and the Arf parity condition in this case has been an insurmountable obstacle for our method. We pose this issue as an open problem in Problem 2.1 and Problem 3.1, the latter being of the most interest.

1.1. **Organization.** We open Section 2 with a review of the definition of Fintushel-Stern surgery and its properties. We review knot traces and analyze their exotic behavior under FS-surgery using Seiberg-Witten theory. We then introduce n-stable diffeomorphisms and the "one-is-enough" conjecture, and then we define simple corks and review the cork theorem. In Section 3, we review Akbulut's method for showing FS-surgered 4-manifolds are 1-stably diffeomorphic following [Akb02]. In Section 4, we prove Theorem 1.2 and Corollary 1.1, and point out the parity issue we mentioned earlier. Where our proofs require explicit Kirby calculus, we give a written description of what is happening with figures together on the following pages. We include numbered captions corresponding to the figures with brief explanations of what is happening. The experienced Kirby calculator may dispense with the verbal descriptions and proceed directly to the figures and their captions.

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2. Background

2.1. **Fintushel-Stern Surgery.** Let X be a smooth, simply-connected 4-manifold and let $T \subset X$ be a smoothly embedded 2-torus with $[T] \cdot [T] = 0$ in $H_2(X;\mathbb{Z})$. Moreover, assume that $\pi_1(X \setminus \nu(T)) = 1$. It follows that $\nu(T) \cong T^2 \times D^2$, so we can parametrize $\partial\nu(T)$ as $S^1_\alpha \times S^1_\beta \times S^1_\mu$ where $S^1_\alpha \times S^1_\beta$ parametrizes T^2 and $S^1_\mu = \partial D^2$. These curves are shown in the standard Kirby diagram of $T^2 \times D^2$ in Figure 2.

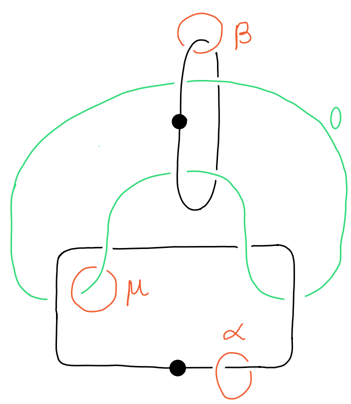

FIGURE 2. A diagram of $T^2 \times D^2$.

Definition 2.1. [FS98] Let (X, T) be as described above with the given parametrization. Let $J \subset S^3$ be a knot, and define X_J, called *Fintushel-Stern surgery along T with companion J*, to be the smooth, simply-connected 4-manifold built according to the following blueprint. Let E_J denote the knot complement of J, and let ∂E_J be parametrized as $S^1_m \times S^1_\ell$, where m is a meridian of J and ℓ is a null-homologous longitude. Consider the 4-manifold $E_J \times S^1$ with boundary $S^1_m \times S^1_\ell \times S^1_s$; the last factor corresponds to the product circle. We define:

$$X_J := (E_J \times S^1) \bigcup_{(m,\ell,s)\mapsto(\alpha,\mu,\beta)} (X \setminus \nu(T))$$

This construction has the potential to change the fundamental group of X, so it is important to establish sufficient conditions for $\pi_1(X_J) \cong \pi_1(X)$.

Lemma 2.2. *Let $S^1_\alpha \times S^1_\beta =: T \subset X$ be a square-zero, smoothly embedded torus. Let μ be a meridian of T in X, and parametrize $\partial(X \setminus \nu(T)) = S^1_\alpha \times S^1_\beta \times S^1_\mu$. If*

$$\mathbb{Z}_\alpha \oplus \mathbb{Z}_\beta < \ker\left(\pi_1(S^1_\alpha \times S^1_\beta \times S^1_\mu) \to \pi_1(X \setminus \nu(T))\right)$$

then $\pi_1(X_J) \cong \pi_1(X)$ for any companion knot J.

Proof. Observe that $\pi_1(X \setminus T)/\{\mu = 1\} \cong \pi_1(X)$ because attaching a 4-dimensional 2-handle to $X \setminus T$ along the curve S^1_μ in the boundary kills exactly μ in the fundamental group. We recognize the resulting manifold as $X \setminus \nu(S^1_\alpha \vee S^1_\beta)$. Removing a boundary sum of $S^1 \times B^3$'s from a 4-manifold never changes the fundamental group, so we have the isomorphism.

When we perform FS-surgery, we identify the two 3-tori together factor-by-factor according to $(m, \ell, s) = (\alpha, \mu, \beta)$. If we assume J is a non-trivial knot, then the boundary inclusion map on $\pi_1((S^3 \setminus \nu(J)) \times S^1)$ is injective. On the other hand, we are assuming that α, β are in the kernel of the boundary inclusion of the torus complement. Applying Seifert-Van-Kampen, we get the following isomorphism:

$$\pi_1(X_J) \cong \Big(\pi_1(X \setminus T) * (\pi_1(S^3 \setminus \nu(J)) \oplus \mathbb{Z}_s)\Big) \ / \ \left\{ \begin{array}{c} m = \alpha = 1, \\ \ell = \mu, \\ s = \beta = 1 \end{array} \right\}$$

Notice that $\pi_1(S^3 \setminus \nu(J))$ is normally generated by m, so the relation $m = 1$ also implies $\ell = 1$. However $\ell = \mu$, so the presentation above reduces to

$$\pi_1(X_J) \cong \pi_1(X \setminus T)/\{\mu = 1\} \ \cong \ \pi_1(X)$$

which proves the claim. □

The key feature of this construction is that it does not alter the intersection form nor the homeomorphism type of X if the result is simply-connected, yet it may change the smooth structure. The classical application of this procedure is to change the smooth structure of an elliptic fibration. The manifolds $E(n)$ for $n \in \mathbb{Z}_{>0}$ are the smooth, simply-connected, symplectic 4-manifolds built as Lefschetz fibrations of the torus over the sphere with monodromy $(\tau_\alpha \tau_\beta)^{6n}$ (for details about these manifolds, see [GS99]). A regular fiber of $E(n)$ satisfies all the necessary hypotheses for performing Fintushel-Stern surgery, and the resulting manifold $E(n)_J$ is known not to be diffeomorphic to $E(n)$ when the Alexander polynomial of J is non-trivial [FS98]. Moreover, Fintushel and Stern conjectured that up to mirroring (since $E_J \times S^1 \cong E_{\overline{J}} \times S^1$), the map from $J \mapsto E(n)_J$ is one-to-one up to diffeomorphisms.

2.2. **Knot Traces.**

Definition 2.3. Let K be a knot in S^3 and $n \in \mathbb{Z}$. The n-trace of K, denoted by $X_n(K)$, is defined as the unique smooth, simply-connected, compact 4-manifold built by attaching a 2-handle to the 4-ball along the knot K with framing n.

This definition has a natural extension to links: given a k-component link $L_1 \cup \ldots \cup L_k$ and framings $n_1, \ldots, n_k \in \mathbb{Z}$, the manifold $X_{(n_1,\ldots,n_k)}(L_1 \cup \ldots \cup L_k)$ is obtained by attaching a 2-handle to the 4-ball along each component of the link with the corresponding framing.

Traces are relevant to the study of FS-surgery because they 'atomize' the problem, since we can visualize FS-surgery as being performed in a knot trace inside the ambient space X. The original hypotheses of the construction require X to contain an embedded, square-zero torus whose complement is simply-connected. Since X is simply connected, the condition on the complement is equivalent to saying that the circles α and β in the boundary of $\nu(T)$ bound immersed disks in the complement which we call D'_α and D'_β. Let m_* denote the signed count of the double points of D'_*. Let n_* denote the respective self-intersection numbers of D'_* defined by taking a push-off which agrees with the product structure of the 3-torus at the boundary. Consider the 4-manifold obtained by attaching two 2-handles to $\nu(T)$ along S^1_α and S^1_β with framings $a := n_\alpha - 2m_\alpha$ and $b := n_\beta - 2m_\beta$. We parametrize (T, α, β) as $(S^1_\alpha \times S^1_\beta, S^1_\alpha, S^1_\beta)$, and we see that this 4-manifold is abstractly defined by Figure 3 (Left). We can cancel the obvious 1-2 handle pairs to recognize this manifold as the 0-trace of the (a, b)-twist knot: $X_0(\mathrm{Tw}(a, b))$ Figure 3 (Right).

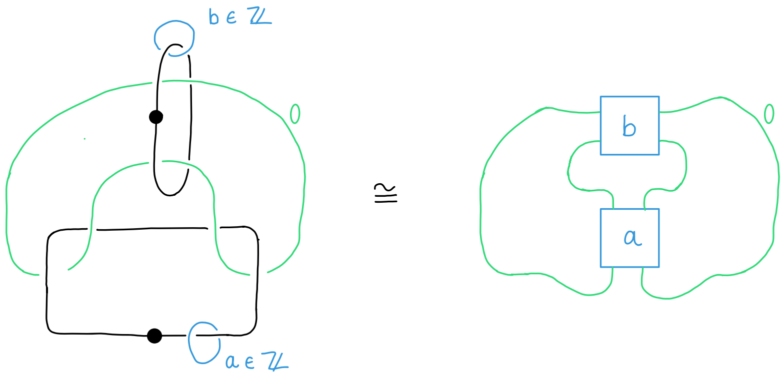

FIGURE 3. $\nu(T) \cup h^2_4(\alpha, n_\alpha - 2m_\alpha) \cup h^2_4(\beta, n_\beta - 2m_\beta) \cong X_0(\mathrm{Tw}(a,b))$

If we consider our original manifold X, we see immediately that there is an immersion $X_0(\mathrm{Tw}(a,b)) \looparrowright X$ such that the composition $\nu(T) \hookrightarrow X_0(\mathrm{Tw}(a,b)) \looparrowright X$ is an embedding. Moreover, the immersed 2-handles can be promoted to embedded 2-handles in the presence of $S^2 \times S^2$-summands. This is usually unnecessary in practice: for elliptic surfaces, the associated trace is $X_0(\mathrm{Tw}(-1,-1))$, which is the 0-trace of the right-handed trefoil, and it is embedded.

2.3. **Distinguishing Fintushel-Stern Surgered Traces.** We now turn to the problem of proving that a knot trace $X_0(K)$ and its FS-surgered counterpart $X_0(K)_J$ are often not diffeomorphic, which implies that there is a non-trivial cork $C(K,J) \hookrightarrow X_0(K)$. This is a fairly easy application of existing technology, and, although we could not find it explicitly written down, it is probably well known.

Theorem 2.4. *Let $a, b \in \mathbb{Z}_{<0}$, and let J be a knot with $\Delta_J(t) \neq 1$. It follows that the manifolds $X_0(\mathrm{Tw}(a,b))$ and $X_0(\mathrm{Tw}(a,b))_J$ are homeomorphic but not diffeomorphic relative to their fixed boundary.*

We will distinguish the trace from its surgered counterpart using Seiberg-Witten theory. The Seiberg-Witten invariant of a smooth 4-manifold X with $b^+ > 1$ and odd is a function defined as follows:

$$SW_X : \{\pm\beta_1, \ldots, \pm\beta_n\} \to \mathbb{Z} \quad \text{where} \quad \{\beta_i\}_{i=1}^{n} \subset H_2(X;\mathbb{Z})$$

This invariant is natural in the sense that, if $\phi : X \to X'$ is an orientation-preserving diffeomorphism, then $SW_{X'}(\phi_*(\beta)) = SW_X(\beta)$, so the image is a diffeomorphism-invariant subset of $\mathbb{Z}$. We can formally extend the invariant as a function to all of $H_2(X;\mathbb{Z})$ by setting it to zero on all other classes. In order to understand the effect of their surgery technique on the Seiberg-Witten invariant of X, Fintushel and Stern encoded the invariant into a polynomial as follows. Let $t_j := \exp(\beta_j)$ be the formal exponentials of the 2-homology classes, with $1 := \exp([0])$, and let $b_j := SW_X(\beta_j)$. We write the Seiberg-Witten invariant as the following formal Laurent polynomial:

$$SW_X = b_0 + \sum_{j=1}^{n} b_j \left(t_j + \frac{(-1)^{(1+b^+)/2}}{t_j} \right)$$

This polynomial is symmetric because $SW_X(\beta) = (-1)^{(1+b^+)/2}\, SW_X(-\beta)$.

Theorem 2.5. [FS98, Thm. 1.5] *Let X be a smooth, closed, simply-connected 4-manifold with $b^+ > 1$ and odd. Let $T \subset X$ be a square-zero, smoothly embedded torus, and assume $\pi_1(X \setminus \nu(T)) = 1$. Choose a knot $J \subset S^3$ and let X_J denote the Fintushel-Stern surgery on X along T with companion J. It follows that*

$$SW_{X_J} = SW_X \cdot \Delta_J(\exp(2[T]))$$

where $\Delta_J(t)$ denotes the symmetrized Alexander polynomial of J, normalized so that $\Delta_J(1) = 1$.

This theorem is very compatible with Kähler surfaces, as these 4-manifolds satisfy $SW_X = 1$. Thus, if one finds a square-zero torus $[T]$ suitable for FS-surgery in a non-trivial homology class of X, the resulting manifold satisfies $SW_{X_J} = \Delta_J(t^2)$ with $t := \exp([T]) \neq 1$. This observation sets us up to prove Theorem 2.4, as we can examine embeddings of $X_0(\mathrm{Tw}(a,b))$ and $X_0(\mathrm{Tw}(a,b))_J$ into simply-connected Kähler 4-manifolds by applying the following theorem of Lisca and Matić:

Theorem 2.6. [LM97, Thm. 3.2] *Let $L_1 \cup \ldots \cup L_n$ be a Legendrian link and consider the link trace W of L where the framing of each component L_i is given by $\mathrm{tb}(L_i) - 1$. Then, there exists a simply-connected, compact, Kähler 4-manifold M with $b^+ > 1$ such that W embeds smoothly in M.*

We proceed with the proof of Theorem 2.4.

Proof. [Theorem 2.4] Let $a, b \in \mathbb{Z}_{<0}$ be given and let $K = \mathrm{Tw}(a,b)$. Let $J \subset S^3$ be given with $\Delta_J(t) \neq 1$. K has a Legendrian diagram with $\mathrm{tb}(K) - 1 = 0$ from which we can obtain a Legendrian diagram of the link $K \cup \mu_K$ where $\mathrm{tb}(\mu_K) - 1 = -2$. Let $W = X_{(0,-2)}(K \cup \mu_K)$ be the associated link trace, and observe that $\partial W \cong S^3_{1/2}(K)$ by performing a slam dunk move on the meridian of K. This is shown in Figure 4.

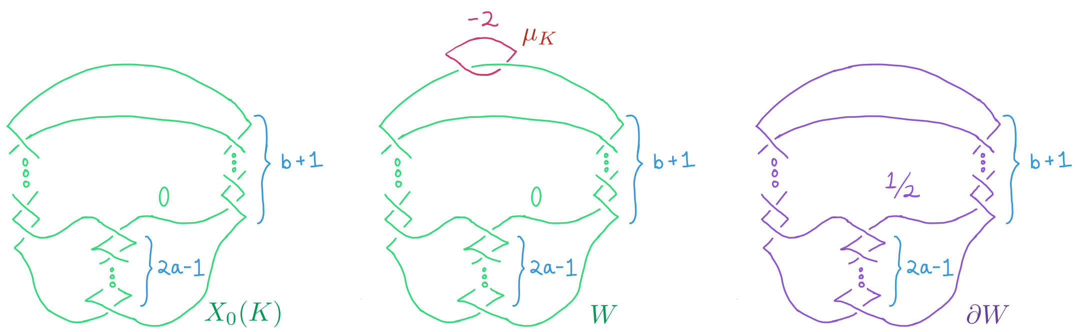

FIGURE 4. Building the manifold W and working out its boundary.

Clearly $X_0(K) \subset W$, and W satisfies the hypotheses of Theorem 2.6 by construction, so let M be a simply-connected, Kähler 4-manifold containing a smoothly embedded W. The torus T which generates $H_2(X_0(K);\mathbb{Z})$ is clearly non-trivial in $H_2(W;\mathbb{Z})$. Moreover, ∂W is an integer-homology-sphere, which implies that the intersection form of M splits off a copy of the intersection form of W. Since $[T]$ is non-trivial in $H_2(W;\mathbb{Z})$, it remains non-trivial in $H_2(M;\mathbb{Z})$.

We now proceed by way of contradiction. Suppose $X_0(K)_J \cong X_0(K)$ rel. boundary. This implies that $M_J \cong M$ when the FS-surgery is performed along $T \subset X_0(K) \subset W \subset M$. Clearly $X_0(K)_J \subset M_J$, so we can define a diffeomorphism $M_J \to M$ by extending a diffeomorphism $X_0(K)_J \to X_0(K)$, which is the identity on the boundary, over the complement.

On the other hand, M is Kähler so $SW_M = 1$, but $SW_{M_J} = \Delta_J(\exp(2[T])) \neq 1$ because $\Delta_J(t) \neq 1$ and $[T] \neq 0 \in H_2(W;\mathbb{Z})$. Thus $M_J \ncong M$, and we conclude that $X_0(K)_J$ is not diffeomorphic rel. boundary to $X_0(K)$. ☐

2.4. **One-Stable diffeomorphisms.** The following theorem is a well-known result of Wall:

Theorem 2.7. [Wal64] *Let X, X' be smooth, simply-connected 4-manifolds with fixed boundary $\partial X = Y = \partial X'$. Moreover, assume that X is homeomorphic to X'. It follows that, for some fixed $n \in \mathbb{Z}_{>0}$,*

$$X \# n(S^2 \times S^2) \cong X' \# n(S^2 \times S^2)$$

When n is minimized, we say X is n-stably diffeomorphic *to X'.*

In the case of closed manifolds, the boundary is generally taken to be S^3, and the theorem is applied to the corresponding punctured manifolds. This was the case which was originally considered by Wall, and his theorem follows from analyzing smooth h-cobordisms between 4-manifolds. It is another result of Wall that such cobordisms always exist between smooth, simply-connected, homotopy-equivalent 4-manifolds. The non-trivial, fixed boundary case follows from a similar analysis of relative h-cobordisms, and it is an elementary reformulation of Wall's original argument.

This result led to the following conjecture:

Conjecture 2.8. *Let X, X' be smooth, closed, and simply-connected 4-manifolds which are homeomorphic but not diffeomorphic. It follows that they are 1-stably diffeomorphic: $X \# S^2 \times S^2 \cong X' \# S^2 \times S^2$.*

The analogue of this conjecture for fixed boundary $Y \neq \#_k(S^1 \times S^2)$ was recently shown to be false [Kan22]. On the other hand, there is still considerable evidence for the conjecture as stated: most of the known constructions of exotic, closed 4-manifolds have been explicitly shown to be 1-stably diffeomorphic.

The "one is enough" conjecture has been extensively studied for exotic pairs produced with Fintushel-Stern surgery. Auckly and Akbulut gave independent proofs [Auc03, Akb02] that, for any exotic pair of the form $E(n), E(n)_J$, there is a diffeomorphism

$$E(n)_J \# S^2 \tilde{\times} S^2 \cong E(n) \# S^2 \tilde{\times} S^2.$$

which extends a diffeomorphism between the surgered and non-surgered associated knot trace inside $E(n)$:

$$X_0(\mathrm{Tw}(-1,-1))_J \# S^2 \tilde{\times} S^2 \cong X_0(\mathrm{Tw}(-1,-1)) \# S^2 \tilde{\times} S^2$$

Moreover, it is well known that $S^2 \tilde{\times} S^2 \cong CP^2 \# \overline{CP}^2$, and a close reading of each proof shows that only the CP^2 term is necessary since we can refine the previous diffeomorphism to

$$X_0(\mathrm{Tw}(-1,-1))_J \# CP^2 \cong X_0(\mathrm{Tw}(-1,-1)) \# CP^2$$

This is not quite the conjecture in general because we are stabilizing with the twisted product of 2-spheres; however, this is equivalent to connect summing with the untwisted product when n is odd because then $E(n)$ is a non-spin manifold.

Akbulut's method also works for some more general X, X_J: those which are spin and which satisfy the following additional, technical hypothesis. Consider the curve $S^1_\alpha \subset \partial(X \setminus \nu(T))$. Since we assume $\alpha, \beta = 1 \in \pi_1(X \setminus \nu(T))$, it follows that there is a properly immersed disk $D^2_\alpha \subset (X \setminus \nu(T))$ with $\partial D^2_\alpha = S^1_\alpha$, so let n_α denote the signed self-intersection number of this disk.

We hasten to clarify what this means, since S^1_α is not null-homologous in $\partial(X \setminus \nu(T^2))$. Let M be a smooth 4-manifold with boundary T^3, and let $\varphi : \partial M \to S^1_p \times S^1_q \times S^1_r$ be an explicit parametrization of its boundary. Assume one of the factors bounds an immersed surface in M, and let it be S^1_p without loss of generality. Let Σ_p be a surface spanning S^1_p in M. We define the *self-intersection number of Σ_p relative to φ* as follows. Pick two generic, nearby points $x_0, x_1 \in S^1_q \times S^1_r$ and consider the pair of circles $S^1_p \times \{x_0\}$ and $S^1_p \times \{x_1\}$. Assume $S^1_p \times \{x_0\} = \partial\Sigma_P$, and let Σ'_P be a push off of Σ_P chosen so that $\partial\Sigma'_P = S^1_p \times \{x_1\}$. Σ'_P inherits an orientation from Σ_P and so we can compute the signed intersection number between them, which we denote $\Sigma_P \cdot_\varphi \Sigma_P$. It is then a fairly standard exercise to show that this number depends only on $[\Sigma_P] \in H_2(M, \partial M; \mathbb{Z})$. In the setting of FS-surgery, we have an explicit parametrization $\varphi : \partial(X \setminus \nu(T)) \to S^1_\alpha \times S^1_\beta \times S^1_\mu$ which allows us to compute $n_\alpha := D^2_\alpha \cdot_\varphi D^2_\alpha$.

If there exists such D^2_α with $n_\alpha \equiv 0 \mod 2$, then it follows from [Akb02] that:

$$X \# S^2 \times S^2 \cong X' \# S^2 \times S^2$$

and if every D^2_α has $n_\alpha \equiv 1 \mod 2$, then:

$$X \# S^2 \tilde{\times} S^2 \cong X' \# S^2 \tilde{\times} S^2$$

Akbulut gives an explicit sequence of Kirby moves which realize this diffeomorphism, with a significant caveat: this sequence relies essentially on having the parity of n_α match the twistedness of the stabilization factor. As far as we are aware, there is no explicit method in the literature for writing down a diffeomorphism in the case where the parities do not match. This is frustrating because it excludes the classical case of the conjecture:

$$E(2) \# S^2 \times S^2 \cong_? E(2)_J \# S^2 \times S^2$$

$E(2)$ is a spin manifold, but when we pass to the complement of a regular fiber, $E(2) \setminus \nu(T^2)$, we see that the standard family of boundary parametrizations, which we use to perform FS-surgery, only produce disks D^2_α with odd n_α. These disks come from pinching off the vanishing cycles of the Lefschetz fibration, and have self-intersection number -1, which means we must use $S^2 \tilde{\times} S^2$ if we are to use the methods of Auckly and Akbulut to construct a 1-stable diffeomorphism.

On the other hand, Wall's result says that because $E(2)$ is spin, there is some finite number of $S^2 \times S^2$'s which are sufficient to undo FS-surgery. Baykur has given an abstract proof, using 5-dimensional cobordism arguments, that $E(2)_J \# S^2 \times S^2 \cong E(2) \# S^2 \times S^2$ for any companion J [Bay18, Thm. 1]. Baykur's argument is a refinement of a similar argument given by him and Sunukjian [BS13, Cor. 16], which uses a blowup in the spin case to ensure the parity of n_α is even. Since Baykur's diffeomorphism is not explicit, we pose the following (open?) problem:

Problem 2.1. Write down an explicit sequence of Kirby moves taking a diagram of $E(2)_J \# S^2 \times S^2$ to $E(2) \# S^2 \times S^2$ for at least one non-trivial knot J.

2.5. **Corks.** We will now formally introduce corks, pointing out that there are two different-yet-equivalent definitions in the literature following the original papers [CFHS96, Mat96]. We review the statements of both versions of the cork theorem:

Theorem 2.9. [Mat96, CFHS96] *Let X, X' be smooth, simply-connected, compact 4-manifolds which are homeomorphic but not diffeomorphic.*

(1) There is a compact, contractible manifold $W \subset X$ equipped with a boundary involution $f : \partial W \to \partial W$ that is not the restriction of any self-diffeomorphism of W such that

$$(X \setminus W) \cup_f W \cong X'$$

The pair (W, f) is called an (involutive) cork *taking $X \to X'$, and the process of removing W from X and regluing it by f is known as* cork twisting. *Moreover, (W, f) have the property that*

$$W \cup_f W \cong S^4$$

(2) There is a pair of compact contractible manifolds (C, C^*) with the same boundary, yet not diffeomorphic rel. boundary, such that

$$(X \setminus C) \cup_{\mathbb{1}} C^* \cong X'$$

and

$$C \cup_{\mathbb{1}_\partial} C^* \cong S^4$$

The pair (C, C^*) is called a doppel cork *taking $X \to X'$, and the process of replacing C by C^* is known as* cork twisting.

Pairs (W, f) are called involutive because f is an involution, whereas pairs (C, C^*) are called doppel because these two 4-manifolds are homeomorphic, though they may not be diffeomorphic. Unless otherwise stated, any cork to which we refer in this paper will be a doppel cork. Still, there is a natural way to construct involutive corks from doppel ones (Proposition 2.12), so the proof of our main theorem will include a "translation" of our results to produce involutive corks for the sake of completeness.

Definition 2.10. A *simple cork* is a pair of contractible, smooth 4-manifolds (C, C^*) with the same boundary built from the following process. Pick a two component link $K_1 \cup K_2 \subset S^3$ which satisfies the following conditions:

(i) $lk(K_1, K_2) = 1$.
(ii) K_1 is smoothly slice with prescribed slice disk D_1.
(iii) K_2 is smoothly slice with prescribed slice disk D_2.

Decompose S^4 as the union of two 4-balls B and B^* and let $K_1 \cup K_2$ sit in the common boundary. Embed D_1 into B and D_2 into B^* so that both disks are embedded into S^4, disjointly. Let

$$C := (B \setminus \nu(D_1)) \cup \nu(D_2) \quad \text{and} \quad C^* = \nu(D_1) \cup (B^* \setminus \nu(D_2))$$

If C is embedded in a smooth 4-manifold X, then define the *cork twist* of C in X to be the manifold:

$$X' := (X \setminus C) \cup_{\mathbb{1}} C^*$$

It is clear from the construction that both C, C^* are contractible and that $C \cup_{\mathbb{1}} C^* \cong S^4$, so this definition gives a doppel cork. Moreover, there is some hope that any two exotic, simply-connected 4-manifolds are related by a simple cork twist in light of Conjecture 2.8 and the following proposition:

Proposition 2.11 (Folklore). *Let X, X' be smooth, simply-connected, closed 4-manifolds which are homeomorphic but not diffeomorphic. If $X \# S^2 \times S^2 \cong X' \# S^2 \times S^2$, then there is a simple cork $(C, C^*) : X \to X'$.*

The proof is an elementary modification of the original proofs of the cork theorem. The essential point is that the manifold $X \# S^2 \times S^2$ contains a plumbing of a single pair of spheres with intersection number one, with the property that surgering one of the spheres recovers X while surgering the other recovers X'. Using the 'rising water principle' [GS99, §6.2], one can show that there is a Morse function on $X \# S^2 \times S^2$ which induces a standard handle decomposition of the plumbed pair of spheres. Since the ambient space is simply connected, one can attach cells from the complement of the plumbing to the plumbing to make the plumbing simply-connected. Performing surgery on either sphere in this enlarged plumbing produces exactly a pair (C, C^*) as described in Definition 2.10.

Simple corks are particularly easy to work with in Kirby diagrams. The common boundary $\partial C = Y = \partial C^*$ is $S^3_{(0,0)}(K_1 \cup K_2)$. We obtain a Kirby diagram of C by dotting K_1 (carving D_1) and attaching a 0-framed 2-handle along K_2, and a Kirby diagram of C^* by carving K_2 along D_2 and attaching a 0-framed 2-handle along K_1. An embedding of $C \hookrightarrow X$ is given by finding a Kirby diagram of X with the property that it contains a framed sub-link equal to the Kirby diagram of C which is itself disjoint from any other dotted components of the main diagram. Given such a diagram of X, we perform the cork twist by exchanging the dot and the zero of the components of the framed sub-link associated to C. We present an example of a simple cork diagram in Figure 5.

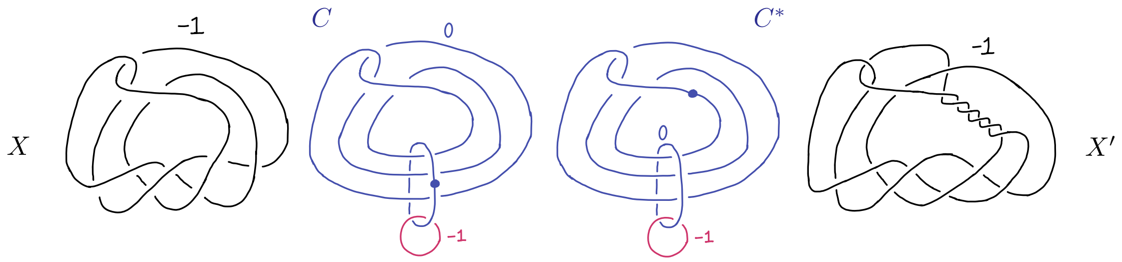

Figure 5. A simple cork twist: the cork is in blue, while the handles of its complement are in red. The cork is embedded because its handles do not pass through any red 1-handles (there are none). The cork twist is performed by exchanging the framing data of the blue handles

We close this section with a method to construct involutive corks from doppel corks:

Proposition 2.12. *Let (C, C^*) be a doppel cork taking $X \to X'$. There is an involutive cork $W = C \natural C^*$ taking $X \to X'$ with boundary involution $f\colon (\partial C \# \partial C^*) \to (\partial C^* \# \partial C)$.*

Proof. By the proof of the cork theorem in [CFHS96], C and C^* can always be constructed to satisfy the following:

$$C \cup_{\mathbb{1}_\partial} C \cong C^* \cup_{\mathbb{1}_\partial} C^* \cong S^4 \cong C \cup_{\mathbb{1}_\partial} C^*$$

This shows that we can twist $S^4 \cong C \cup_{\mathbb{1}_\partial} C^*$ along either half of the doppel cork, without changing the smooth structure of S^4. Let's consider the manifold $X \# S^4$, which is clearly diffeomorphic to X. If we simultaneously twist the cork (C, C^*) taking $X \to X'$ and the trivial cork (C^*, C) taking $S^4 \to S^4$ on each of its summands, we get the manifold $X' \# S^4 \cong X'$. We can consider both cork twists as a single procedure: we remove $C \natural C^*$ from $X \# S^4$, and then we glue $C^* \natural C$ into the complement to obtain $X' \# S^4$. Therefore, $W = C \natural C^*$ is an involutive cork that takes $X \to X'$. Its boundary is $\partial C \# \partial C^*$ which is the connected sum of two copies of the same 3-manifold. The involution is the map that that exchanges the two summands, preserving orientations. We note that none of the involutive corks that we construct this way will be simple because the involutive cork's diagram is the disjoint union of the diagrams of C and C^*. A schematic of this procedure is given below in Figure 6. ☐

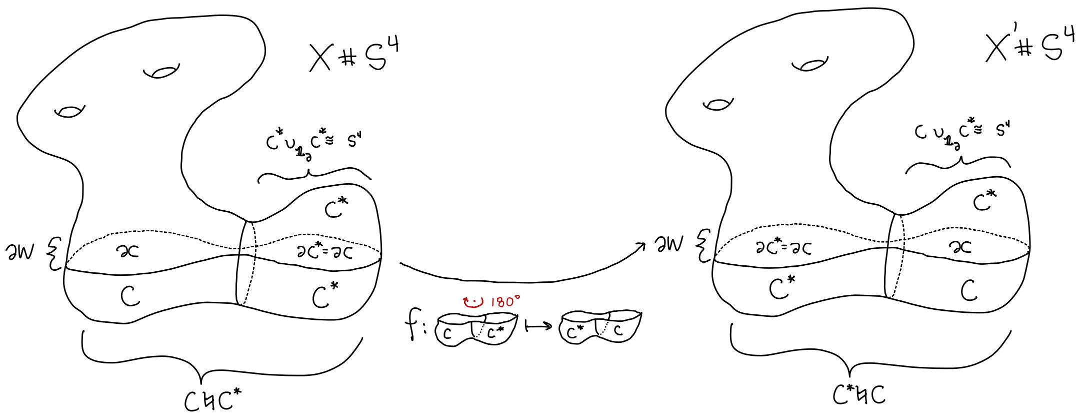

Figure 6. The involutive cork twist taking $X \# S^4 \to X' \# S^4$, where f swaps the components of $\partial W = \partial C \# \partial C^*$.

3. One-Stable Diffeomorphisms Between Traces

In this section, we will use the method of [Akb02, Thm. 2.1] to prove the following theorem:

Theorem 3.1. *Let $K = Tw(a,b) \subset S^3$ for $a, b \in \mathbb{Z}$ and let $J \subset S^3$ be given. Let $X_0(K)_J$ be the FS-surgered trace along the obvious capped off Seifert surface for K with companion J. It follows that*

$$\begin{aligned} \textit{if } a \equiv 0 \mod 2, \textit{ then } & \quad X_0(K)_J \# S^2 \times S^2 \cong X_0(K) \# S^2 \times S^2 \\ \textit{if } a \equiv 1 \mod 2, \textit{ then } & \quad X_0(K)_J \# S^2 \tilde{\times} S^2 \cong X_0(K) \# S^2 \tilde{\times} S^2 \\ \textit{if } a = 1, \textit{ then } & \quad X_0(K)_J \# \overline{CP}^2 \cong X_0(K) \# \overline{CP}^2 \\ \textit{if } a = -1, \textit{ then } & \quad X_0(K)_J \# CP^2 \cong X_0(K) \# CP^2 \end{aligned}$$

Notice that the natural Seifert surface for K has a symplectic basis α, β with the self-linking of α equal to a and that of β equal to b. Moreover, we see from Figure 3 that D_α is embedded and $n_\alpha = a$ for the torus along which we will perform FS-surgery.

3.1. The Crossing Change Lemma.

The proof of Theorem 3.1 will rely on the following *Crossing Change Lemma*, which is present, but not explicitly stated, in [Akb02]. We give an explicit proof of it here, since we believe it is relatively unknown among experts.

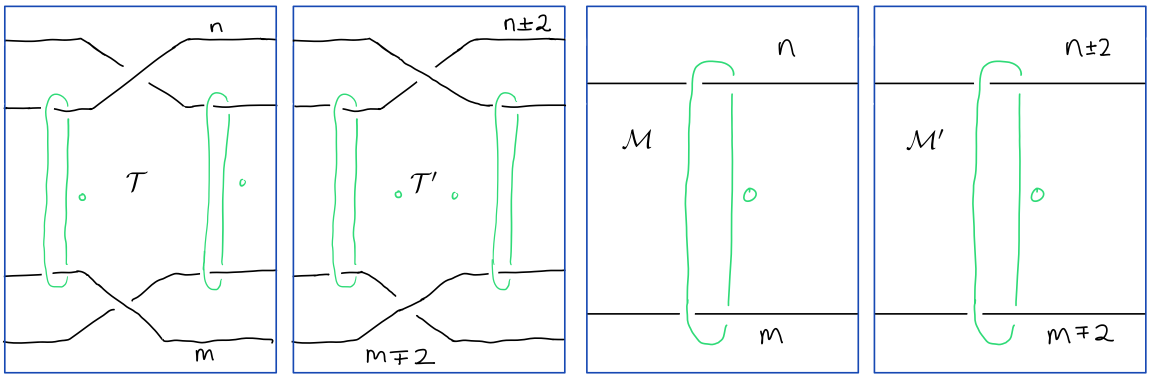

Figure 7. The tangles $\mathcal{T}, \mathcal{T}'$ and $\mathcal{M}, \mathcal{M}'$ in a Kirby diagram of a 4-manifold

Lemma 3.2 (Crossing Change Lemma)**.** *Let X be a 4-manifold with a Kirby diagram $\mathscr{D}$ which contains either the tangle $\mathcal{T}$ or $\mathcal{M}$ shown in Figure 7. Let X' be the 4-manifold obtained by replacing the tangle $\mathcal{T}$ or $\mathcal{M}$ in the diagram $\mathscr{D}$ with the tangle $\mathcal{T}'$ or $\mathcal{M}'$ respectively. It follows that $X \cong X'$.*

Proof. The argument is a direct calculation using Kirby calculus. All the slides and isotopies involved live in the 3-ball which contains the tangle. The calculation is given in Figure 8. We remark that the proof still works if we replace the black strand with any number of parallel strands. In each step of the diagram on the following page, we simply slide the entire bunch of black strands over each of the green 2-handles, and carry the entire bunch through the sequence of Reidemeister moves. □

The $\mathcal{M} \to \mathcal{M}'$ version is necessary when we use the lemma to change crossings between multiple parallel strands of satellite knots: we use it when we are unknotting the companion knot in a satellite construction. Each multi-crossing we change effectively adds twists to the satellite knot according to the change in writhe of the diagram of the companion. We can undo these extra twists either before or after unknotting the companion by some number of $\mathcal{M} \to \mathcal{M}'$ replacements.

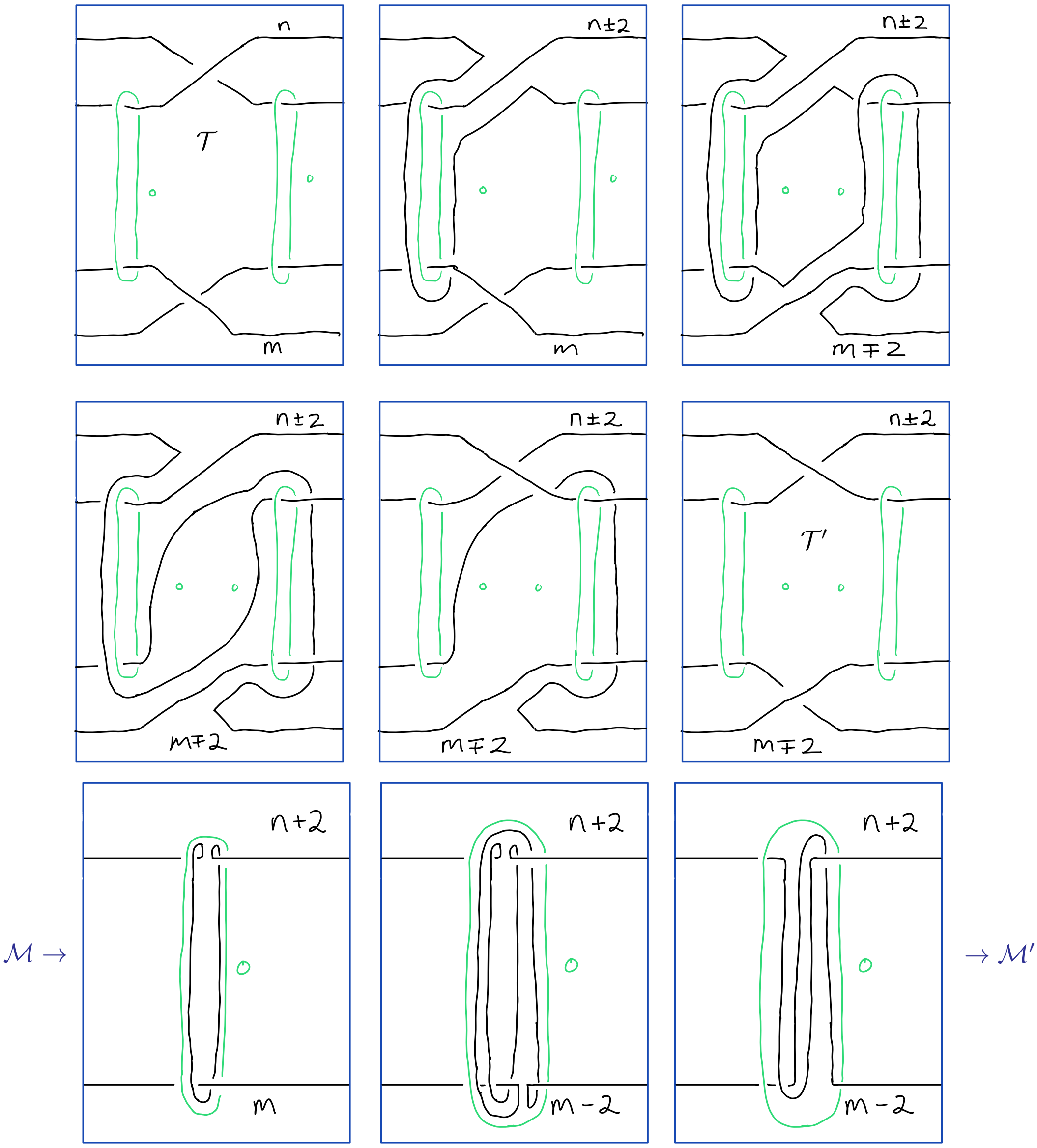

FIGURE 8. The proof of the Crossing Change Lemma: the first two rows take $\mathcal{T}$ to $\mathcal{T}'$. The third row shows the intermediate steps taking $\mathcal{M}$ to $\mathcal{M}'$.

3.2. **Kirby Diagrams of FS-Surgered Traces.** Let K, J be given as in Theorem 3.1, and let b_J be the bridge number of J. We can find a braid Br_J on $2b_J$-strands such that its closure shown in Figure 9 (top) gives J. We say that a knot is in *plat position* if its diagram has this form. Using this diagram for J, we obtain a Kirby diagram of $X_0(K)_J$ (bottom).

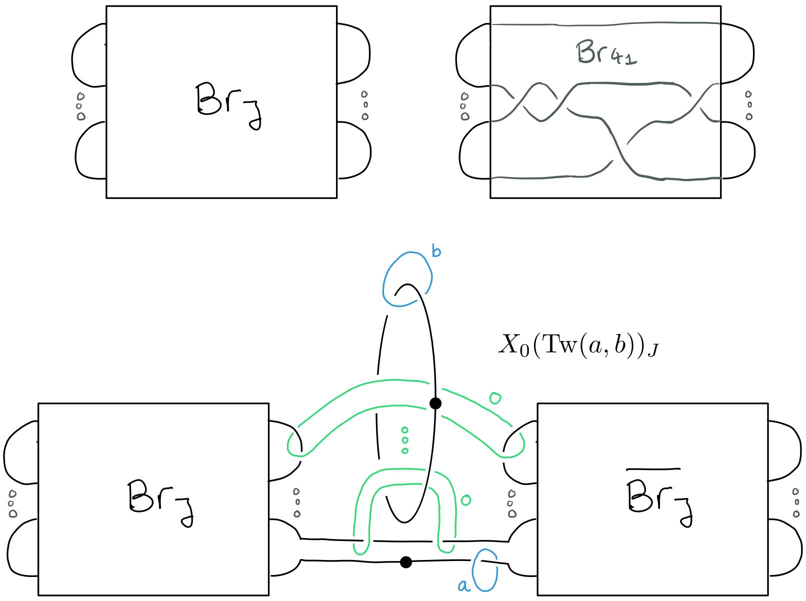

FIGURE 9. The knot J as a closure of the braid Br_J (top), and a Kirby diagram of $X_0(\mathrm{Tw}(a,b))_J$ (bottom)

We wish to point out three things. First, the carved copy of $J \# \overline{J}$ (dotted in the diagram) is carved along the spun disk for J. Second, we chose our gluing conventions for FS-surgery so that the 1-handle whose meridian represents the circle α is the one which gets tied in the knot $J \# \overline{J}$. Third, observe that if the black handle was an n-framed 2-handle instead of a carved slice disk, then we could apply the Crossing Change Lemma repeatedly between the black handle and the green handles to untangle the copies of Br_J and $\overline{Br_J}$ in the diagram by changing one pair of crossings at a time. This would untie the black handle and allow us to detach and cancel all but one of the green handles against implicit 3-handles. This observation is originally due to Akbulut, and it is the core of his argument in [Akb02].

3.3. **The Diffeomorphisms.** We will now prove Theorem 3.1 using the method of [Akb02, Thm. 2.1]. We will then point out how this method fails when the parity of a disagrees with the parity of the S^2-bundle summand ($S^2 \times S^2$ counts as even, $S^2 \tilde{\times} S^2$ as odd). We can use $\mp CP^2$ for the cases when $a = \pm 1$ exactly, which will become clear from the method of the proof.

Proof. [Theorem 3.1] Let $K = \mathrm{Tw}(a,b)$ and $J \subset S^3$ be given, and assume that we are given a diagram of J in plat-position. Let H denote the right-handed Hopf link, and note that the 4-manifold $X_{(-a,0)}(H)$ is diffeomorphic to $S^2 \times S^2$ if $a \equiv 0 \mod 2$ and $S^2 \tilde{\times} S^2$ if $a \equiv 1 \mod 2$. We perform a direct Kirby calculation in Figure 10 to show

$$X_0(\mathrm{Tw}(a,b))_J \# X_{(-a,0)}(H) \cong X_0(\mathrm{Tw}(a,b)) \# X_{([a]_{\mathrm{mod}\ 2},0)}(H).$$

The proof of the remaining statements is essentially the same. When $a = \pm 1$ we introduce a summand of $X_{\mp 1}(U)$ which we use to blow-up the a-framed 2-handle. This reduces the framing to zero, so we can cancel the resulting 1-2 pair leaving an ∓ 1 framed 2-handle where the carved copy of $J \# \overline{J}$ was. We then apply the Crossing Change Lemma to undo the knot, and proceed as in the previous case (see Figure 11).

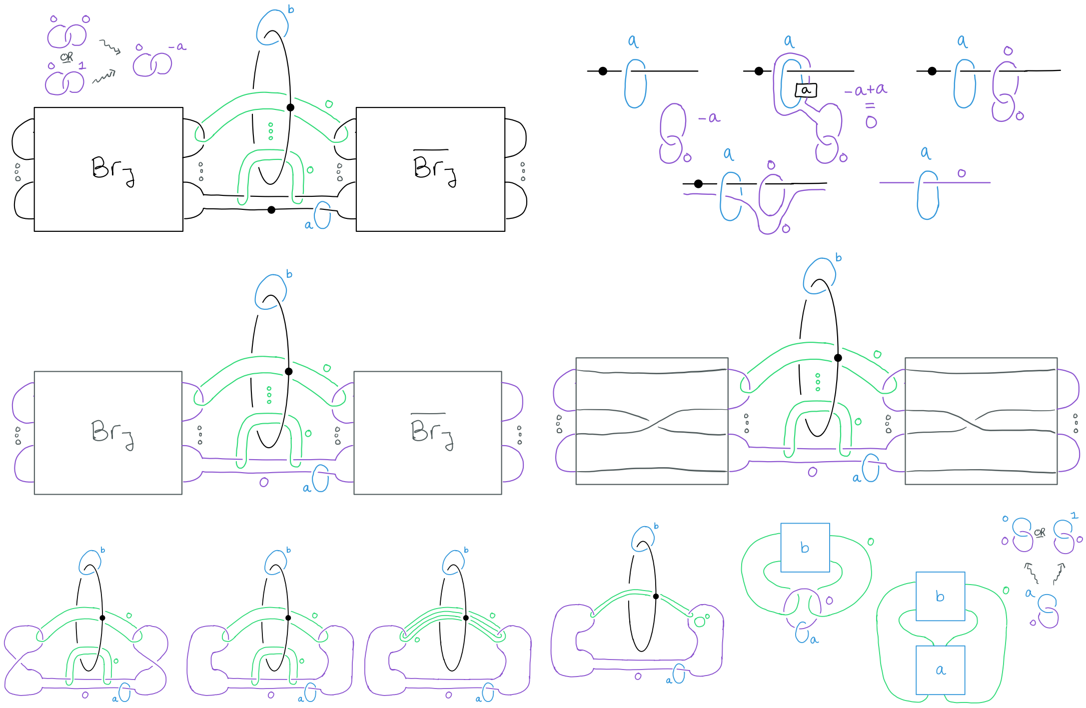

FIGURE 10. Undoing FS-surgery in presence of an $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$ summand. The step between the middle two pictures involves simplifying Br_J and $\overline{Br_J}$, one crossing at a time, using repeated applications of the Crossing Change Lemma.

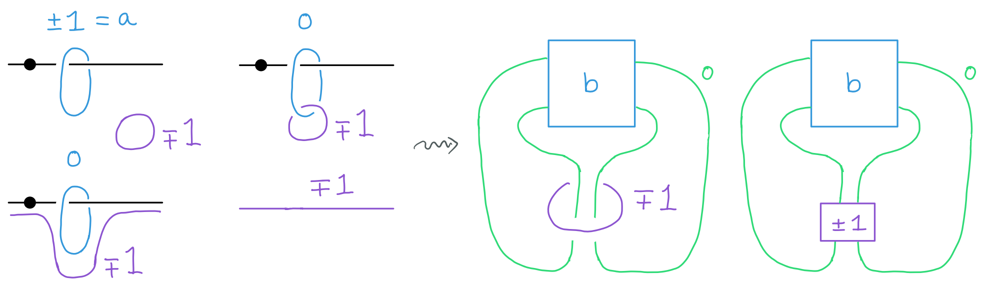

FIGURE 11. Undoing FS-surgery in presence of a $\mp CP^2$ summand. The grey arrow represents the same intermediate steps shown in Figure 10.

□

3.4. **The Parity Issue.** What happens when the parity of a does not match the parity of the S^2-bundle summand? Consider the right handed trefoil $K = \overline{3_1} = \mathrm{Tw}(-1,-1)$, whose 0-trace is associated to the elliptic surfaces $E(n)$. Clearly $X_0(K)$ is spin, and so $X_0(K)_J \# S^2 \times S^2$ is spin. In order to apply the Crossing Change Lemma to undo the FS-surgery, we need to show that this manifold has a Kirby diagram which looks like a diagram of $X_0(K)_J$ after turning the carved $J \# \overline{J}$ into a 2-handle (its exact framing does not matter). However, this is impossible! In such a diagram, the (-1)-framed 2-handle attached to the meridian of $J \# \overline{J}$ could be blown down, which proves the manifold in that diagram splits off a $\overline{CP}^2$-summand. Thus the manifold is not spin and cannot be diffeomorphic to $X_0(K)_J \# S^2 \times S^2$. We show this explicitly in Figure 12 (top).

Another route would be to try to carry out the method of the proof of Theorem 3.1 despite the framing difference. In this case, we are working with $X_0(\mathrm{Tw}(a,b))_J \# X_{(-a-1,0)}(H)$ at the start. After performing the same initial set of slides, we obtain Figure 12 (bottom). Observe that the red pair of handles do not cancel, unlike the previous case. This is because they give an embedded copy of the (-1)-framed blowdown of the slice disk for $J \# \overline{J}$. Although this manifold is contractible, it is not the 4-ball for any non-trivial J. This can be seen since its boundary is $S^3_{+1}(J \# \overline{J}) \not\approx S^3$ by the Property-P conjecture. It follows that we cannot obviously eliminate the carved disk from the Kirby diagram, and so it is very uncertain whether we can apply the Crossing Change Lemma to undo the FS-surgery. We therefore pose this as a problem:

Problem 3.1. Write down an explicit sequence of Kirby moves taking:

$$X_0(\mathrm{Tw}(-1,-1))_{3_1} \# S^2 \times S^2 \longrightarrow X_0(\mathrm{Tw}(-1,-1)) \# S^2 \times S^2$$

Baykur's argument [Bay18] proves that the manifolds in Problem 3.1 are diffeomorphic avoiding the parity issue we have run into here. Therefore, it should be possible in theory to read Baykur's argument carefully, and follow it along using Kirby calculus to find the explicit diffeomorphism we are missing. Our attempts at this have been unsuccessful so far, so we pose this problem to the greater community. Once an explicit diffeomorphism has been found, the methods of this paper will be able to find corks in $E(2n)_J$ which undo the Fintushel-Stern surgery and solve Problem 1.1.

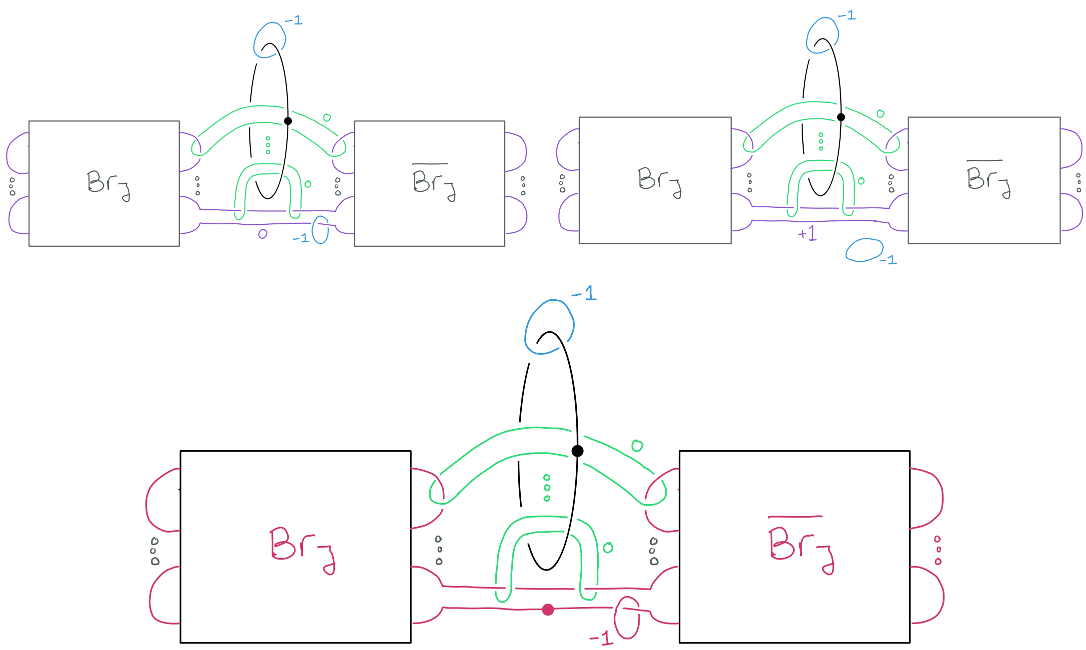

Figure 12. Replacing the carved knot with a 2-handle is not equivalent to summing with $S^2 \times S^2$ in $X_0(\mathrm{Tw}(-1,-1))_J$ (top). Finding the blowdown of $J \# \overline{J}$ inside $X_0(\mathrm{Tw}(-1,-1))_J$ (bottom).

4. Finding Corks in Fintushel-Stern Surgered Traces

We now turn in earnest to the proof of Theorem 1.2 and Corollary 1.1.

4.1. **Proof of Theorem 1.2.** By the hypotheses of Theorem 1.2, K must satisfy $g_3(K) = 1$ and $\mathrm{Arf}(K) = 0$. The first condition means that $X_0(K)$ will have a Kirby diagram as in Figure 13, where the tangle τ may be any four ended tangle with the connectivity indicated by the dashed lines. Meanwhile, the $\mathrm{Arf}(K) = 0$ condition implies that the framing of at least one of handles attached along α or β is even. We assume without loss of generality that the framing of α is even, and we perform the FS-surgery in such a way that the 1-handle linked by α is the one replaced by $J \# \overline{J}$ as described in Figure 9.

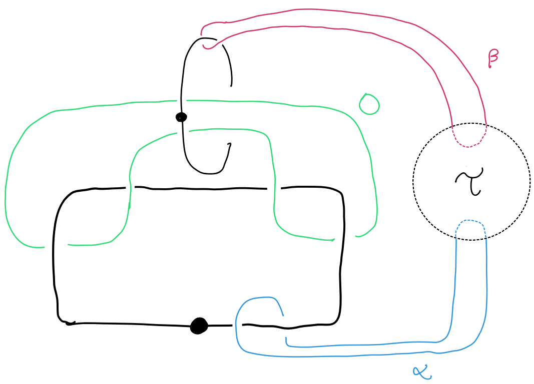

Figure 13. A Kirby diagram of $X_0(K)$ for any knot with 3-genus one. The box labelled τ will contain a non-unique 4-ended tangle which connects the two red ends together and the two blue ends together.

The simplest knots K satisfying these hypotheses are the twist knots $\mathrm{Tw}(a, b)$, for which α and β are disjoint. We will describe a single example of the procedure: we will find the cork twist for $K = \mathrm{Tw}(4, b)$ and $J = 4_1$. We claim this example contains all the ideas needed for the general case.

Proof. [Theorem 1.2]

Let $K = \mathrm{Tw}(4, b)$ and let $J = 4_1$; observe that K satisfies the theorem's hypotheses. We will perform a sequence of Kirby moves on a the diagram of $X_0(K)_J$ until we see an embedded sub-diagram which we can recognize as $C(K, J)$. We will perform the cork twist by swapping the dot and zero on the two components of $C(K, J)$ in the diagram, which corresponds to cutting out $C(K, J)$ and replacing it with $C^*(K, J)$ in the manifold. We will then continue doing Kirby calculus until we obtain a diagram which is almost $X_0(K)$. Here, we will use a cute trick: there will be a 1-2 handle pair with *geometric* linking number one, with the framing of the 2-handle equal to zero, and with both handles attached along unknots. These handles give an embedded 4-ball in the manifold, and since 4-balls are uniquely attached to 4-manifolds, we can swap the dot and the zero on these two handles without changing

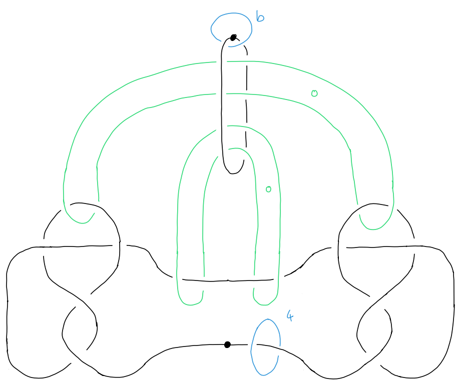

Figure 14. The starting diagram for $X_0(\mathrm{Tw}(4, b))_{4_1}$. Note we may cancel the b-framed 2-handle against the upper 1-handle in favor of a twist box.

the diffeomorphism type of our manifold (effectively performing a cork twist on the trivial cork: B^4). We will then arrive at the standard Kirby diagram of $X_0(K)$, and the theorem will be proved.

We start with Figure 14 and name the handles as follows: the green 0-framed 2-handles which pass through the twist box are called 'ropes', after Akbulut; the remaining carved knot is called R, a reminder of 'Ribbon'; and the 4-framed 2-handle is called α, just like its attaching circle. The remaining diagrams will be on two pages following the description of the steps.

We slide α twice over R to reduce its framing to 0. In general, we need $\mathrm{Arf}(K) = 0$ for this step. The α and β curves arise as a symplectic basis for the linking form of a Seifert torus for K, and the framings of the corresponding 2-handles of the $X_0(K)$ are given by their self-linking numbers. It is a classical fact that $\mathrm{Arf}(K) = 0$ if and only if at least one of these two basis curves has even self-linking number. Therefore, we may assume that $a = \mathrm{fr}(\alpha) \equiv 0 \mod 2$, which allows us to slide α over R until its framing is 0. Had we done the FS-surgery with a is odd, then this method would not work. See Figure 15 **(1)**. Next, we slide the double-strands of α over the ropes to make α unknotted as an individual component in the diagram **(2)**.

We can now recognize our cork as the sub-diagram given by $\alpha \cup R$:

$$C(\mathrm{Tw}(4, b), 4_1) = (B^4 \setminus \nu(D^2_R)) \cup h^2_4(\alpha, 0)$$

If we were to cancel the b-framed 2-handle against the obvious 1-handle, then we would introduce b full twists where the 1-handle used to be, and this gives an explicit embedding of $C(\mathrm{Tw}(4, b), 4_1) \hookrightarrow X_0(\mathrm{Tw}(4, b))_{4_1}$. Next, we swap the dot and the zero, thereby performing the cork twist Figure 15 **(3)**, and then we push R, which is now a 0-framed 2-handle, over the ropes **(4)**. We then perform the isotopy from the proof of the Crossing Change Lemma, and stretch out the ropes **(5,6,7)**.

Next comes the trick: notice that R and α are attached to a Hopf link if one erases all the other components from Figure 16 **(8)**. It follows that this diagram decomposes the manifold it represents into a 4-ball consisting of R and α with all the other handles attached on top. If we reverse the dot and zero on R and α, then their attaching link is still a Hopf link, so they still contribute a 4-ball to the manifold. Since 4-balls glue uniquely up to diffeomorphism, we can perform this swap without changing the diffeomorphism type of the manifold represented by the diagram **(9)**. We see that the ropes are now perfectly parallel to one another, so we can slide one over the other and cancel it against a 3-handle in the background, removing it from the diagram. Lastly, we slide α back over R to undo the slides we did at the beginning and return its framing to 4, and then we cancel the pair α and R to obtain $X_0(\mathrm{Tw}(4, b))$. Thus we have shown that the twist of $C(\mathrm{Tw}(4, b)) \subset X_0(\mathrm{Tw}(4, b))_{4_1}$ is diffeomorphic to $X_0(\mathrm{Tw}(4, b))$ **(12)**.

Modifying the previous steps to reverse FS-surgery on a general $X_0(K)_J$ satisfying the theorem's hypotheses is straightforward. If the companion knot J is different, we simply need to repeatedly apply the Crossing Change Lemma to unknot it and proceed as before. (In practice, the number of strands passing over the ropes becomes enormous.) The modifications required for knots K which are not twist knots are slightly more complicated. If K is not a twist knot, then in general the 2-handles attached along α, β may link each other and be knotted in any way *a priori* in the tangle τ. However, both 2-handles still have meridional 1-handles, and neither 2-handle passes through the other's 1-handle (β is disjoint from a Seifert disk for R). We can essentially ignore the 2-handle attached along β: at the end, it will only determine where the strands of α which have gone over the ropes end up. We can start the proof by sliding α over R to 'unknot it', then we slide it more times over R to reduce its framing to zero, and then continue with the rest of the proof as usual. To illustrate this point, we include the first part of the proof when α is attached along a 0-framed trefoil and β along a -1-framed unknot ($K = Wh^+_0(3_1)$) Figure 16 (bottom).

□

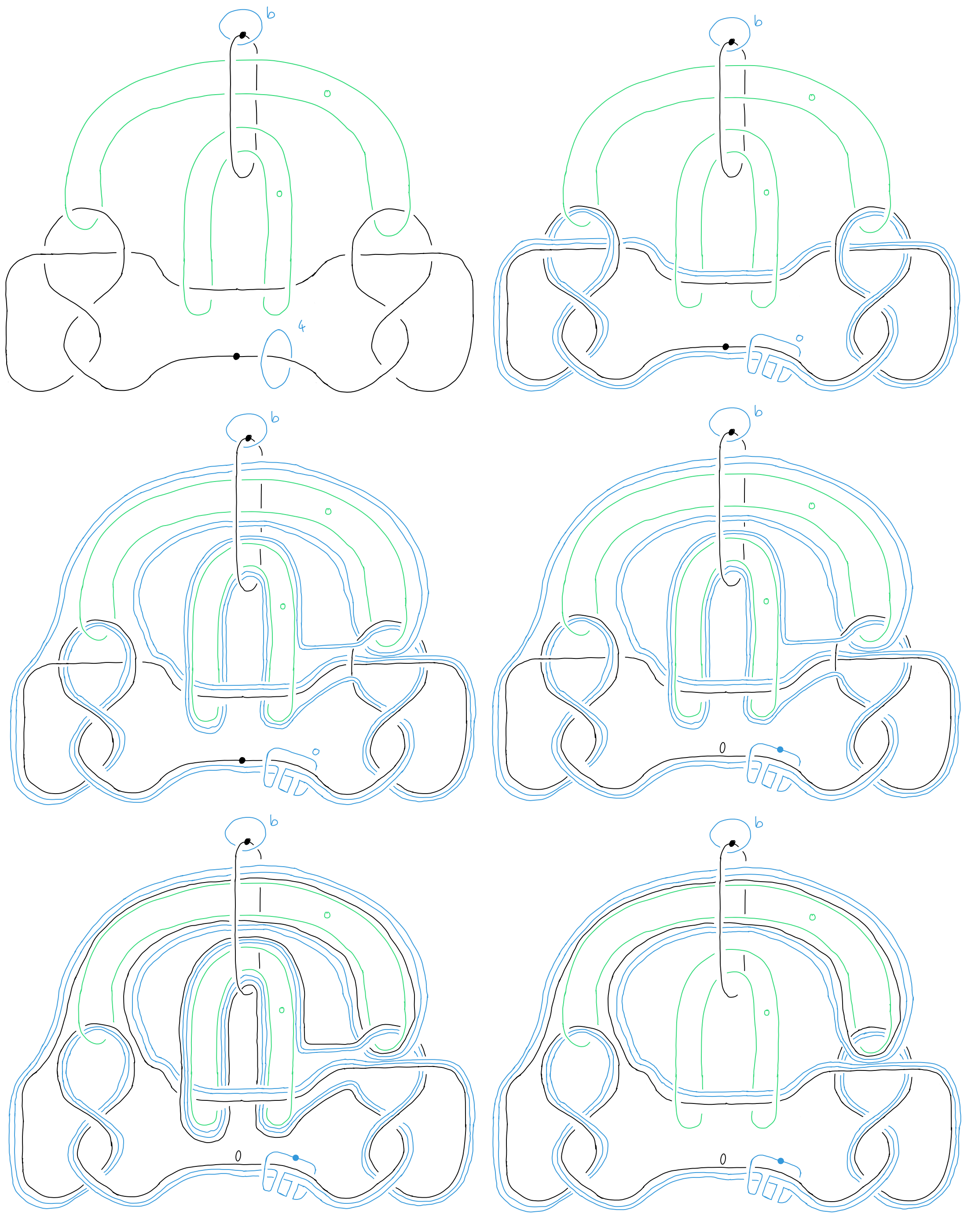

Figure 15. Proof that $C(\mathrm{Tw}(4,b),4_1) : X_0(\mathrm{Tw}(4,b))_{4_1} \to X_0(\mathrm{Tw}(4,b))$. Steps:

- **(0)** Starting diagram of $X_0(\mathrm{Tw}(4,b))_{4_1}$.
- **(1)** Slide α over R until $\mathrm{fr}(\alpha) = 0$.
- **(2)** Slide α over the 'ropes' to unknot it.
- **(3)** Twist the resulting cork: $\alpha \cup R$.
- **(4)** Slide R over the 'ropes'.
- **(5)** Simplify by an isotopy.

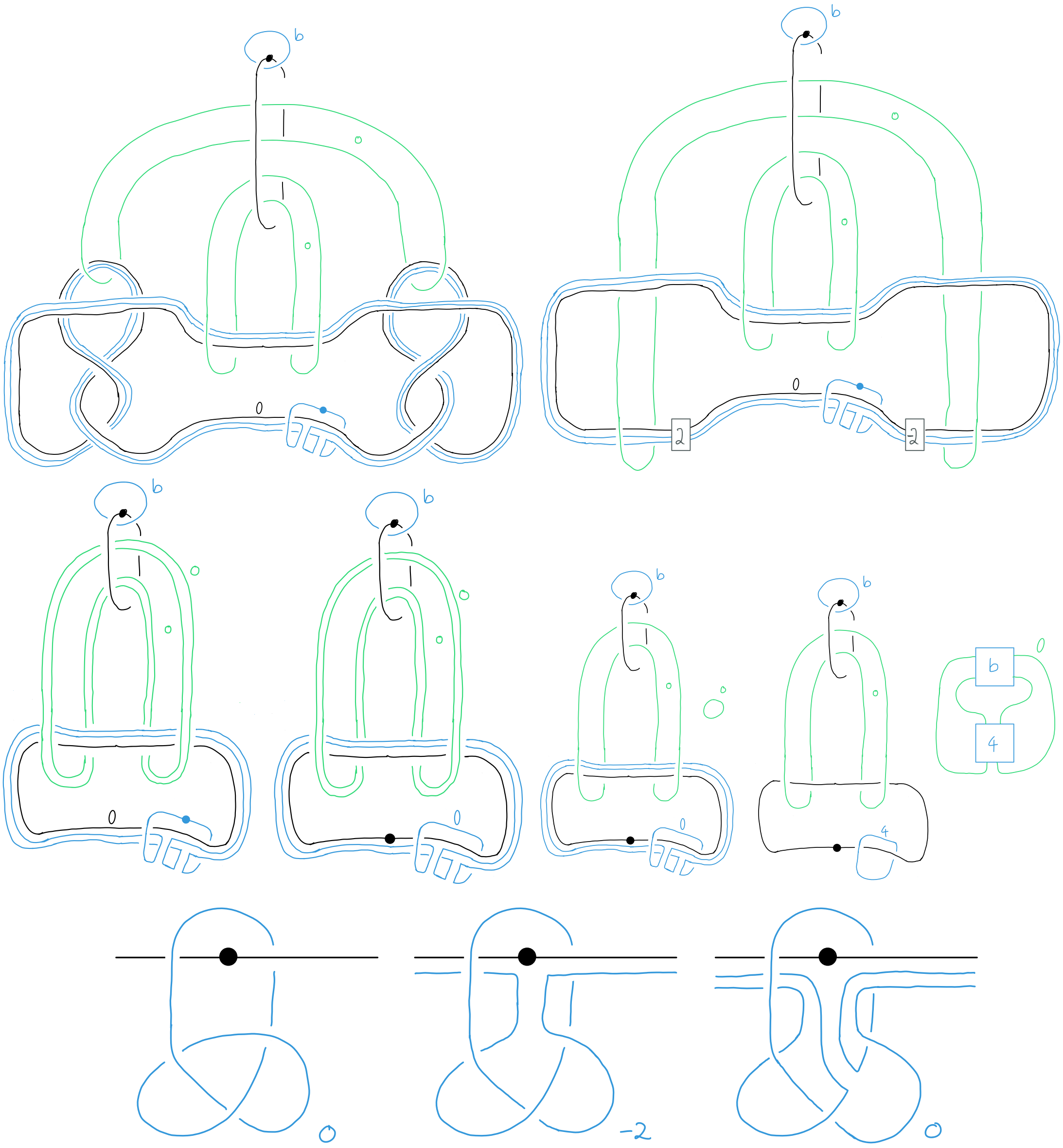

FIGURE 16. Steps:

(6) Another standard isotopy.
(7) Make the unknot round.
(8) Isotope the 'ropes' into parallel position.
(9) Perform a trivial ($\cong B^4$) cork twist on $\alpha \cup R$.
(10) Slide a rope over the other and cancel it against a 3-handle.
(11) Slide α over R to restore its original framing and position.
(12) Cancel both 1-handles to recover $X_0(K)$.
(*) If α starts out knotted, then slide it over R to 'unknot' it, and then do more slides to get $\mathrm{fr}(\alpha) = 0$. The rest follows as in (1)-(12).

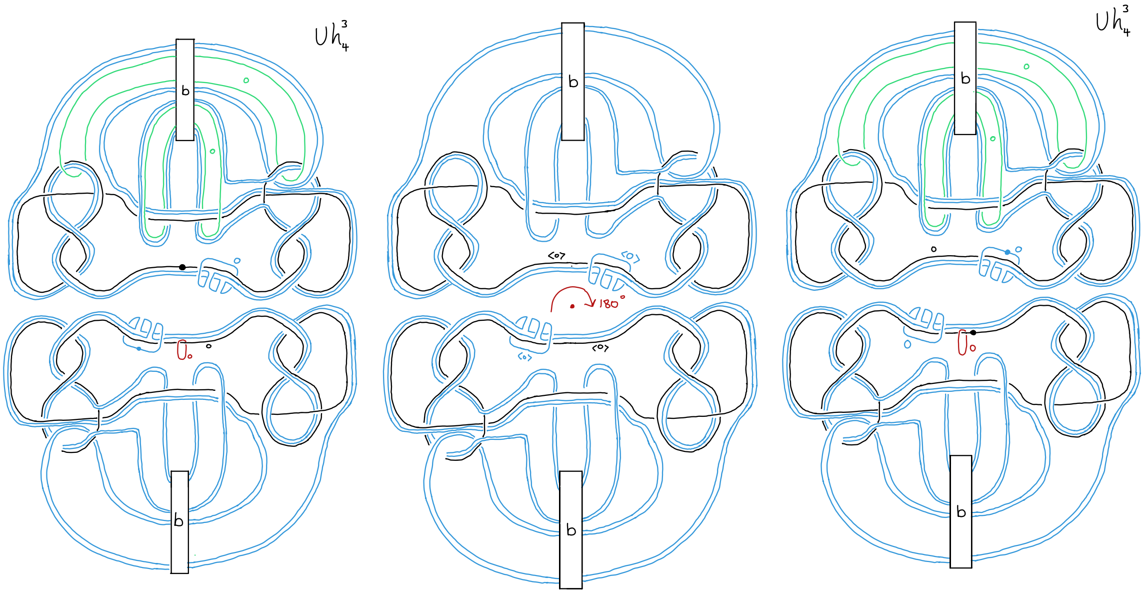

FIGURE 17. Finding an involutive cork in a trace. Steps:
(3a) Connect sum an S^4 with handle structure $C^* \cup_{\mathbb{1}_\partial} C^*$ (lower components).
(3b) The boundary involution $f\colon \partial W \to \partial W$.
(3c) Twist the involutive cork (W, f).

From the proof of Theorem 1.2, we obtain explicit descriptions of doppel corks connecting the homeomorphic manifolds $X_0(K)_J$ and $X_0(K)$. We can obtain embedded involutive corks, with an equivalent twist, by applying Proposition 2.12. Let (C, C^*) be the embedded doppel cork taking $X_0(K)_J \to X_0(K)$. Immediately before twisting this cork in $X_0(K)_J$ in Figure 15 **(2)**, we trivially connect sum our manifold with an S^4, to which we give the handle structure corresponding to $C^* \cup_{\mathbb{1}_\partial} C^*$ Figure 17 **(3a)**. Our involutive cork W will be the boundary sum of the original doppel cork C and the C^* component of S^4 given by the 1-2 handle pair. If we visualize ∂W as a 3-dimensional surgery diagram, the boundary involution $f\colon \partial W \to \partial W$ corresponds to rotating this diagram 180° around the distinguished point in red Figure 17 **(3b)**. Twisting the cork (W, f) corresponds to cutting out W and then gluing it back according to this rotation, which produces Figure 17 **(3c)**. Note how the lower connected component of this diagram is again $S^4 \cong C \cup_{\mathbb{1}_\partial} C^*$, so we can cancel all its handles and get back to Figure 15 **(3)**—the aftermath of twisting the original doppel cork. Therefore, both the doppel cork (C, C^*) and the involutive cork (W, f) yield twists that take $X_0(K)_J \to X_0(K)$.

4.2. **Proof of Corollary 1.1.** The proof of the corollary is essentially the same as that of the main theorem, except here we are working with the trace of the right-handed trefoil in $E(1)$. Thus, the framing of α is -1. However, the section of $E(1)$ also has framing -1, and it has linking number zero with R. Therefore, we can slide α over the section to make its framing even and then proceed as in the previous proof.

Proof. [Corollary 1.1] We start with a diagram of $E(1)$ showing a cusp neighborhood, which is diffeomorphic to $X_0(\mathrm{Tw}(-1,-1))$, and the section of the fibration (the purple 2-handle). Since this is all we need, we ignore the remaining handles (in orange) to get $X_0(\mathrm{Tw}(-1,-1)) \cup S \subset E(1)$, and perform FS-surgery with companion 4_1 to obtain $E(1)_{4_1}$. We name the purple handle S for 'section' and name the remaining handles as before (see Figure 18).

We slide α once over R and then once over S to reduce its framing to zero. Then we slide α over the ropes to unknot it, and we obtain our cork Figure 20 **(14,15)**. We leave it to the reader to check that α is unknotted, and that the cork diagram given here is isotopic to that of Figure 1. Then we perform the twist, and slide R thrice and S twice over the ropes. The intuition here is that we need to apply the crossing change lemma to one skein of R and S and a corresponding skein of R and α. Doing so will add extra pairs of twists in each skein, which do not automatically cancel via isotopy, because the skeins contain different handles. We remedy this by performing a trivial pair of crossing changes (effectively transforming a pair of Reidemeister I moves into their mirrors) using only the inner rope. This introduces the ± 2 twist boxes which appear in **(19)**. These twists offset those created by the isotopy in **(22,23)**.

The ropes are now parallel to one-another, which means we can slide one over the other and cancel it against a 3-handle **(25,26,27)**. Notice that α and R once more make a Hopf link, so we can swap their framings without changing the manifold: we used this trick earlier in the proof of Theorem 1.2. We do this and then reverse the sequence of slides of α over R and S by which we set $\mathrm{fr}(\alpha)$ to zero. We continue simplifying until we recover $E(1)$ **(34)**.

Once again, it should be clear that something similar can be done for any J and $E(2n-1)$ because there will always be a handle S attached to the same knot with odd framing. The trick is that after we slide α over S at the beginning, the framing of α will become $-2n-2$, which means we will need to slide α over R $(n+1)$ times. All those strands will have to slide over the ropes at least as many times as the unknotting number of J, which means the resulting simple cork will be terribly complicated. □

4.3. **Further Work.** The techniques of the previous section should provide ample motivation to study Problem 3.1. Once that problem is solved, our techniques will be able to find corks in the classical case $E(2)_J$, which is currently outside the scope of our results.

The figures to which we refer in the proof of Corollary 1.1 are given on the following pages, along with brief descriptions of what is happening in each.

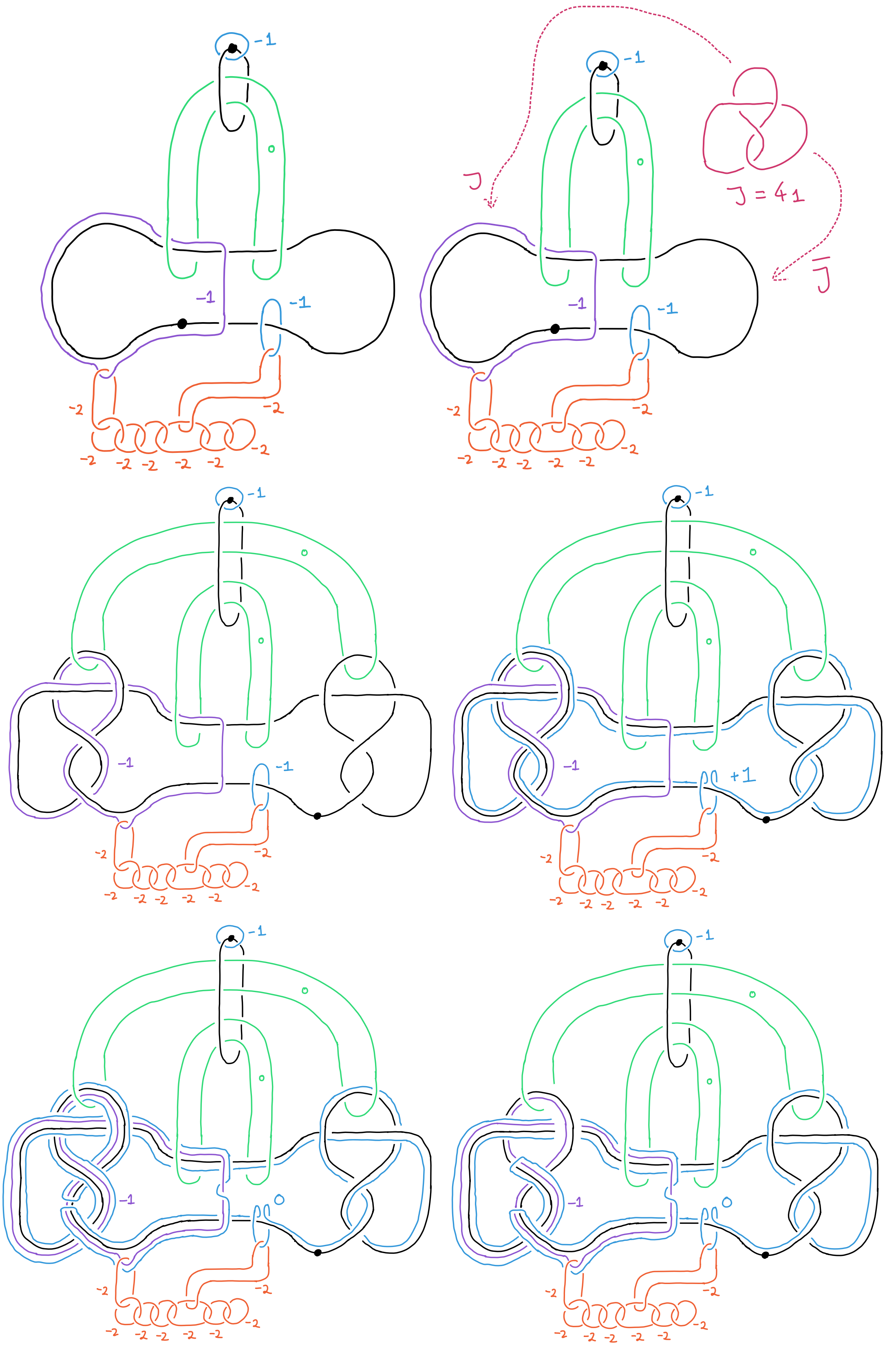

FIGURE 18. The proof of Corollary 1.1. Steps:

(0) Diagram of $E(1)$.
(1) Apply FS-surgery to $E(1)$ along a regular fiber with companion 4_1.
(2) Diagram of $E(1)_{4_1}$.
(3)-(4) Slide α over R and then over S to make $\mathrm{fr}(\alpha) = 0$.
(5)-(8) Retract the fingers of α along R.

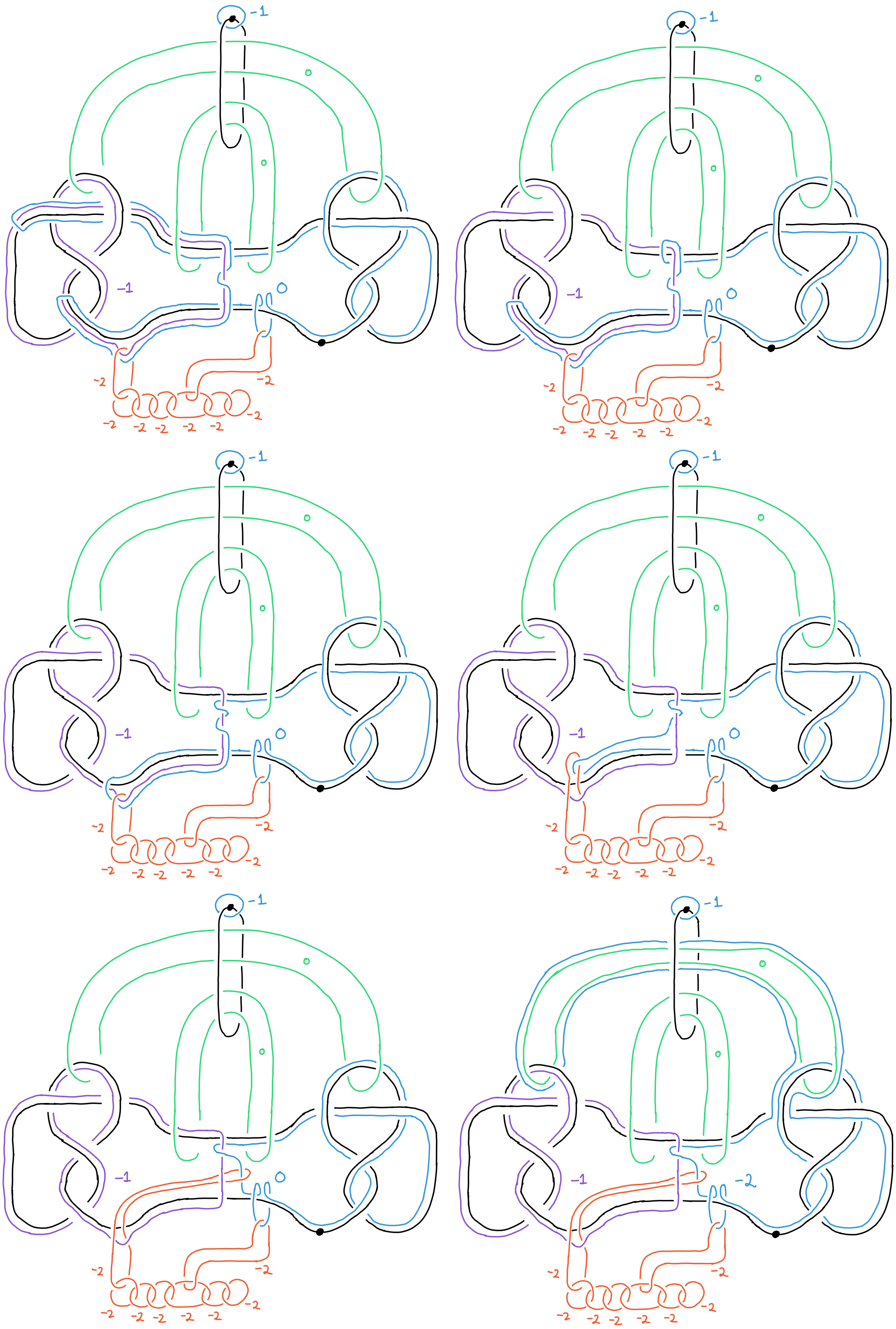

FIGURE 19.

(5)-(8) Retract the fingers of α along R.

(9)-(10) Retract the finger of α while extending a finger of orange.

(11)-(12) Slide α over the ropes as in the $\mathcal{M}$ version of Lemma 3.2.

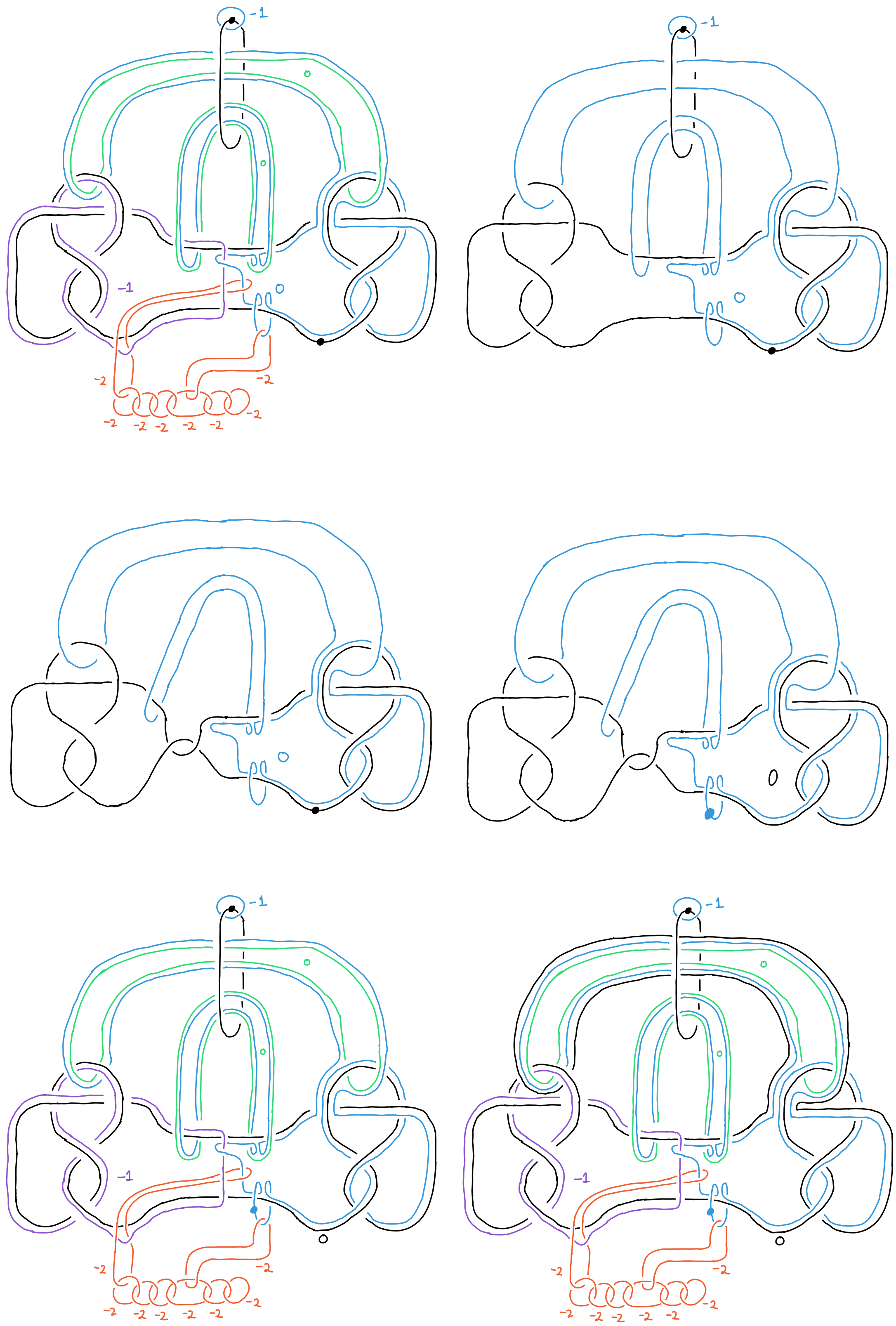

Figure 20. Steps:

(11)-(12) Slide α over the ropes as in the $\mathcal{M}$ version of Lemma 3.2.

(13)-(14) Temporarily remove S, the ropes, and the orange handles. Cancel the 1-2 pair on the top.

(15) Twist the cork formed by α and R by swapping their framing information.

(16) Bring back the temporarily removed handles, and uncancel the 1-2 pair on the top.

(17) Slide R over the outer rope.

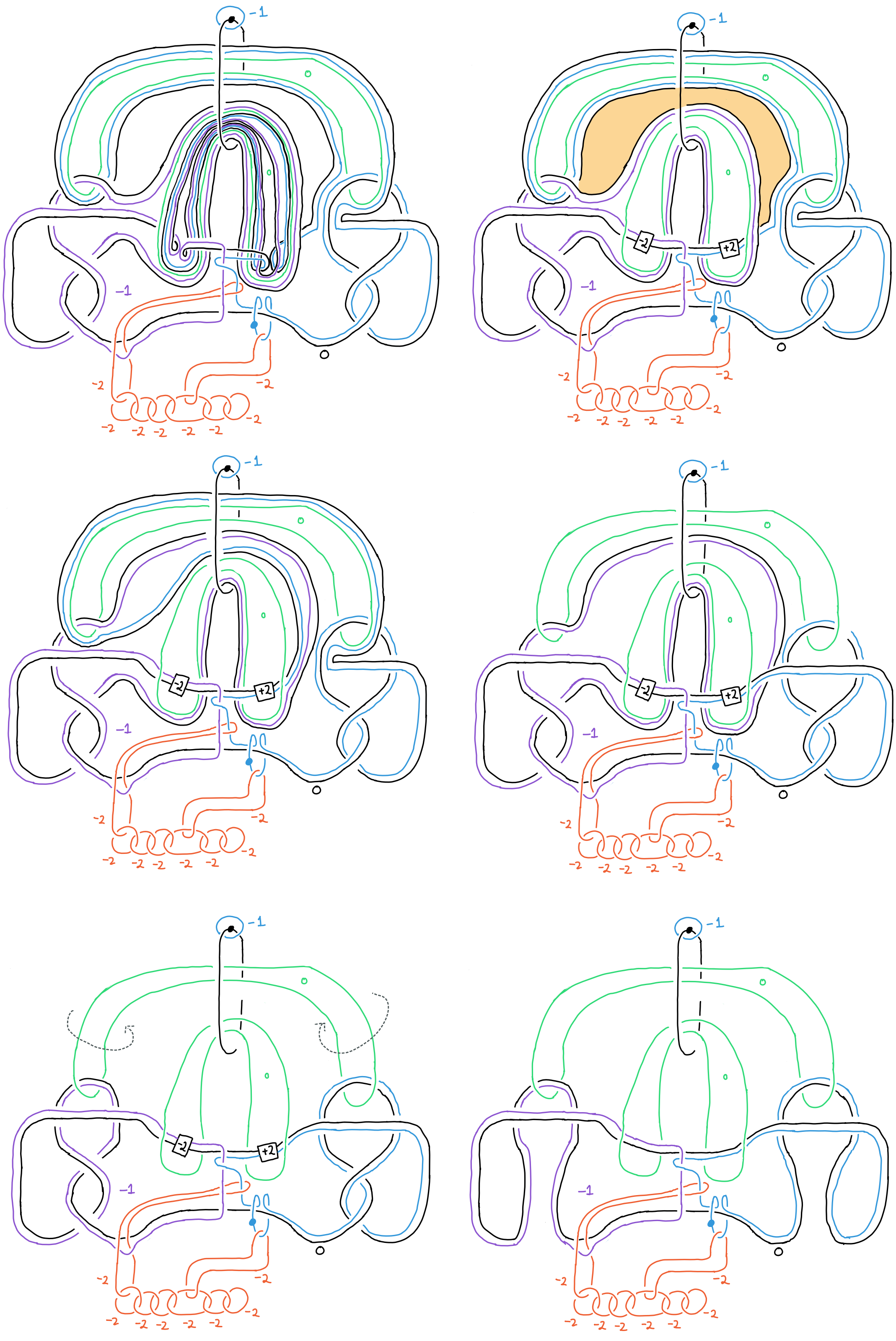

FIGURE 21. Steps:

(18) Slide R thrice and S twice over the inner rope.

(19) Use the isotopy of Lemma 3.2 to simplify the black-purple skein on the inner rope, adding twist boxes.

(20) Do a Reidmeister-2 move along the rectangle on the previous figure.

(21) Use the isotopy of Lemma 3.2 to simplify the black-blue skein on the outer rope.

(22) Retract the finger of R and S along the inner rope.

(23) Rotate the outer rope over itself.

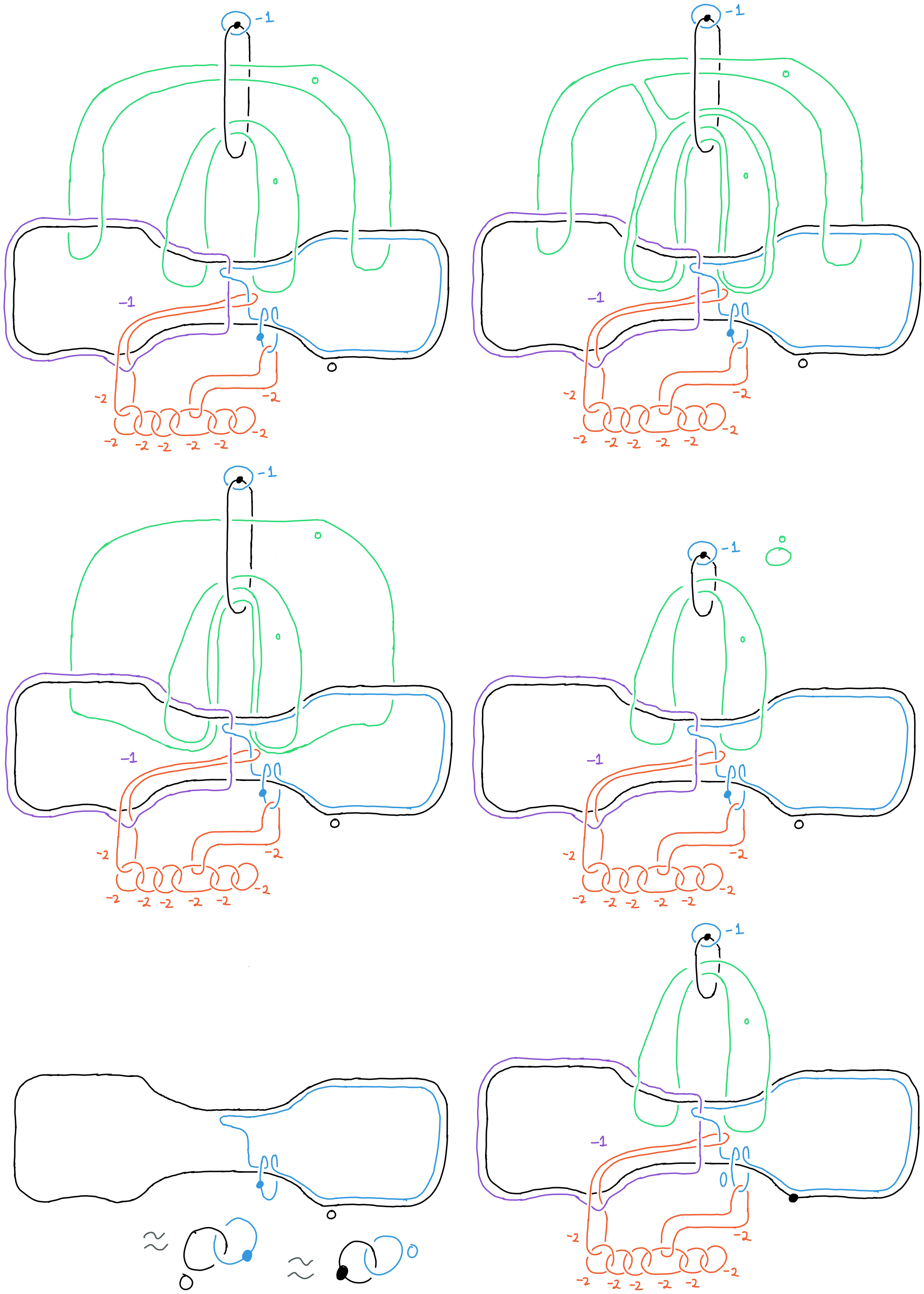

FIGURE 22. Steps:

- **(24)** Isotope the skeins of R with α and S.
- **(25)** Slide the outer rope over the inner rope.
- **(26)-(27)** Isotope the outer rope away from the diagram, and cancel it with a 3-handle.
- **(28)-(29)** Note that R and α form a B^4, so swap their framings.

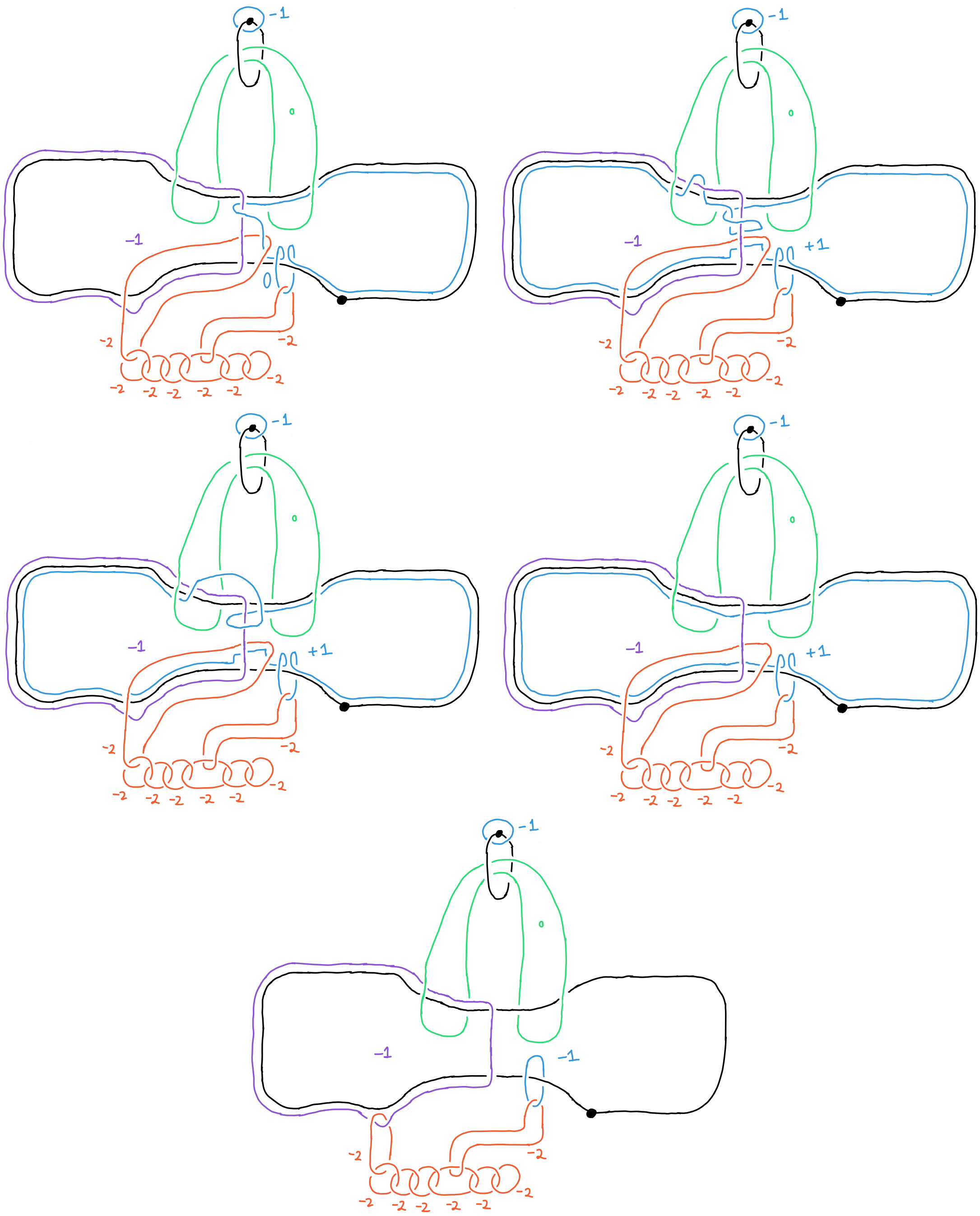

FIGURE 23. Steps:

(30)-(32) Slide α over S and then over R, reverting the slides on **(3)-(4)**.

(33) Retract the finger of orange.

(34) Recognize the original diagram of $E(1)$!

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