UROP+ final paper

Roni Edwin

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Abstract

We present a proof of conjecture 4.4 in [1] about the bounds in the exponent for the Fractal Uncertainty principle.

Contents

1. Introduction

The Fractal Uncertainty Principle (FUP) is a statement which vaguely speaking states that no function can be localised in position and frequency close to a fractal set. Before diving into the Fractal Uncertainty Principle, it is perhaps useful to review the standard uncertainty principle, which has to do with the Fourier transform of a function on the real line. The standard uncertainty principle we have in mind is the one used in [1]: Choose $h > 0$ and define the following unitary fourier transform $\mathcal{F}_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ given by, for an integrable function f

$$\mathcal{F}_h f(\xi) = \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} f(x) e^{-\frac{ix\xi}{h}} dx \tag{1.1}$$

In the case where f is not integrable, $\mathcal{F}_h$ can be given by extension from the restriction of $\mathcal{F}_h$ to the dense subspace $\mathcal{S}(\mathbb{R})$, Schwartz space. With this in mind, the continuous version of the FUP might be stated as so: Both f and $\mathcal{F}_h f$ cannot have large mass on the interval $[0, h]$. Intuitively, this saying that the fourier transform 'smears out' functions. This is seen in the standard formula

$$\mathcal{F}_h(x \mapsto f(ax))(\xi) = \frac{1}{|a|} \mathcal{F}_h f\left(\frac{\xi}{a}\right)$$

We can make the statement about rigorous. For $X \subset \mathbb{R}$, denote the operator $\mathbb{1}_X$ as the multiplication by the indicator function for X. So

$$\mathbb{1}_X(f)(t) = \begin{cases} f(t), & \text{if } t \in X \\ 0, & \text{otherwise} \end{cases}$$

Then the standard version of the uncertainty principle can be formulated precisely as

$$\left|\left|\mathbb{1}_{[0,h]}\mathcal{F}_h\mathbb{1}_{[0,h]}\right|\right| \leq \frac{h^{\frac{1}{2}}}{\sqrt{2\pi}} \tag{1.2}$$

where $||\cdot||$ is denotes the operator norm on $L^2(\mathbb{R})$.

The uncertainty principle discussed in this paper involves localisation on more esoteric sets than intervals. Specifically, subsets of $\mathbb{Z}_N$, and under the discrete fourier transform $\mathcal{F}_N$ acting as an operator on $\mathbb{C}^N$. Similar to the continuous case, our work will involve obtaining an upper bound for the operator norm

$$||\mathbb{1}_X\mathcal{F}_N\mathbb{1}_X||$$

for certain fractal sets $X \subset \{0, 1, ..., N-1\}$ of interest, called *Cantor sets.* The Fractal uncertainty principle in this case gives an upper bound for the operator norm in terms on the integer N, typically exponential. An example bound might look something like

$$||\mathbb{1}_X\mathcal{F}_N\mathbb{1}_X|| \text{ is } O(N^{-\beta}) \text{ as } N \to \infty$$

for an exponent β.

FUP has seen some applications in fields like quantum chaos, the study of quantum systems in situations where the underlying classical system has chaotic behavior. For a more in-depth review of Fractal sets and FUP and its applications, see [1].

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2. Review

We will start by defining the discrete Fourier transform and outlining it's basic properties. We will then introduce discrete cantor sets, which will be play an important role in the study of the Fractal Uncertainty Principle presented in this work.

2.1. Discrete Fourier Transform

Let N be a natural number. For a vector $u = (u(l))_{l=0}^{N-1} \in \mathbb{C}^N$, the discrete fourier transform $\mathcal{F}_N : \mathbb{C}^N \to \mathbb{C}^N$ is given by

$$(\mathcal{F}_N u)(j) = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} \exp\left(-\frac{2\pi i j l}{N}\right) u(l) \tag{2.1}$$

Equivalently, as a matrix with entries $(\mathcal{F})_{jl}$ with $0 \leq j, l \leq N-1$, we have $(\mathcal{F})_{jl} = \frac{1}{\sqrt{N}} \exp\left(-\frac{2\pi i j l}{N}\right)$. An important property of $\mathcal{F}_N$ is that it's unitary: this means that it preserves the euclidean norm $|\cdot|$ of u. So for each $u \in \mathbb{C}^N$, we have

$$|\mathcal{F}_N(u)| = |u|$$

Throughout this paper, we'll use $||\cdot||$ to denote, unless otherwise stated, the operator norm of a linear operator (on the appropriate vector space). Recall the operator norm $||T||$ of a linear operator T on a vector space V is given by

$$||T|| := \sup_{||u||=1} ||Tu||_V$$

An associated norm is the Hilbert-Schmidt norm, for an operator T on a Hilbert space H, given by

$$||T||_{HS}^2 = \sum_{i=1}^{\infty} |\langle Te_i, e_i \rangle|^2$$

Where $\langle \cdot, \cdot \rangle$ is the inner product.

2.2. Fractal and Discrete Cantor Sets

For an integer M, we define $\mathbb{Z}_M := \{0, 1, ..., M-1\}$. For a given M and an alphabet $A \subset \mathbb{Z}_M$, the discrete cantor set $\mathcal{C}_k$ is a subset of $\mathbb{Z}_N$, where $N = M^k$, of the following form

$$\mathcal{C}_k = \left\{ \sum_{i=0}^{k-1} a_i M^i : a_i \in A \right\}$$

An important quantity associated with these sets is their dimension δ, given by

$$\delta = \frac{\log |A|}{\log M}$$

Importantly, $N^\delta = |A|^k = |\mathcal{C}_k|$.

3. Fractal Uncertainty Principle for Discrete Cantor Sets

With the background information in hand, we're ready to state and prove the theorem of the Fractal Uncertainty Principle for discrete cantor sets, following the same argument presented in [1]. We define $\mathbb{1}_{\mathcal{C}_k} : \mathbb{C}^N \to \mathbb{C}^N$ to be the indicator operator of $\mathcal{C}_k$. That is to say

$$\mathbb{1}_{\mathcal{C}_k}(u)(j) = \begin{cases} u(j), & \text{if } j \in \mathcal{C}_k \\ 0, & \text{otherwise} \end{cases}$$

Theorem 3.1. *Let $M \geq 3$, $N = M^k$, and A be a non-empty and strict subset of $\mathbb{Z}_M$.Then $\exists \beta > \max\left(0, \frac{1}{2} - \delta\right)$ and a constant C, depending only on M and A such that*

$$||\mathbb{1}_{\mathcal{C}_k} F_N \mathbb{1}_{\mathcal{C}_k}||_{\mathbb{C}^N \to \mathbb{C}^N} \leq CN^{-\beta} \tag{3.1}$$

Recall δ is the dimension of A defined above.

Define

$$r_k := ||\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}||_{\mathbb{C}^N \mapsto \mathbb{C}^N}$$

One important idea in the proof of this theorem is the observation that $\log r_k$ is subadditive, that is,

$$\log r_{k_1+k_2} \leq \log r_{k_1} + \log r_{k_2}$$

as shown in the lemma below

Lemma 3.2. *For all $k_1, k_2 \geq 1$ we have $r_{k_1+k_2} \leq r_{k_1} \cdot r_{k_2}$*

Proof. Let $k = k_1 + k_2$, $N_j = M^{k_j}$, $N = M^k$. Define

$$\ell^2(\mathcal{C}_k) = \{u \in \mathcal{C}^N : \operatorname{supp} u \subset \mathcal{C}_k\}$$

So that r_k is the norm of the operator

$$\mathcal{G}_k : \ell^2(\mathcal{C}_k) \to \ell^2(\mathcal{C}_k)$$

given by $\mathcal{G}_k u = \mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N u$ for $u \in \ell^2(\mathcal{C}_k)$.

The proof of this lemma relies on writing $\mathcal{G}_k$ in terms of $\mathcal{G}_{k_1}$ and $\mathcal{G}_{k_2}$ by using the fact that

$$\mathcal{C}_k = \mathcal{C}_{k_1} + N_1 \mathcal{C}_{k_2} = \mathcal{C}_{k_2} + N_2 \mathcal{C}_{k_1}$$

Consider $u \in \ell^2(\mathcal{C}_k)$ and let $v = \mathcal{G}_k u$. Associate to u, v the $|A|^{k_1} \times |A|^{k_2}$ matrices U,V given by

$$U_{ab} = u(N_1 b + a), \ V_{ab} = v(N_2 a + b) \text{ for } a \in \mathcal{C}_{k_1} \text{ and } b \in \mathcal{C}_{k_2}$$

For $j = N_2 p + q \in \mathcal{C}_k$, where $p \in \mathcal{C}_{k_1}$ and $q \in \mathcal{C}_{k_2}$, we have

$$V_{pq} = v(j) = \mathcal{F}_N u(j) = v(N_2 p + q) = \frac{1}{\sqrt{N}} \sum_{a \in \mathcal{C}_{k_1}, b \in \mathcal{C}_{k_2}} \exp\left(-\frac{2\pi i (N_2 p + q)(N_1 b + a)}{N}\right) U_{ab} \tag{3.2}$$

Notice that $N_1 N_2 = N$, so we can write (3.2) as

$$\begin{aligned} V_{pq} &= \frac{1}{\sqrt{N}} \sum_{a \in \mathcal{C}_{k_1}, b \in \mathcal{C}_{k_2}} \exp\left(-\frac{2\pi i p a}{N_1}\right) \exp\left(-\frac{2\pi i b q}{N_2}\right) \exp\left(-\frac{2\pi i a q}{N}\right) U_{ab} \\ &= \sum_{a \in \mathcal{C}_{k_1}} \frac{1}{\sqrt{N_1}} \exp\left(-\frac{2\pi i p a}{N_1}\right) \left(\exp\left(-\frac{2\pi i a q}{N}\right) \sum_{b \in \mathcal{C}_{k_2}} \frac{1}{\sqrt{N_2}} \exp\left(-\frac{2\pi i b q}{N_2}\right) U_{ab} \right) \end{aligned}$$

Define the intermediate $|A|^{k_1} \times |A|^{k_2}$ matrix U' obtained by replacing each row U_r of U by $\mathcal{G}_{k_2} U_r$, that is,

$$U'_{aq} = \frac{1}{\sqrt{N_2}} \sum_{b \in \mathcal{C}_{k_2}} \exp\left(-\frac{2\pi i b q}{N_2}\right) U_{ab}$$

Then define the intermediate $|A|^{k_1} \times |A|^{k_2}$ matrix V' obtained by multiplying pointwise each entry of U' by $\exp\left(-\frac{2\pi i a q}{N}\right)$, so

$$V'_{aq} = \exp\left(-\frac{2\pi i a q}{N}\right) U'_{aq}$$

Then V is obtained by replacing each column V'_c of V' by $\mathcal{G}_{k_1} V'_c$. Specifically,

$$V_{pq} = \frac{1}{\sqrt{N_1}} \sum_{a \in \mathcal{C}_{k_1}} \exp\left(-\frac{2\pi i p a}{N_1}\right) V'_{aq}$$

Now note from the definitions of U and V that the norms of u,v are the Hilbert-schmidt norms of U,V respectively:

$$||u||^2 = \sum_{a \in \mathcal{C}_{k_1}, b \in \mathcal{C}_{k_2}} |U_{ab}|^2, \ ||v||^2 = \sum_{a \in \mathcal{C}_{k_1}, b \in \mathcal{C}_{k_2}} |V_{ab}|^2$$

For each row U'_r of U' and corresponding row U_r of U, we have

$$U'_r = \mathcal{G}_{k_2} U_r \implies |U'_r| \leq r_{k_2} |U_r| \implies ||U'||_{HS} \leq r_{k_2} ||U||_{HS}$$

Additionally, $||V'||_{HS} = ||U'||_{HS}$ because $|V'_{aq}| = |U'_{aq}|$. For each column V_c of V and corresponding column V'_c of V'

$$V_c = \mathcal{G}_{k_1} V'_c \implies |V_c| \leq r_{k_1} |V'_c| \implies ||V||_{HS} \leq r_{k_1} ||V'||_{HS}$$

and putting this together implies

$$||V||_{HS} \leq r_{k_1} r_{k_2} ||U||_{HS}$$

□

We use lemma 3.2 to give a proof of Theorem 3.1.

Proof. Lemma 3.2 allows us to use Fekete's lemma, which says that for a sub-additive sequence $\{a_n\}_{n=1}^{\infty}$, that is $a_{n+m} \leq a_n + a_m$,

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n}$$

By lemma 3.2, the sequence $\left\{\frac{\log r_k}{\log M}\right\}_{k=1}^{\infty}$ is subadditive. If $\exists k'$ such that

$$\frac{\log r_{k'}}{k' \log M} < \min\left(0, \delta - \frac{1}{2}\right)$$

Then

$$\lim_{k\to\infty} \frac{\log r_k}{k \log M} < \min\left(0, \delta - \frac{1}{2}\right)$$

So we can choose $\beta > 0$ so that

$$\lim_{k\to\infty} \frac{\log r_k}{k \log M} = \inf_k \frac{\log r_k}{k \log M} < -\beta < \min\left(0, \delta - \frac{1}{2}\right)$$

Then for large enough S, $k \geq S$ would imply that

$$\frac{\log r_k}{k \log M} \leq -\beta$$

And then by choosing a large enough constant C we can then get for all k

$$\frac{\log r_k}{k \log M} \leq \frac{\log C}{k \log M} - \beta \implies r_k \leq C N^{-\beta}$$

We just need to prove that $\exists k$ such that

$$\frac{\log r_k}{k \log M} < \min\left(0, \delta - \frac{1}{2}\right) \iff r_k < \min\left(1, N^{\delta - \frac{1}{2}}\right)$$

The proof of this consists of two parts:

Claim 1. *There exists k such that $r_k < 1$.*
We proceed by contradiction. Since $\mathcal{F}_N$ is unitary, we have $||\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}|| \leq 1$. Assume that $r_k = 1$, which implies that $\exists u \in \mathbb{C}^N \setminus \{0\}$ such that

$$|\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} u| = |u|$$

Note that this implies $\operatorname{supp} u \subset \mathcal{C}_k$ and $\operatorname{supp}(\mathcal{F}_N u) \subset \mathcal{C}_k$.

Define the polynomial p by

$$p(z) = \sum_{l=0}^{N-1} u(l) z^l$$

so that

$$\mathcal{F}_N u(j) = \frac{1}{\sqrt{N}} p\left(e^{\frac{-2\pi i j}{N}}\right)$$

Since $\operatorname{supp}(\mathcal{F}_N u) \subset \mathcal{C}_k$, it means that for $j \in \mathbb{Z}_N \setminus \mathcal{C}_k$

$$\mathcal{F}_N u(j) = 0$$

and so p has at least

$$|\mathbb{Z}_N \setminus \mathcal{C}_k| = |\mathbb{Z}_N| - |\mathcal{C}_k| = M^k - |A|^k \geq M^k - (M-1)^k$$

roots. The last inequality above is true because $|A|$ is at most $M-1$ as A is a non-empty strict subset of $\mathbb{Z}_M$.

On the other hand, the set $\mathbb{Z}_N \setminus \mathcal{C}_k$ contains M^{k-1} consecutive numbers:

$$aM^{k-1}, aM^{k-1}+1, \cdots, aM^{k-1} + \left(M^{k-1} - 1\right)$$

for $a \in \mathbb{Z}_M \setminus A$. We can 'shift' $\mathcal{C}_k$ circularly, which does not change the norm r_k, to map these numbers to

$$(M-1)M^{k-1}, (M-1)M^{k-1}+1, \cdots, M^k - 1$$

Using the fact that $\operatorname{supp}(u) \in \mathcal{C}_k$, it follows that $u(l) = 0$ for $(M-1)M^{k-1} \leq l \leq M^k - 1$, which gives that

$$\deg p \leq (M-1)M^{k-1}$$

Combining this with the lower bound on the number of zeros of p gives

$$M^k - (M-1)^k \leq \deg p \leq (M-1)M^{k-1}$$

However, for large enough k, we have

$$(M-1)^k < M^{k-1} \implies M^k - (M-1)^k > (M-1)M^{k-1} = M^k - M^{k-1}$$

which is a contradiction. So $\exists k$ such that $r_k < 1$.

Claim 2. *For $k \geq 2$, we have $r_k < N^{\delta - \frac{1}{2}}$*

Once again, we proceed by contradiction. Note that $N^{\delta-\frac{1}{2}}$ is the Hilbert-Schmidt norm of $\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}$. Letting $(e_l)_{l=0}^{N-1}$ denote the standard orthonormal basis

$$||\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}||_{HS}^2 = \sum_{l=1}^{N} |\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k} e_l|^2 = \sum_{l \in \mathcal{C}_k} |\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N e_l|^2 = \sum_{l \in \mathcal{C}_k} \sum_{j \in \mathcal{C}_k} |(\mathcal{F}_N)_{jl}|^2 = \frac{|\mathcal{C}_k|^2}{N} = N^{2\delta - 1}$$

Note then that if $r_k = N^{\delta - \frac{1}{2}}$, then $\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}$ is a rank 1 operator. In fact, $||\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}||_{HS}$ is the square root of the sum of the norm squared of the eigenvalues of $\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}$, while $||\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}||$ is the norm of the eigenvalue with the largest norm. Then the determinant of any 2×2 minor of $\mathbb{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbb{1}_{\mathcal{C}_k}$ is 0, that is

$$\det\begin{pmatrix} e^{-\frac{2\pi ijl}{N}} & e^{-\frac{2\pi ijl'}{N}} \\ e^{-\frac{2\pi ij'l}{N}} & e^{-\frac{2\pi ij'l'}{N}} \end{pmatrix} = 0 \text{ for all } j, j', l, l' \in \mathcal{C}_k$$

and evaluating the above expression gives

$$\frac{(j-j')(l-l')}{N} \in \mathbb{Z}\ \forall j, j', l,' l \in \mathcal{C}_k$$

However, for $k \geq 2$, we can take $j = l$, $j' = l' \in \mathcal{C}_k$ so that $l \neq l'$ and $|l - l'| < M \leq \sqrt{N}$. This choice of $j = l$ and $j' = l'$ gives a contradiction, which finishes the proof of the FUP. ☐

4. Bounding the exponent

Theorem 3.1 gives an upper bound on the operator norm of localizing the fourier transform on discrete cantor sets. This is achieved by getting a lower bound on the exponent β, that is $\beta > \max(0, \frac{1}{2} - \delta)$ (3.1) holds. This next section is about the tightness of this bound. Specifically you might wonder if that bound may be improved, or if the lower bound of $\max(0, \frac{1}{2} - \delta)$ is the best you can do, in that for any given δ, you can construct alphabets and M so the best β tends to $\max(0, \frac{1}{2} - \delta)$. We can ask the following question:

Question. *For a given $\delta \in (0, 1)$, can we exhibit a sequence of pairs (M_j, A_j) such that*

$$\delta(M_j, A_j) := \frac{\log|A_j|}{M_j} \to \delta,\ \beta(M_j, A_j) \to \max\left(\frac{1}{2} - \delta, 0\right)$$

Where $\beta(M, A) := -\limsup_{k\to\infty} \frac{\log||\mathbb{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbb{1}_{\mathcal{C}_k}||}{\log N}$, $N = M^k$ (represents the best β in (3.1))?

We'll see that this is indeed the case. We break this into parts. For $\delta < \frac{1}{2}$, the answer is in the affirmative, as in Proposition 3.17 in [2]. For $\delta > \frac{1}{2}$, this is believed to be true, illustrated in the following conjecture (Conjecture 4.4 in [1]). The conjecture is as follows:

Conjecture 4.1. *Fix $\delta \in \left(\frac{1}{2}, 1\right)$. Then there exists a sequence of pairs (M_j, A_j) such that*

$$\delta(M_j, A_j) \to \delta,\ \beta(M_j, A_j) \to 0$$

Here $\delta(M, A) = \frac{\log|A|}{M}$ and

$$\beta(M, A) := -\limsup_{k\to\infty} \frac{\log||\mathbb{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbb{1}_{\mathcal{C}_k}||}{\log N} = -\inf_k \frac{\log||\mathbb{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbb{1}_{\mathcal{C}_k}||}{\log N},\ N = M^k$$

Informally, this is saying that the 'best' β in (3.1) can be made arbitrarily small for a given $\delta \in \left(\frac{1}{2}, 1\right)$. The main theme of this paper has been building up to proving this conjecture. Before going into the proof, we will look at the case for when $\delta < \frac{1}{2}$ for inspiration. For $\delta < \frac{1}{2}$, we will prove the following:

Proposition 4.2. *Assume that $\delta < \frac{1}{2}$, and $2 \leq L \leq 2\sqrt{M}$. Consider the alphabet*

$$A = \{0, 1, ..., L-1\}$$

Then for M large enough, there exists some global constant K such that $\beta(M, A)$ is bounded above by

$$\beta \leq \frac{1}{2} - \delta + \frac{KL^2}{M \log M} \tag{4.1}$$

The proof of proposition 4.2 relies on exploiting the structure of $\mathcal{C}_k$ to get a lower bound for $||\mathbb{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbb{1}_{\mathcal{C}_k}||$. For an alphabet $A \subset M$, we define a function G_A given by

$$G_A(x) = \frac{1}{\sqrt{M}} \sum_{a \in A} \exp(-2\pi i a x)$$

Getting a lower bound for G_A is going to help us find a lower bound for $||\mathbb{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbb{1}_{\mathcal{C}_k}||$, as shown in the lemma below.

Lemma 4.3. *We have the following equality:*

$$\mathcal{F}_N(\mathbf{1}_{\mathcal{C}_k})(j) = \prod_{s=1}^{k} G_A\left(\frac{j}{M^s}\right) \tag{4.2}$$

Proof. For a set $S \subset \mathbb{Z}_N$, define the vector $\mathbf{1}_S$ by

$$\mathbf{1}_S(j) = \begin{cases} 1, & \text{if } j \in S \\ 0, & \text{otherwise} \end{cases}$$

Observe that $n \in \mathcal{C}_k \iff n$ can be written as

$$n = \sum_{i=0}^{k-1} n_k$$

with each $n_i \in AM^i = \{aM^i : a \in A\}$. Mathematically we can express this as

$$\mathbf{1}_{\mathcal{C}_k}(n) = \sum_{n_0 + \cdots + n_{k-1} = n} \left(\prod_{s=0}^{k-1} \mathbf{1}_{AM^s}(n_s) \right)$$

So,

$$\begin{aligned}
\mathcal{F}_N\left(\mathbf{1}_{\mathcal{C}_k}\right)(j) &= \frac{1}{\sqrt{N}}\sum_{l=0}^{N-1}\exp\left(-\frac{2\pi ijl}{N}\right)\left(\sum_{n_0+\cdots+n_{k-1}=l}\left(\prod_{s=0}^{k-1}\mathbf{1}_{AM^s}(n_s)\right)\right)\\
&= \frac{1}{\sqrt{N}}\sum_{l=0}^{N-1}\left(\sum_{n_0+\cdots+n_{k-1}=l}\left(\prod_{s=0}^{k-1}\exp\left(-\frac{2\pi in_sj}{N}\right)\mathbf{1}_{AM^s}(n_s)\right)\right)\\
&= \sum_{n_0\in A}\cdots\sum_{n_{k-1}\in AM^{k-1}}\left(\frac{1}{\sqrt{M}}\prod_{s=0}^{k-1}\exp\left(-\frac{2\pi in_sj}{M^k}\right)\right)\\
&= \sum_{n_0\in A}\cdots\sum_{n_{k-1}\in A}\left(\frac{1}{\sqrt{M}}\prod_{s=0}^{k-1}\exp\left(-\frac{2\pi in_sj}{M^{k-s}}\right)\right)\\
&= \prod_{s=1}^{k}\left(\frac{1}{\sqrt{M}}\sum_{n_s\in A}\exp\left(-\frac{2\pi in_sj}{M^s}\right)\right)\\
&= \prod_{s=1}^{k}G_A\left(\frac{j}{M^s}\right)
\end{aligned}$$

□

Below, we present the proof of proposition 4.2.

Proof. Put $u = \mathbf{1}_{\mathcal{C}_k}$, so that

$$r_k^2 \geq \frac{|\mathbb{1}_{\mathcal{C}_k}\mathcal{F}_N u|^2}{|u|^2}$$

Recall that $A = \{0, 1, ..., L-1\}$. From lemma 4.3, we have

$$\mathcal{F}_N(u)(j) = \prod_{s=1}^{k}G_A\left(\frac{j}{M^s}\right)$$

which gives

$$r_k^2 \geq \frac{||\mathbb{1}_{\mathcal{C}_k}F_N\mathbb{1}_{\mathcal{C}_k}u||^2}{||u||^2} = \frac{1}{L^k}\sum_{j\in\mathcal{C}_k}|F_N(\mathbf{1}_{\mathcal{C}_k})(j)|^2 = \frac{1}{L^k}\sum_{j\in\mathcal{C}_k}\left(\prod_{s=1}^{k}\left|G_A\left(\frac{j}{M^s}\right)\right|^2\right) \tag{4.3}$$

Since A is an arithmetic series, a quick calculation shows that

$$G_A(x) = \frac{\exp(-2\pi iLx) - 1}{\sqrt{M}\left(\exp(-2\pi ix) - 1\right)}$$

Take $L \geq 9$ and let

$$\Omega = \left\{\sum_{r=0}^{k-1}b_rM^r : b_i \in \left\{0, 1, ..., \left\lfloor\frac{L}{9}\right\rfloor\right\}\right\}$$

For $s \in \mathbb{N}$, $j \in \Omega$, consider the fractional part of $\frac{j}{M^s}$, $\left\{\frac{j}{M^s}\right\}$. We have

$$\left\{\frac{j}{M^s}\right\} = \sum_{i=0}^{s-1}\frac{b_iM^i}{M^s} \leq \left\lfloor\frac{L}{9}\right\rfloor\left(\frac{M^s-1}{M^s(M-1)}\right)$$

and since $L \leq 2\sqrt{M}$

$$\frac{2L^2}{9} \cdot \frac{M^s - 1}{M^s(M-1)} \leq 1 \implies \left\{\frac{j}{M^s}\right\} \leq \frac{1}{2L}$$

So for such $j \in \Omega$ and $s \in \mathbb{N}$, we have

$$\left|G_A\left(\frac{j}{M^s}\right)\right| = \left|G_A\left(\left\{\frac{j}{M^s}\right\}\right)\right| = \frac{1}{\sqrt{M}}\left|\frac{\sin\left(L\pi\left\{\frac{j}{M^s}\right\}\right)}{\sin\left(\pi\left\{\frac{j}{M^s}\right\}\right)}\right| \geq \frac{L}{\sqrt{M}}\left|\frac{\sin\left(L\pi\left\{\frac{j}{M^s}\right\}\right)}{L\pi\left\{\frac{j}{M^s}\right\}}\right|$$

where

$$0 \leq L\left\{\frac{j}{M^s}\right\} \leq \frac{1}{2}$$

The function $\left[0, \frac{1}{2}\right] \ni x \mapsto \left|\frac{\sin(\pi x)}{\pi x}\right|$ is positive on its domain, so for some global constant K, we have

$$x \in \left[0, \frac{1}{2}\right] \implies \left|\frac{\sin(\pi x)}{\pi x}\right| \geq \frac{1}{K} \implies \left|G_A\left(\frac{j}{M^s}\right)\right| \geq \frac{L}{K\sqrt{M}}$$

In particular,

$$\begin{aligned} r_k^2 &\geq \frac{1}{L^k}\sum_{j \in \mathcal{C}_k} |F_N(\mathbf{1}_{\mathcal{C}_k})(j)|^2 \\ &\geq \frac{1}{L^k}\sum_{j \in \Omega} |F_N(\mathbf{1}_{\mathcal{C}_k})(j)|^2 \\ &\geq \frac{|\Omega|}{L^k}\frac{L^{2k}}{K^{2k}M^k} \\ &\geq \frac{L^{2k}}{(3K)^{2k}M^k} \end{aligned}$$

This lower bound on r_k^2 gives

$$\frac{\log r_k}{k\log M} \geq \delta - \frac{1}{2} - \frac{\log 3K}{\log M}$$

which implies that

$$\beta \leq \frac{1}{2} - \delta + \frac{\log(3K)}{\log M}$$

Now if for some $c > 0$ we have $L \geq c\sqrt{M}$, then we can choose a constant D such that

$$c^2 D \geq \log(3K) \implies \frac{DL^2}{M\log M} \geq \frac{\log(3K)}{\log M}$$

which gives the bound in (4.1).

Next we consider the case where $\frac{L}{\sqrt{M}}$ is small. We have

$$|G_A(x)| = |G_A(\{x\})| \geq \frac{L}{\sqrt{M}}\left|\frac{\sin(L\pi\{x\})}{L\pi\{x\}}\right|$$

Note that for x small, we can pick some positive constant H such that

$$\left|\frac{\sin(\pi x)}{\pi x}\right| > 1 - Hx > 0 \text{ for } x > 0$$

and so

$$\left|G_A\left(\frac{j}{M^s}\right)\right| \geq \frac{L}{\sqrt{M}}\left(1 - HL\left\{\frac{j}{M^s}\right\}\right)$$

In general we have

$$\left\{\frac{j}{M^s}\right\} \leq L\left(\frac{M^s - 1}{M^s(M-1)}\right)$$

and if M is big enough $\frac{1}{M-1} \leq \frac{q}{M}$ for some constant q so

$$\left|G_A\left(\frac{j}{M^s}\right)\right| \geq \frac{L}{\sqrt{M}}\left(1 - \frac{\mathcal{H}L^2}{M}\right)$$

for some constant $\mathcal{H} > 0$. We finally get that

$$r_k^2 \geq \frac{L^{2k}}{M^k}\left(1 - \frac{\mathcal{H}L^2}{M}\right)^{2k}$$

which implies that

$$\begin{aligned}\frac{\log r_k}{k \log M} &\geq \frac{\log L}{\log M} - \frac{1}{2} + \frac{\log\left(1 - \frac{\mathcal{H}L^2}{M}\right)}{\log M} \\ &\geq \delta - \frac{1}{2} + \frac{\log\left(1 - \frac{\mathcal{H}L^2}{M}\right)}{\log M}\end{aligned}$$

When $x > 0$ is small, we have $0 \leq -\log(1-x) \leq Dx$, for some constant $D > 0$. Since $\frac{L^2}{M}$ is small, or specifically as $\frac{\mathcal{H}L^2}{M}$ is small, we have

$$\beta \leq \frac{1}{2} - \delta + \frac{D\mathcal{H}L^2}{M \log M}$$

□

The proof for Conjecture 4.1 is similar to the one presented above. We tweak the choice of Ω.

Proposition 4.4. *Let $\delta \in \left(\frac{1}{2}, 1\right)$. For an integer M, alphabet $A \subset \mathbb{Z}_M$, let β be as defined above, so*

$$\beta = -\lim_{k\to\infty} \frac{\log r_k}{k \log M} = -\inf_k \frac{\log r_k}{k \log M}$$

where $r_k = ||\mathbb{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbb{1}_{\mathcal{C}_k}||$. Suppose $L = \lfloor M^\delta \rfloor$ with M big, and $A = \{0, 1, ..., L-1\}$. Then for some constant $\mathbf{W}$, we have

$$\beta \leq \frac{\mathbf{W}}{\log M} \tag{4.4}$$

Proof. Say $L \sim M^\delta$, with $\delta > \frac{1}{2}$. Specifically, let $L = \lfloor M^\delta \rfloor$. We consider the alphabet $A = \{0, 1, ..., L-1\}$. Recall that

$$G_A(x) = \frac{1}{\sqrt{M}} \sum_{a\in A} \exp\left(-2\pi i a x\right)$$

For this choice of A we have

$$G_A(x) = \frac{1}{\sqrt{M}} \frac{\exp(-2\pi i L x) - 1}{\exp(-2\pi i x) - 1}$$

We once again put $u = \mathbf{1}_{\mathcal{C}_k}$, so then

$$r_k^2 \geq \frac{||\mathbb{1}_{\mathcal{C}_k} F_N \mathbb{1}_{\mathcal{C}_k} u||^2}{||u||^2} = \frac{1}{L^k} \sum_{j \in \mathcal{C}_k} |F_N\left(\mathbf{1}_{\mathcal{C}_k}\right)(j)|^2$$

Recall that

$$F_N\left(\mathbf{1}_{\mathcal{C}_k}\right)(j) = \prod_{s=1}^{k} G_A\left(\frac{j}{M^s}\right)$$

This time, let

$$\Omega = \left\{\sum_{r=0}^{k-1} b_r M^r : b_i \in \mathbb{Z} \cap \left[0, \frac{M^{1-\delta}}{3}\right]\right\} \subset \mathcal{C}_k$$

and pick M large enough so that $\lfloor \frac{M^{1-\delta}}{3} \rfloor$ is non-zero. For $s \in \mathbb{N}$, $j \in \Omega$, consider the fractional part of $\frac{j}{M^s}$, $\left\{\frac{j}{M^s}\right\}$. We have (with $b_i \in \mathbb{Z} \cap \left[0, \frac{M^{1-\delta}}{3}\right]$)

$$\left\{\frac{j}{M^s}\right\} = \sum_{i=0}^{s-1} \frac{b_i M^i}{M^s} \leq \frac{M^{1-\delta}}{3}\left(\frac{M^s - 1}{M^s(M-1)}\right)$$

Now for $M \geq 5$, we have

$$\frac{2M}{3(M-1)} \leq 1 \implies \frac{2M}{3}\left(\frac{M^s - 1}{M^s(M-1)}\right) \leq 1$$

and

$$L = \lfloor M^\delta \rfloor \implies \frac{2LM^{1-\delta}}{3}\left(\frac{M^s - 1}{M^s(M-1)}\right) \leq 1$$

which gives that

$$\left\{\frac{j}{M^s}\right\} \leq \frac{1}{2L}$$

For $j \in \Omega$ and $s \in \mathbb{N}$, we have

$$\left|G_A\left(\frac{j}{M^s}\right)\right| = \left|G_A\left(\left\{\frac{j}{M^s}\right\}\right)\right| = \frac{1}{\sqrt{M}}\left|\frac{\sin\left(L\pi\left\{\frac{j}{M^s}\right\}\right)}{\sin\left(\pi\left\{\frac{j}{M^s}\right\}\right)}\right| \geq \frac{L}{\sqrt{M}}\left|\frac{\sin\left(L\pi\left\{\frac{j}{M^s}\right\}\right)}{L\pi\left\{\frac{j}{M^s}\right\}}\right|$$

Note that

$$L\left\{\frac{j}{M^s}\right\} \leq \frac{1}{2}$$

Since the function $\left[0, \frac{1}{2}\right] \ni x \mapsto \left|\frac{\sin(\pi x)}{\pi x}\right|$ is strictly positive on the domain, it follows that there exists some global constant $K > 1$ such that

$$\left|G_A\left(\frac{j}{M^s}\right)\right| \geq \frac{L}{K\sqrt{M}} \text{ for } j \in \Omega, s \in \mathbb{N}$$

So

$$j \in \Omega \implies |F_N\left(\mathbf{1}_{\mathcal{C}_k}\right)(j)|^2 \geq \frac{L^{2k}}{K^{2k}M^k} \implies r_k^2 \geq \frac{1}{L^k}|\Omega|\frac{L^{2k}}{K^{2k}M^k} \sim \frac{1}{L^k}\left(\frac{M^{1-\delta}}{3}\right)^k \frac{L^{2k}}{K^{2k}M^k}$$

For some other global constant $1 \geq c$ approximately equal to 1, we have

$$r_k^2 \geq \frac{c^k}{L^k}\left(\frac{M^{1-\delta}}{3}\right)^k \frac{L^{2k}}{K^{2k}M^k}$$

Let $1 \geq d$ approximately equal to 1 be s.t. $L \geq dM^\delta$. We get

$$r_k^2 \geq \frac{1}{K^{2k}M^k}\left(\frac{Mcd}{3}\right)^k = \left(\frac{cd}{3K^2}\right)^k$$

which gives

$$\beta = -\inf_k \frac{\log r_k}{k \log M} \leq \frac{\log\left(\frac{3K^2}{cd}\right)}{2\log M}$$

□

Observe how Proposition 4.4 implies the truth of Conjecture 4.1. For $\delta \in \left(\frac{1}{2}, 1\right)$, we create the pair (M_j, A_j) by setting

$$(M_j, A_j) = (j, \{0, 1, ..., \lfloor j^\delta \rfloor - 1\})$$

Then

$$\delta(M_j, A_j) = \frac{\log\lfloor j^\delta \rfloor}{\log j} \to \delta \text{ as } j \to \infty$$

From Proposition 4.4, we get some constant $\mathbf{W} > 0$ such that

$$0 \leq \beta(M_j, A_j) \leq \frac{\mathbf{W}}{\log j} \implies \beta(M_j, A_j) \to 0 \text{ as } j \to \infty$$

5. Further questions

A natural question to ask is how tight the bound on beta (4.4) from Proposition 4.4 is. One possible way to improve this bound might be to obtain a better choice of u.

Our initial choice of u was $u = \mathbf{1}_{\mathcal{C}_k}$. Observe that we can write

$$\mathbf{1}_{\mathcal{C}_k} = \mathbf{1}_A * \mathbf{1}_{AM} * \cdots * \mathbf{1}_{AM^{k-2}} * \mathbf{1}_{AM^{k-1}}$$

where $*$ is the convolution operator.

To get an improved bound on β, we can systematically look for u that optimizes this upper bound. As a start, we can choose

$$u = g * g_M * \cdots * g_{M^{k-2}} * g_{M^{k-1}}$$

where $g \in \mathbb{C}^M$ with $\operatorname{supp} g \subset \mathcal{A}$ and

$$g_{M^s}(n) = \begin{cases} g\left(\frac{n}{M^s}\right), & \text{if } n = aM^s \text{ for } a \in A \\ 0, & \text{otherwise} \end{cases}$$

Here we view g as a function on C^{M^k} by extending it by zero. Notice that g_{M^s} is supported in AM^s.

With this much more general choice of u, the hope is to be able to systematically find the most optimal upper bound on the exponent β.

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