BOUNDING GAPS IN MULTIDIMENSIONAL WEYL SEQUENCES

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Abstract. For a fixed $N \in \mathbb{N}$ and a polynomial $P(x) \in \mathbb{R}[x]$, a Weyl sequence is a sequence of the form $\{P(n) \bmod 1\}_{n=1}^{N}$. Weyl sequences have natural properties to study such as the sizes of gaps. We first give a brief exposition of known results for Weyl sequences. Then we generalize definitions and formulate an approach to bound the number of gaps of higher-dimensional Weyl sequences.

1. Introduction

Given $N \in \mathbb{N}$ and d polynomials $P_1(x), \dots, P_d(x) \in \mathbb{R}[x]$, a d-dimensional Weyl sequence in $\mathbb{R}^d/\mathbb{Z}^d$ is a sequence of the form

$$S = S(P_1, \dots, P_d; N) := \left\{(P_1(n), \dots, P_d(n)) \bmod [0,1]^d\right\}_{n=1}^{N}$$

Over the past century, one-dimensional Weyl sequences have been extensively studied. In particular, the distribution of points and the length of gaps have long been of interest, and we naturally generalize this study to higher-dimensional Weyl sequences. We say that an open region $T \subset \mathbb{R}^d/\mathbb{Z}^d$ is a *gap* in S if $S \cap T = \varnothing$.

When $d = 1$, we are essentially studying the distribution of fractional parts of Weyl sequences for some polynomial $P(x) \in \mathbb{R}[x]$, of the form $S(P; N) = \{P(n) \bmod 1\}_{n=1}^{N}$. In this case, one classical result by Weyl [7, 8] was his work on uniform distribution of degree one Weyl sequences.

Fix some $N \in \mathbb{N}$, and suppose $P(n) = \alpha n$. If $\alpha \in \mathbb{Q}$, say $\alpha = a/q$, then there are only finitely many points in $S(P; N)$ that are unique. In particular, the sequence is periodic after q terms because $(a/q)n \equiv (a/q)(n+q) \pmod 1$. However, if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then all N points in $S(P; N)$ are distinct. In fact, if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the sequence $\{\alpha n \bmod 1\}_{n=1}^{N}$ appears uniformly distributed. More precisely, S is *equidistributed*, as described by [3] and [5] and formally formulated by Weyl [7, 8] in 1914.

Definition 1. A sequence $S(P; N) = \{P(n) \bmod 1\}_{n=1}^{N} \subset \mathbb{R}/\mathbb{Z}$ is *equidistributed* if for any $\delta \in [0, 1)$ and any interval I of measure δ,

$$\lim_{N \to \infty} \frac{\#(S(P; N) \cap I)}{N} = \delta.$$

Figure 1 shows the equidistribution of points in $S(P; N) = \left\{\sqrt{2}n \bmod 1\right\}_{n=1}^{N}$ for various values of N. Definition 1 captures what happens in fixed intervals $I \subset \mathbb{R}/\mathbb{Z}$ as

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$N \to \infty$ for a Weyl sequence. For fixed but large N, it is also interesting to consider the gaps in a Weyl sequence. A gap in $S(P;N)$ is an open interval (a,b) such that $S(P;N) \cap (a,b) = \varnothing$. One natural question is bounding the number of gaps of length at least σ in $S(P;N)$ for a fixed $\sigma > 0$. The points in $S(P;N)$ appear uniformly spaced, and the gaps have size approximately $1/N$.

FIGURE 1. The distribution of $S(P;N) = \left\{\sqrt{2}n \bmod 1\right\}_{n=1}^{N}$, for (from top to bottom) $N = 30, 50, 100$.

Unlike the case when $\deg P = 1$, when $\deg P \geq 2$, $S(P;N)$ exhibits behavior similar to that of a uniformly random sequence. For example, when $P(n) = \sqrt{2}n^2$ and $S(P;N) = \{P(n) \bmod 1\}_{n=1}^{N}$, Weyl proved that

$$|\#(S_N \cap [0,1/2]) - N/2| \leq 10N^{1/2} \log N.$$

Furthermore, numerical estimates suggest that for any $k \geq 2$ and for all $N \in \mathbb{N}$ with $P(n) = \sqrt{2}n^k$, the sequence $S(P;N) = \{P(n) \bmod 1\}_{n=1}^{N}$ satisfies

$$|\#(S_N \cap [0,1/2]) - N/2| \leq C_k N^{1/2} \log N,$$

for some constant C_k depending on k, though this has not been proven. Figure 2 compares the distribution of $S(P;N) = \left\{\sqrt{2}n^k\right\}_{n=1}^{N}$ for $k = 2,3$ and that of a random sequence for $N = 50$.

FIGURE 2. The distribution of $S(P;50) = \left\{\sqrt{2}n^k \bmod 1\right\}_{n=1}^{50}$, for (from top to bottom) $k = 2,3$ and a random sequence.

Following Vinogradov, we define $e(x) := e^{2\pi i x}$. Weyl's results on the equidistribution of degree 1 Weyl sequences used Fourier analysis to study the Weyl sums

$$\left|\sum_{n=1}^{N} e(P(n))\right|.$$

These sums are easier to analyze because we can use Fourier analysis to study them. These sums are also useful for studying higher degree Weyl sequences.

This paper is organized as follows. In Section 2 we introduce terminology motivated from our analysis of higher-dimensional Weyl sequences and discuss past results for 1-dimensional sequences. In Section 3 we give a discussion on finding a bound for higher-dimensional Weyl sequences. In Appendix A, we briefly discuss estimating the length of the longest gap in a one-dimensional Weyl sequence.

2. Bounding Weyl sums and gaps in Weyl sequences

We say $A \lesssim B$ if $A \leq CB$ for some constant C not depending on A or B. If $A \lesssim B$ and $B \lesssim A$, we say $A \sim B$. If E is a set, $|E|$ stands for the measure or the cardinality of E as appropriate.

To make our problem precise, we make the following definition. In the sequel, we fix a rectangular parallelepiped $T \subset \mathbb{R}^d/\mathbb{Z}^d$ with axes $e_1, \ldots, e_d$ and dimensions $\sigma_1, \ldots, \sigma_d$, so $|T| \sim \sigma_1 \cdots \sigma_d$. We refer to such a T as a tube.

Definition 2. We say that two tubes T_1 and T_2 with the same orientations and sides are *A-comparable* if there exists a fat tube $T^\dagger$ of the same orientation, but sides $A > 1$ times larger, such that $T_1 \cup T_2 \subseteq T^\dagger$. Otherwise, we say T_1 and T_2 are *A-incomparable.* A collection $\mathscr{G}$ is *pairwise A-incomparable* if no two members of $\mathscr{G}$ are A-comparable.

Given a d-dimensional Weyl sequence $S(P_1, \ldots, P_d; N) \subset \mathbb{R}^d/\mathbb{Z}^d$, we pick an arbitrary maximal collection $\mathscr{G}$ of pairwise A-incomparable tubes T, each contained in $S^{\complement}$. Our goal is to bound $|\mathscr{G}|$. We may pick an arbitrary maximal collection by the following lemma.

Lemma 1. *Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two maximal collections of pairwise A-incomparable tubes. Then $|\mathscr{G}_1| \sim_{A,d} |\mathscr{G}_2|$.*

The general idea of the proof is as follows. Consider an element $G \in \mathscr{G}_1$, and let $G' \subset \mathscr{G}_2$ be the set of elements in $\mathscr{G}_2$ that are pairwise A-comparable to G. The set

$$\mathscr{G}_2 \setminus \bigcup_{G \in \mathscr{G}_1} G'$$

has elements that are pairwise A-incomparable to elements in $\mathscr{G}_1$. Adding these elements to $\mathscr{G}_1$ contradicts the maximality of $\mathscr{G}_1$.

2.1. **Finding an upper bound for $|\mathscr{G}|$.** To find an upper bound for $|\mathscr{G}|$, we first let $\eta_T \colon \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}$ be an L^1-normalized smooth bump function on $0.9T$. In other words, η_T is supported on $0.9T$ and integrates to 1 over $\mathbb{R}^d/\mathbb{Z}^d$. Next we construct a measure μ on $\mathbb{R}^d/\mathbb{Z}^d$ defined by

$$\mu = \sum_{n=1}^{N} \delta_{(P_1(n),\ldots,P_d(n)) \bmod [0,1]^d}.$$

Define the function $f_T\colon \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{C}$ by $f_T(\mathbf{x}) \coloneqq (\mu * \eta_T)(\mathbf{x}) - N$. Then we have the lower bound

$$|\mathscr{G}||T|N^2 \lesssim \int_{\mathbb{R}^d/\mathbb{Z}^d} |f_T|^2. \tag{1}$$

This lower bound follows from the observation that there are $|\mathscr{G}|$ A-incomparable tubes, such that on each tube we have $(\mu * \eta_T)(\mathbf{x}) = 0$, so

$$|f_T(\mathbf{x})|^2 = |(\mu * \eta_T)(\mathbf{x}) - N|^2 = |-N|^2 = N^2.$$

2.2. **Finding an upper bound for $\int |f_T|^2$.** Now we describe an approach to find an upper bound to the integral in (1). We accomplish this using Fourier analysis and the theory of Weyl sums. In particular, by Parseval's theorem we have

$$\int_{\mathbb{R}^d/\mathbb{Z}^d} |f_T|^2 = \sum_{(z_1,\ldots,z_d)\in\mathbb{Z}^d} \left|\widehat{f_T}(z_1,\ldots,z_d)\right|^2. \tag{2}$$

Moreover,

$$\widehat{f_T}(0,\ldots,0) = \widehat{\mu}(0,\ldots,0)\widehat{\eta_T}(0,\ldots,0) - N \cdot \mathbf{1}_{(0,\ldots,0)}(0,\ldots,0) = 0.$$

Therefore, the sum in (2) is

$$\begin{aligned} \sum_{(z_1,\ldots,z_d)\in\mathbb{Z}^d} \left|\widehat{f_T}(z_1,\ldots,z_d)\right|^2 &= \sum_{(z_1,\ldots,z_d)\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \left|\widehat{f_T}(z_1,\ldots,z_d)\right|^2 \\ &= \sum_{(z_1,\ldots,z_d)\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \left|\widehat{\mu}(z_1,\ldots,z_d)\right|^2 \left|\widehat{\eta_T}(z_1,\ldots,z_d)\right|^2. \end{aligned} \tag{3}$$

Let $\mathbf{P} = (P_1,\ldots,P_d)$ and fix $\mathbf{q} = (z_1,\ldots,z_d)$. Then

$$\widehat{\mu}(z_1,\ldots,z_d) = \sum_{n=1}^{N} e((\mathbf{q}\cdot\mathbf{P})(n)),$$

so the sum in (3) becomes

$$\sum_{\mathbf{q}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \left|\sum_{n=1}^{N} e((\mathbf{q}\cdot\mathbf{P})(n))\right|^2 |\widehat{\eta_T}(z_1,\ldots,z_d)|^2. \tag{4}$$

To bound the sum in (3) we need to bound the Weyl sum in (4). To do this, we define the following two terms. For the sake of simplicity, we consider when the polynomials $P_1,\ldots,P_d$ are monomials and have the same degree.

Definition 3. Let $\alpha \in \mathbb{R}$. The *norm* of α is denoted by $\|\alpha\|$ and is defined by

$$\|\alpha\| \coloneqq \min_{z \in \mathbb{Z}} |\alpha - z|.$$

By definition, $\|\alpha\| \in [0, 0.5]$.

Definition 4. Let k be the degree of the monomials of $P_1, \ldots, P_d$. Let $\mathbf{P_k}$ be the column vector consisting of the coefficients of the kth-degree term in $P_1, \ldots, P_d$. Then

$$R_{T,N}(\mathbf{P_k}) \coloneqq \sum_{\mathbf{q} \in T^*} \min\left(N, \|\mathbf{q} \cdot \mathbf{P_k}\|^{-1}\right).$$

To get more intuition for these two definitions, we consider the case when $d = 1$.

2.3. **Theory of 1-dimensional Weyl sums.** Suppose then that $P(x)$ is a polynomial with leading term αx^k. We are interested in bounding the Weyl sum

$$\left|\sum_{n=1}^{N} e(P(n))\right|. \tag{5}$$

When $k = 1$, (5) is essentially a geometric series and the sum is precisely

$$\left|\sum_{n=1}^{N} e(P(n))\right| = \left|e(\alpha_0) \sum_{n=1}^{N} e(\alpha n)\right| = \left|\frac{e(\alpha) - e(\alpha)^{N+1}}{1 - e(\alpha)}\right| \leq \frac{2}{|1 - e(\alpha)|}. \tag{6}$$

Note that the bound in (6) is large when $|1 - e(\alpha)|$ is close to 0, and this occurs when α is close to an integer. This motivates Definition 3 from [3].

Because $|1 - e(\alpha)| \geq \|\alpha\|/5$, (6) implies the following theorem.

Theorem 1. *Let $P(x) = \alpha x + \alpha_0$ for some $\alpha, \alpha_0 \in \mathbb{R}$. Then*

$$\left|\sum_{n=1}^{N} e(P(n))\right| \lesssim \|\alpha\|^{-1}.$$

Essentially, Theorem 1 states that the size of a degree 1 Weyl sum depends on how close α is to an integer, up to a constant. Now we consider the case when $k = 2$. Suppose $P(x) = \alpha x^2 + \alpha_1 x + \alpha_0$. Note that

$$\left|\sum_{n=1}^{N} e(P(n))\right|^2 = \left(\sum_{n=1}^{N} e(P(n))\right)\left(\sum_{m=1}^{N} e(-P(m))\right) = \sum_{n,m=1}^{N} e(P(n) - P(m)). \tag{7}$$

We can write the sum in (7) as

$$\sum_{|d| \leq N} \sum_{n=N_1(d)}^{N_2(d)} e(P(n+d) - P(n)) \tag{8}$$

where $N_1(d)$ and $N_2(d)$ are bounds depending on d and $N_2(d)-N_1(d) \leq N$. However, $P(n+d)-P(n) = 2d\alpha n + \alpha_1 d$, so Theorem 1 implies that (7) is bounded by

$$\left|\sum_{n=1}^{N} e(P(n))\right|^2 = \sum_{|d|\leq N} \sum_{n=N_1(d)}^{N_2(d)} e(P(n+d)-P(n)) \lesssim \|2d\alpha\|^{-1}.$$

Moreover, the inner sum in (8) is bounded by N because

$$\left|\sum_{n=N_1(d)}^{N_2(d)} e(P(n+d)-P(n))\right| \leq \sum_{n=N_1(d)}^{N_2(d)} |e(P(n+d)-P(n))| = N_2(d)-N_1(d) \leq N$$

by the triangle inequality. Combining these two bounds yields

$$\left|\sum_{n=1}^{N} e(P(n))\right|^2 \lesssim \sum_{|d|\leq N} \min\left(N, \|2d\alpha\|^{-1}\right) \leq \sum_{|d|\leq 2N} \min\left(N, \|d\alpha\|^{-1}\right). \tag{9}$$

Now (9) motivates the following adapted definition from [6].

Definition 5. Let $\alpha \in \mathbb{R}$. Then define

$$R_{D,N}(\alpha) = \sum_{|d|\leq D} \min\left(N, \|d\alpha\|^{-1}\right).$$

Hence, we have the following theorem.

Theorem 2. *Let $P(x)$ be a polynomial with leading term αx^2 for some $\alpha \in \mathbb{R}$. Then*

$$\left|\sum_{n=1}^{N} e(P(n))\right|^2 \lesssim R_{2N,N}(\alpha).$$

Now we briefly discuss bounding Weyl sums of degree greater than 2. The main mechanism here is *Weyl differencing* [7,8], which we have used in (8) to compute the bound in Theorem 2. This mechanic essentially lowers the degree of the considered polynomial $P(n)$ from k to $k-1$, and we can inductively apply Weyl differencing a total of $k-1$ times to reduce $P(n)$ to a degree 1 polynomial, which can be fairly approximated because it is a geometric series.

Lemma 2 (Weyl differencing [7,8])**.** *Let $P(x) = \alpha x^k$ and let $P_d(x) := P(x+d)-P(x)$. Note that $P_d(x) = kd\alpha x^{k-1}$. If $N_2 - N_1 \leq N$, then*

$$\left|\sum_{n=N_1}^{N_2} e(P(n))\right|^2 = \sum_{|d|\leq N} \sum_{n=N_1(d)}^{N_2(d)} e(P_d(n))$$

for some integers $N_1(d), N_2(d)$ satisfying $N_2(d) - N_1(d) \leq N$.

By repeated Weyl differencing we obtain Theorem 3.

Theorem 3. *Let $P(x)$ be a polynomial with leading term αx^k. For any $\delta > 0$, there exists some C_δ depending on δ such that*

$$\left|\sum_{n=1}^{N} e(P(n))\right|^{2^{k-1}} \leq C_\delta N^\delta N^{2^{k-1}-k} R_{k!N^{k-1},N}(\alpha).$$

If α is close to a rational number of the form a/d, then $\|d\alpha\|^{-1}$ can be very large. We make this formulation more precise with the following definition.

Definition 6. A number α is (C,ε)-Diophantine if for all $a/q \in \mathbb{Q}$, $(a,q) = 1$,

$$\left|\alpha - \frac{a}{q}\right| \geq \frac{C}{q^{2+\varepsilon}}.$$

For example, $\sqrt{2}$ is $(1/10, 0)$-Diophantine. We can further bound $R_{D,N}(\alpha)$ if α is (C,ε)-Diophantine.

Proposition 1. *Let α be (C,ε)-Diophantine. Then*

$$R_{D,N}(\alpha) \lesssim C^{-1} N^{1+\varepsilon} \log N + D \log N.$$

Another useful bound for $R_{D,N}(\alpha)$ is the following.

Proposition 2. *For any $\delta > 0$,*

$$\sum_{0<|a|\leq A} R_{D,N}(a\alpha) = C_\delta A^\delta D^\delta R_{AD,N}(\alpha)$$

for some constant C_δ depending on δ.

Propositions 1 and 2 can be used to prove a bound on the number of gaps of length at least σ in a degree 2 Weyl sequence. This statement is formulated more precisely in the following theorem.

Theorem 4. *Let α be (C,ε)-Diophantine. Let $P(n) = \alpha n^2$, and consider the Weyl sequence $S_N = \{P(n) \bmod 1\}_{n=1}^N \subset \mathbb{R}/\mathbb{Z}$. The number of gaps in S_N with length at least σ is bounded by $(\sigma^{-2}N^{-1+\varepsilon})^{1+o(1)}$.*

The proof of Theorem 4 involves constructing a measure μ, defined by

$$\mu = \sum_{n=1}^{N} \delta_{P(n) \bmod 1} \tag{10}$$

on $\mathbb{R}/\mathbb{Z}$, where δ is the Dirac delta. Using Propositions 1 and 2, one can show the following lemma.

Lemma 3. *Let μ be the measure constructed in* (10)*. Then*

$$\sum_{0<|a|\leq A} |\widehat{\mu}(a)|^2 \leq C_\delta A^\delta N^\delta R_{2AN,N}(\alpha). \tag{11}$$

In particular, if α is (C,ε)-Diophantine, then (11) *becomes*

$$\sum_{0<|a|\leq A} |\widehat{\mu}(a)|^2 \leq C_\delta A^\delta N^\delta C^{-1} A N^{1+\varepsilon}. \tag{12}$$

Now we are ready to prove Theorem 4.

Proof of Theorem 4. Let G be the number of gaps in S of length at least σ. Let η_w be an L^1-normalized smooth bump function of width w over $\mathbb{R}/\mathbb{Z}$. In other words, η_w is supported on $[-w, w]$ and integrates to 1 over $\mathbb{R}/\mathbb{Z}$. Define the function $f_w \colon \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ by $f_w(x) := (\mu * \eta_w)(x) - N$. We want to bound the quantity

$$\int_{\mathbb{R}/\mathbb{Z}} |f_w(x)|^2 \, dx.$$

Let $w = \sigma/10$. On one hand,

$$GN^2\sigma \lesssim \int_{\mathbb{R}/\mathbb{Z}} |f_w(x)|^2 \, dx \tag{13}$$

because there are G gaps of length at least $4\sigma/5$ where $|f_w(x)|^2 = N^2$. We obtain the value N^2 by an analysis of the gaps. Let $(a - \sigma/2, a + \sigma/2)$ be a gap of length σ in S. Therefore, for any $x \in (a - 2\sigma/5, a + 2\sigma/5)$, we have

$$f_w(x) = (\mu * \eta_w)(x) - N = \sum_{n=1}^{N} \eta_w(x - P(n)) - N = -N,$$

and so $|f_w(x)|^2 = N^2$. On the other hand,

$$\int_{\mathbb{R}/\mathbb{Z}} |f_w(x)|^2 \, dx \leq \left(w^{-1}N\right)^{1+o(1)}. \tag{14}$$

To obtain (14), we apply Parseval's theorem and get

$$\int_{\mathbb{R}/\mathbb{Z}} |f_w(x)|^2 \, dx = \sum_{a \in \mathbb{Z}} \left|\widehat{f_w}(a)\right|^2 = \sum_{a \in \mathbb{Z}\setminus\{0\}} |\widehat{\mu}(a)|^2 \, |\widehat{\eta_w}(a)|^2.$$

We partition the sum

$$\sum_{a \in \mathbb{Z}\setminus\{0\}} |\widehat{\mu}(a)|^2 \, |\widehat{\eta_w}(a)|^2 = \left(\sum_{0<|a|\leq w^{-1}} + \sum_{|a|>w^{-1}} \right) |\widehat{\mu}(a)|^2 \, |\widehat{\eta_w}(a)|^2$$

and evaluate each sum separately. By (11), the first sum is bounded by

$$\sum_{0<|a|\leq w^{-1}} |\widehat{\mu}(a)|^2 \, |\widehat{\eta_w}(a)|^2 \leq \sum_{0<|a|\leq w^{-1}} |\widehat{\mu}(a)|^2 \leq C_\delta w^{-\delta-1} N^\delta C^{-1} N^{1+\varepsilon}.$$

The second sum is bounded by

$$\begin{aligned}\sum_{|a|>w^{-1}} |\widehat{\mu}(a)|^2 \, |\widehat{\eta_w}(a)|^2 &\lesssim \sum_{k=0}^{\infty} \frac{1}{w^2} \sum_{\frac{2^k}{w}<|a|\le\frac{2^{k+1}}{w}} \frac{1}{|a|^2} \, |\widehat{\mu}(a)|^2 \\ &\le \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \sum_{0<|a|\le\frac{2^{k+1}}{w}} |\widehat{\mu}(a)|^2 \\ &\le \sum_{k=0}^{\infty} \frac{1}{2^{2k}} C_\delta \frac{2^{k\delta+\delta+k+1}}{w^{\delta+1}} N^\delta C^{-1} N^{1+\varepsilon} \\ &\le C_\delta w^{-\delta-1} N^\delta C^{-1} N^{1+\varepsilon} \sum_{k=0}^{\infty} 2^{k(\delta-1)}.\end{aligned}$$

Combining these two bounds yields that

$$\sum_{a\in\mathbb{Z}\setminus\{0\}} |\widehat{\mu}(a)|^2 \, |\widehat{\eta_w}(a)|^2 \le C_\delta C^{-1} \frac{N^{1+\delta+\varepsilon}}{w^{\delta+1}}.$$

Hence, (13) and (14) imply that

$$GN^2\sigma \lesssim \int_{\mathbb{R}/\mathbb{Z}} |f_w(x)|^2 \, dx \le C_\delta C^{-1} \frac{N^{1+\delta+\varepsilon}}{w^{\delta+1}}, \tag{15}$$

in which we obtain our desired bound for G. □

3. Higher-dimensional Weyl sequences

We now return to the higher-dimensional Weyl sequences. We fix an $N \in \mathbb{N}$ and let $P_1(x), \dots, P_d(x) \in \mathbb{R}[x]$. Consider the d-dimensional Weyl sequence

$$\left\{(P_1(n), \dots, P_d(n)) \bmod [0,1]^d\right\}_{n=1}^{N} \subset \mathbb{R}^d/\mathbb{Z}^d.$$

Recall that we constructed the measure μ by

$$\mu = \sum_{n=1}^{N} \delta_{(P_1(n),\dots,P_d(n)) \bmod [0,1]^d},$$

the L^1-normalized bump function η_T, and the function $f_T := (\mu * \eta_T) - N$. Let T be a rectangular parallelpiped with axes $e_1, \dots, e_d$ and dimensions $\sigma_1, \dots, \sigma_d$, and let $\mathscr{G}$ be an arbitrary maximal collection of parallelepipeds that are pairwise A-incomparable and contained in $S^{\complement}$.

For simplicity, suppose the polynomials $P_1, \dots, P_d$ are all monomials of degree k, and let $\mathbf{P_k}$ be the column vector of the leading coefficients. We first prove an important result on the average value of $R_{T,N}$ for any $\mathbf{P_k}$.

Lemma 4. *Let $\mathbf{P_k} = (k_1, \dots, k_d)$. The average value of $R_{T,N}(\mathbf{P_k})$ is $|T| \log N + N$.*

Proof. Let $d\mathbf{k} = dk_1 \cdots d_{k_d}$. We have

$$\int_{\mathbb{R}^d/\mathbb{Z}^d} R_{T,N}(\mathbf{P_k})\, d\mathbf{k} = \sum_{\mathbf{q}\in T^*} \int_{\mathbb{R}^d/\mathbb{Z}^d} \min\left(N, \|\mathbf{q}\cdot\mathbf{P_k}\|^{-1}\right)\, dk_1\, \cdots\, dk_d.$$

For a fixed $\mathbf{q} = (q_1, \ldots, q_d) \in T^* \setminus \{\mathbf{0}\}$ we have

$$\int_{\mathbb{R}^d/\mathbb{Z}^d} \min\left(N, \|\mathbf{q}\cdot\mathbf{P_k}\|^{-1}\right)\, dk_1\, \cdots\, dk_d$$

$$= \sum_{0\le n_1\le q_1-1} \cdots \sum_{0\le n_d\le q_d-1} \left(\int_{n_1 q_1^{-1}}^{(n_1+1)q_1^{-1}} \cdots \int_{n_d q_d^{-1}}^{(n_d+1)q_d^{-1}} \min\left(N, \|\mathbf{q}\cdot\mathbf{P_k}\|^{-1}\right)\, d\mathbf{k} \right)$$

and after a change of variables, this becomes

$$q_1^{-1}\cdots q_d^{-1} \sum_{0\le n_1\le q_1-1} \cdots \sum_{0\le n_d\le q_d-1} \int_{\mathbb{R}^d/\mathbb{Z}^d} \min\left(N, \|\mathbf{P_k}\cdot(1,\ldots,1)\|^{-1}\right) d\mathbf{k}.$$

This quantity simplifies to

$$\int_{\mathbb{R}^d/\mathbb{Z}^d} \min\left(N, \|\mathbf{P_k}\cdot(1,\ldots,1)\|^{-1}\right) d\mathbf{k} \sim \log N.$$

This analysis holds for any $\mathbf{q} \in T^* \setminus \{\mathbf{0}\}$, so the total contribution from $\mathbf{q} \in T^* \setminus \{\mathbf{0}\}$ and $\mathbf{0}$ is $|T|\log N + N$. Thus,

$$\int_{\mathbb{R}^d/\mathbb{Z}^d} R_{T,N}(\mathbf{P_k})\, d\mathbf{k} \sim |T|\log N + N,$$

as required. □

We now focus on the case where the polynomials have degree $k = 2$. Let $\mathbf{P_2}$ be the column vector of its coefficients. Recall that to find an upper bound in (2), we need to bound the Weyl (inner) sum in (4). This sum is precisely

$$\sum_{(z_1,\ldots,z_d)\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} |\widehat{\mu}(z_1,\ldots,z_d)|^2\, |\widehat{\eta}(z_1,\ldots,z_d)|^2 = \sum_{\mathbf{q}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \left|\sum_{n=1}^{N} e\left((\mathbf{q}\cdot\mathbf{P_2})\, n^2\right)\right|^2 |\widehat{\eta}(\mathbf{q})|^2$$

This sum can be partitioned to

$$\left(\sum_{\substack{\mathbf{q}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}\\ \mathbf{q}\in T^*}} + \sum_{\substack{\mathbf{q}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}\\ \mathbf{q}\notin T^*}} \right) \left|\sum_{n=1}^{N} e\left((\mathbf{q}\cdot\mathbf{P_2})\, n^2\right)\right|^2 |\widehat{\eta}(\mathbf{q})|^2.$$

We now focus on bounding the first sum, as this leads to a bound for the second sum as well. For dyadic λ and for all $\mathbf{q} \in 2\lambda T^* \setminus \lambda T^*$ we have $|\widehat{\eta}(\mathbf{q})| < 1/\lambda^N$. Hence, we bound the first sum as

$$\sum_{\substack{\mathbf{q}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}\\ \mathbf{q}\in T^*}} \left|\sum_{n=1}^{N} e\left((\mathbf{q}\cdot\mathbf{P_2})\, n^2\right)\right|^2 |\widehat{\eta}(\mathbf{q})|^2 \le \sum_{\substack{\mathbf{q}\in 2\lambda T^*\setminus\lambda T^*\\ \lambda\,\text{dyadic}}} \left|\sum_{n=1}^{N} e\left((\mathbf{q}\cdot\mathbf{P_2})\, n^2\right)\right|^2 \cdot \frac{1}{\lambda^{2N}}.$$

To further bound the sum, we want to bound the sum

$$\sum_{\substack{\mathbf{q}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}\\ \mathbf{q}\in T^*}} \left| \sum_{n=1}^{N} e\left(\left(\mathbf{q}\cdot\mathbf{P_2}\right)n^2\right)\right|^2$$

Because none of the polynomials $P_1,\dots,P_d$ have degree more than 2, this sum is bounded by

$$\sum_{\substack{\mathbf{q}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}\\ \mathbf{q}\in T^*}} \left| \sum_{n=1}^{N} e\left(\left(\mathbf{q}\cdot\mathbf{P_2}\right)n^2\right)\right|^2 \le R_{(2N)^{-1}T,N}(\mathbf{P_2}) - N|T|. \tag{16}$$

One approach to bounding the sum of $R_{T,N}$ terms in (16) is to visualize the dot product $\mathbf{q}\cdot\mathbf{P_2}$. Consider the line L spanned by $\mathbf{P_2}$ and the orthogonal projection of $(P_1(n),\dots,P_d(n))$ mod $[0,1]^d$ on L for each $n=1,\dots,N$. Denote this collection of N points by Q. The sum of $R_{T,N}$ terms essentially calculates how close the elements of Q are to an integer on L. Here, Q is an example of a quasicrystal.

Quasicrystals are important objects in mathematics and physical sciences. Many properties of quasicrystals are well-known; see [1] and [4] for discussion. One natural question pertinent to our question is the distribution of the points in Q on L.

Question 1. *What $\mathbf{q}\in T^*$ are such that $\|\mathbf{q}\cdot\mathbf{P_2}\|\sim\lambda$ for dyadic λ?*

If we know more about the distribution of points in Q on L, then we may be able to find an upper bound for the sum of $R_{T,N}$ terms in (16). This will lead to an upper bound for the integral in (2) as discussed in Section 2.

Let $m := \max\{\deg(P_1),\dots,\deg(P_d)\}$. Similar to the one-dimensional case, the Weyl sequence appears uniformly distributed when $m=1$. However, when $m>1$, the Weyl sequence appears more uniform random. This can be seen in Figure 3. Fix $N=10000$ and consider the sequences

$$S_1 := \left\{\left(\sqrt{2}n,\sqrt{7}n\right) \bmod [0,1]^2\right\}_{n=1}^{N},$$
$$S_2 := \left\{\left(\sqrt{2}n^2,\sqrt{7}n\right) \bmod [0,1]^2\right\}_{n=1}^{N},$$
$$S_3 := \left\{\left(\sqrt{2}n^2,\sqrt{7}n^2\right) \bmod [0,1]^2\right\}_{n=1}^{N}.$$

As Figure 3 depicts, the sequence S_1 has a more distinct pattern, whereas S_2 and S_3 appear like randomly distributed points.

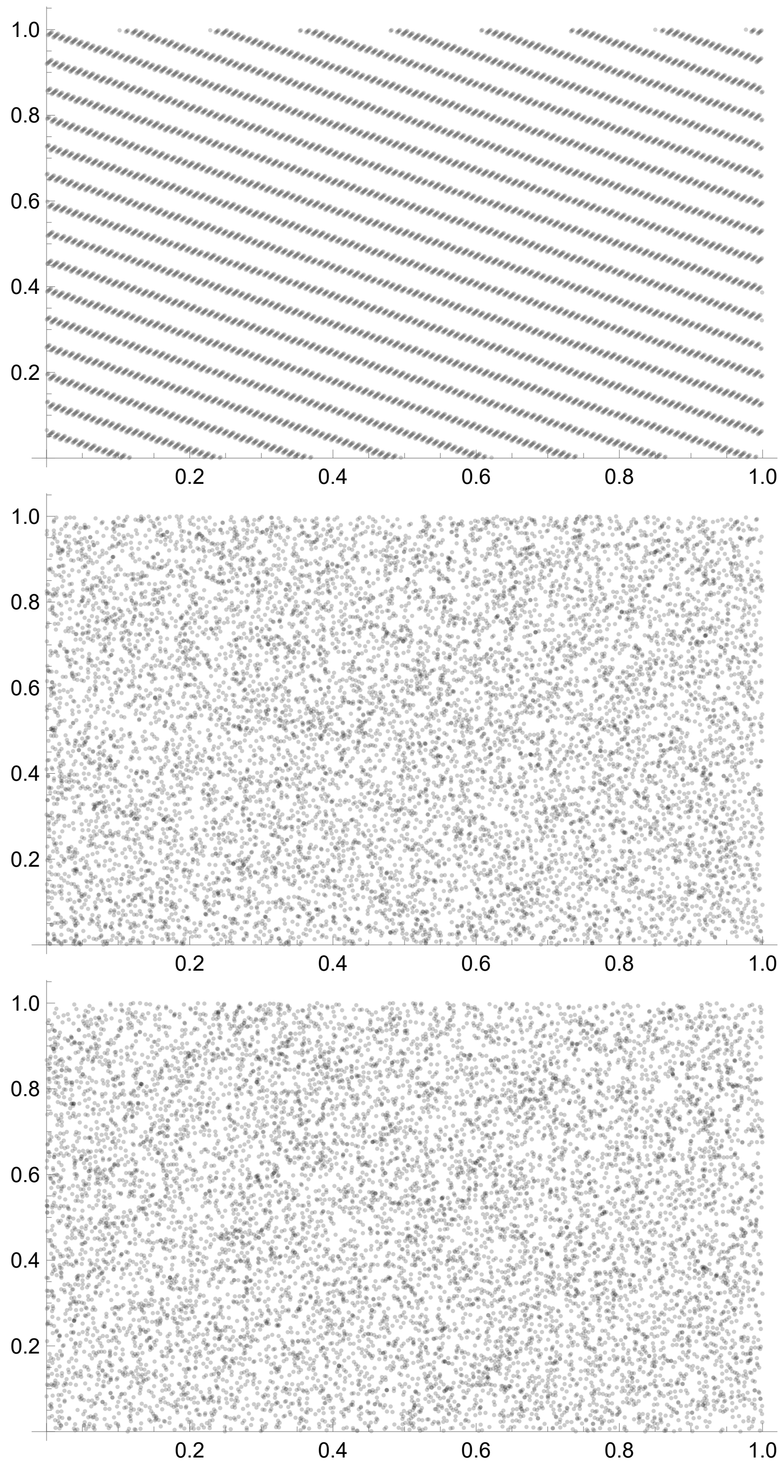

FIGURE 3. The distribution of S_1, S_2, and S_3 for $N = 10000$.

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Appendix A. Estimating Length of Longest Gap

Let $P(n) = \alpha n^2$ and let $S_N = \{P(n) \bmod 1\}_{n=1}^N$. To estimate the length of the longest gap, we consider a bump function ψ_w of width w and height 1 instead of η_w. In other words, ψ_w is supported on $[-w, w]$ and has height 1. Here we aim to estimate

$$\sum_{n=1}^{N} \psi_w(P(n) \bmod 1). \tag{17}$$

The sum in (17) counts the number of points of S_N that are in an interval of length w. If the sum is 0, then there is a gap of length w in S_N. Roughly,

$$\widehat{\psi_w}(a) = \begin{cases} w & |a| \le w^{-1} \\ 0 & |a| > w^{-1} \end{cases}.$$

Thus, the sum in (17) essentially becomes

$$\begin{aligned} \left| w \sum_{|a| \le w^{-1}} \sum_{n=1}^{N} e(aP(n)) \right| &= w \left| \sum_{n=1}^{N} \sum_{|a| \le w^{-1}} e(aP(n)) \right| \\ &= w \left| \sum_{n=1}^{N} \frac{e\left(-w^{-1}P(n)\right) - e\left(\left(w^{-1}+1\right)P(n)\right)}{1 - e(P(n))} \right|. \end{aligned} \tag{18}$$

Note that

$$\left| \sum_{n=1}^{N} \frac{e\left(-w^{-1}P(n)\right) - e\left(\left(w^{-1}+1\right)P(n)\right)}{1 - e(P(n))} \right| = \left| \sum_{n=1}^{N} \frac{\sin\left(\left(w^{-1}+\frac{1}{2}\right)2\pi P(n)\right)}{\sin\left(\pi P(n)\right)} \right|,$$

which is the sum of Dirichlet kernels

$$D_{w^{-1}}(2\pi P(n)) = \frac{\sin\left(\left(w^{-1}+\frac{1}{2}\right)2\pi P(n)\right)}{\sin\left(\pi P(n)\right)} = U_{2w^{-1}}\left(\cos\left(\pi P(n)\right)\right),$$

where U_n is the Chebyshev polynomial of the second kind.

By the triangle inequality, (18) is bounded by

$$2w \sum_{n=1}^{N} \frac{1}{|1 - e(P(n))|} = 2w \sum_{n=1}^{N} \frac{1}{|1 - e(\alpha n^2)|}.$$

The summand is large whenever $e(\alpha n^2)$ is close to 1, or when $\|\alpha n^2\|$ is small. Understanding the number theoretic properties of α may lead to better understanding of $\|\alpha n^2\|$ and thus a bound to the sum in (17).

Another form of analysis is to approximate the integral and bounding the error term in

$$\left| \int_0^N \frac{\sin\left(\left(w^{-1}+\frac{1}{2}\right)2\pi P(x)\right)}{\sin\left(\pi P(x)\right)} \, dx \right| \le \int_0^N \left| \frac{\sin\left(\left(w^{-1}+\frac{1}{2}\right)2\pi P(x)\right)}{\sin\left(\pi P(x)\right)} \right| dx.$$

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