ON THE $P = W$ CONJECTURE FOR A MULTIPLICATIVE SLICE

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Abstract. In [5], Seidel and Smith conjecture that a particular slice of nilpotents may be used to recover Khovanov's arc algebras, a generalization of his namesake homology to tangles. This was eventually proven in [1]. In this paper, we investigate properties of this slice and a related multiplicative analogue as a specific case of the $P = W$ conjecture. More specifically, we investigate the cohomology of the slices, the number of points of the slices over a finite field, and their relation to Springer fibers and affine Springer fibers.

Contents

1. Introduction

Let $G = \mathrm{SL}_{2n}(\mathbb{C})$ be a complex algebraic group and $\mathfrak{g} = \mathfrak{sl}_{2n}(\mathbb{C})$ its Lie algebra. Consider the affine subspace $\mathcal{S}_n$ of $\mathfrak{g}$ consisting of matrices of the form

$$T = \begin{pmatrix} 0 & & & & a_1 & & & & & b_1 \\ 1 & 0 & & & a_2 & & & & & b_2 \\ & 1 & 0 & & a_3 & & & & & b_3 \\ & & \ddots & \ddots & \vdots & & & & & \vdots \\ & & & 1 & a_n & & & & & b_n \\ & & & & c_1 & 0 & & & & d_1 \\ & & & & c_2 & 1 & 0 & & & d_2 \\ & & & & c_3 & & 1 & 0 & & d_3 \\ & & & & \vdots & & & \ddots & \ddots & \vdots \\ & & & & c_n & & & & 1 & d_n \end{pmatrix}$$

and let $\chi\colon \mathfrak{g} \to \mathfrak{g}/G \cong \mathbb{C}^{2n-1}$ be the adjoint quotient map which takes a matrix to the coefficients of its characteristic polynomial. For a monic polynomial $p(x) \in \mathbb{C}[x]$ of degree $2n$ with distinct roots, the fiber $\chi^{-1}(p)$ is a smooth manifold. We fix such a polynomial p. The object of interest is the fiber $\chi|_{\mathcal{S}_n}^{-1}(p)$, which we denote by

$\mathcal{Y}_n = \mathcal{Y}_{n,p}$. It carries the structure of both a $4n$-dimensional smooth manifold and an affine variety over $\mathbb{C}$. For complex polynomials

$$\begin{aligned}
A(x) &= x^n - a_n x^{n-1} + \cdots + (-1)^n a_1 \\
B(x) &= b_n x^{n-1} - b_{n-1} x^{n-1} + \cdots + (-1)^{n-1} b_1 \\
C(x) &= c_n x^{n-1} - c_{n-1} x^{n-1} \cdots + (-1)^{n-1} c_1 \\
D(x) &= x^n - d_n x^{n-1} + \cdots + (-1)^n d_1,
\end{aligned}$$

the defining polynomial equations $\det(T - xI) = p(x)$ of $\mathcal{Y}_n$ may be rewritten as

$$A(x)D(x) - B(x)C(x) = p(x). \tag{1.1}$$

We will often think of the points of $\mathcal{Y}_n$ as such a tuple (A, B, C, D).

In [5], Seidel and Smith conjectured that Khovanov's arc algebras, a generalization of Khovanov homology to tangles, may be recovered using $\mathcal{Y}_n$. This was later proven by Abouzaid and Smith in [1]. In this paper, we focus on properties of $\mathcal{Y}_n$ itself, and that of an open subvariety $\mathcal{Y}_n^\times$ given by

$$\{(A, B, C, D) \in \mathcal{Y}_n \colon a_1 \neq 0\}$$

where we additionally require $p(0) \neq 0$. More specifically, motivated by the Weil conjectures, we investigate the connection between the cohomology of $\mathcal{Y}_n$ and $\mathcal{Y}_n^\times$ and the number of points of each variety over a finite field. This is a special case of the deep $P = W$ conjecture, which asserts that for certain pairs of algebraic varieties there is a close connection between the weight filtration on the cohomology of one variety and the perverse filtration on the cohomology of the other. In this case, the other varieties are certain (affine) Springer fibers. The point counting reflects the weight filtration on the cohomology of $\mathcal{Y}_n$ and $\mathcal{Y}_n^\times$, and we attempt to relate this to the cohomology of the (affine) Springer fibers.

In the case of $\mathcal{Y}_n$, its cohomology is obtained via a homotopy equivalence between $\mathcal{Y}_n$ and the Springer fiber $\mathcal{B}_{n,n}$ of complete flags in $\mathbb{C}^{2n}$ preserved by a two-block nilpotent of type (n, n). In counting the number of solutions $N_{n,q}$ to 1.1 over a finite field $\mathbb{F}_q$, we make the following conjecture.

Conjecture 1.2. *Let $c_{n,k} = \binom{n}{k} - \binom{n}{k-1}$. Then*

$$N_{n,q} = \sum_{k=0}^{n} c_{2n,k} q^{2n-k}.$$

The connection lies in the fact that the coefficients $c_{2n,k}$ appear as the nonzero Betti numbers of $\mathcal{B}_{n,n}$ and $\mathcal{Y}_n$. We give a way to verify Conjecture 1.2 for small n.

The natural analog of $\mathcal{B}_{n,n}$ for the multiplicative slice $\mathcal{Y}_n^\times$ is a certain affine Springer fiber $\mathcal{X}_\gamma = \mathcal{X}_{\gamma,n}$ whose points are rank $2n$ $\mathbb{C}[[t]]$-submodules of $\mathbb{C}(\!(t)\!)^{2n}$

fixed by the operator

$$\gamma = \begin{pmatrix} 0 & & & t^n & & & & \\ 1 & 0 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 1 & 0 & & & & \\ & & & & 0 & & & at^n \\ & & & & a & 0 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & a & 0 \end{pmatrix}$$

with $a^n \neq 1$. The structure of $\mathcal{X}_\gamma$ is less well-understood in general. We give an explicit description of $\mathcal{X}_\gamma$ in the case $n = 2$ and we propose the following conjecture for the general case.

Conjecture 1.3. *For a fixed n, we have $\mathcal{X}_\gamma = \bigcup_{d \in \mathbb{Z}} \mathcal{C}_d$, where each $\mathcal{C}_d \cong \mathbb{P}^n$ and for $1 \leq k \leq n$, $\mathcal{C}_i \cap \mathcal{C}_j \cong \mathbb{P}^{n-k}$ if $|i - j| = k$ and empty otherwise.*

The structure of the paper is as follows. In Section 2, we describe the cohomology of $\mathcal{Y}_n$ in more detail using $\mathcal{B}_{n,n}$. In Section 3, we count the number of points of $\mathcal{Y}_n$ and $\mathcal{Y}_n^\times$ over a finite field $\mathbb{F}_q$. Finally, in Section 4, we recall some facts about affine Springer fibers and attempt to use them to understand the cohomology of $\mathcal{Y}_n^\times$. In the process, we describe some relevant affine Springer fibers explicitly, and conjecture a description for the general case.

2. Cohomology of $\mathcal{Y}_n$

We begin by examining the cohomology of $\mathcal{Y}_n$, which is well-understood.

Proposition 2.1. *The cohomology ring of $\mathcal{Y}_n$ is*

$$H^*(\mathcal{Y}_n; \mathbb{Z}) = \mathbb{Z}[x_1, \dots, x_{2n}]/I,$$

where $|x_i| = 2$ and I is the ideal generated by the x_i^2 and $\sum_{|J|=k} \prod_{j \in J} x_j$ for $1 \leq i, k \leq 2n$ and $J \subset \{1, \dots, 2n\}$.

Proposition 2.1 is proved by showing that $\mathcal{Y}_n$ is homotopy equivalent to the Springer fiber $\mathcal{B}_{n,n}$ of complete flags in $\mathbb{C}^{2n}$ preserved by a two-block nilpotent of type (n, n). This is done in [2]. Then the cohomology ring of $\mathcal{B}_{n,n}$ is as claimed by Theorem 1 in Section 1 of [3].

Corollary 2.2. *The cohomology of $\mathcal{Y}_n$ is even, and for $0 \leq k \leq n$ we have*

$$H^{2k}(\mathcal{Y}_n; \mathbb{Z}) = \mathbb{Z}^{c_{2n,k}},$$

where $c_{2n,k} = \binom{2n}{k} - \binom{2n}{k-1}$.

Proof. For a subset $J \subset \{1, \dots, 2n\}$, let x_J denote $\prod_{j \in J} x_j$. Following [3], we say J is *admissible* if for each $m \in \{1, \dots, 2n\}$ the intersection $J \cap \{1, \dots, m\}$ contains at most $m/2$ elements. By Corollary 1 in Section 4 of [3], as J ranges across all admissible k-element subsets of $\{1, \dots, 2n\}$, the x_J form a basis of $H^{2k}(\mathcal{Y}_n; \mathbb{Z})$. The set of such J may be put in bijection with the lattice paths from $(0, 0)$ to $(2n, 2n-2k)$ moving in the directions $(1, \pm 1)$ which do not cross the x-axis. A Catalan-style reflection argument shows the number of such paths is precisely $c_{2n,k}$. $\square$

3. Point counting over finite fields

Motivated by the Weil conjectures, we count the number of points of $\mathcal{Y}_n$ and $\mathcal{Y}_n^\times$ over a finite field $\mathbb{F}_q$. We illustrate examples for $n = 1, 2$, then conjecture the answer for arbitrary n. Let $N_{n,q}$ and $M_{n,q}$ denote the number of points of $\mathcal{Y}_n$ and $\mathcal{Y}_n^\times$ over $\mathbb{F}_q$, respectively.

Let us first describe the strategy using an analogy over $\mathbb{C}$. Let $\Psi \colon \mathcal{Y}_n \to \mathbb{C}^n$ be the map sending (A, B, C, D) to (the nontrivial coefficients of) A, and assume A has distinct roots. If A shares no roots with p, by evaluating x at the roots r_i of A, we must have $B(r_i)C(r_i) = p(r_i)$, which is nonzero. Each choice of $B(r_i) \in \mathbb{C}^\times$ determines $C(r_i)$, and the polynomials B and C are themselves determined by the choices of $B(r_i)$ and $C(r_i)$. This in turn fixes D, so we conclude that the fiber $\Psi^{-1}(A)$ is isomorphic to $(\mathbb{C}^\times)^n$. If instead a root r_i of A is also a root of p, then a copy of $\mathbb{C}^\times$ is replaced by the variety $\{xy = 0 \colon x, y \in \mathbb{C}\}$ for each such r_i.

The analogy is not perfect, since we have not accounted for what happens when A has roots with repeated multiplicity. Furthermore, A may not split completely over $\mathbb{F}_q$, i.e. it may have irreducible factors of degree larger than 1. Nevertheless, it gives us a way to compute $N_{n,q}$ and $M_{n,q}$.

Proposition 3.1. *We have $N_{1,q} = q^2 + q$ and $M_{1,q} = q^2 + 1$.*

Proof. We begin with $N_{1,q}$. We wish to count the number of solutions to

$$(x + a)(x + d) - bc = p(x), \tag{3.2}$$

where $a, b, c, d \in \mathbb{F}_q$ and $p(x) \in \mathbb{F}_q[x]$ is a fixed quadratic with two distinct roots. There are two cases.

If $-a$ is a root of $p(x)$, then evaluating at $x = -a$ shows that we must have $bc = 0$. For each of the $2q - 1$ pairs (b, c) with $bc = 0$ there is a unique $d \in \mathbb{F}_q$ satisfying 3.2. Otherwise, if $-a$ is not a root of $p(x)$, then there are $q - 1$ pairs (b, c) with $bc = p(-a)$. As before, each such pair (b, c) determines a unique d satisfying 3.2. In total, we have

$$N_{1,q} = 2(2q - 1) + (q - 2)(q - 1) = q^2 + q.$$

To obtain $M_{1,q}$ from $N_{1,q}$, we subtract out the solutions with $a = 0$. There are $q - 1$ such solutions since 0 is not a root of $p(x)$. Thus

$$M_{1,q} = N_{1,q} - (q - 1) = q^2 + 1.$$

□

Proposition 3.3. *For q an odd prime power, we have $N_{2,q} = q^4 + 3q^3 + 2q^2$ and $M_{2,q} = q^4 + 2q^3 + 2q + 1$.*

Proof. We first count $N_{2,q}$. We split into two cases, depending on whether $A(x) = x^2 + a_1 x + a_2$ is irreducible or not.

- *Case 1: $A(x)$ has two distinct roots.* Suppose A has distinct roots r_1 and r_2. The analogy over $\mathbb{C}$ shows that the number of elements in $\Psi^{-1}(A)$ depends only on whether the r_i are roots of p or not. If both are roots of p, there are $(2q - 1)^2$ solutions. If exactly one r_i is a root of p, there are $(q - 1)(2q - 1)$ solutions. If neither are roots of p, then there are $(q - 1)^2$ solutions. In total, across all A with distinct roots, there are

$$\binom{4}{2}(2q - 1)^2 + 2(4)(q - 4)(q - 1)(2q - 1) + \binom{q - 4}{2}(q - 1)^2$$

solutions.

- *Case 2: $A(x)$ has a double root.* Let $A(x) = (x-r)^2$. As we before, we have $B(r)C(r) = p(r)$. Differentiating 1.1 and evaluating at $x = r$, we obtain

$$B'(r)C(r) + B(r)C'(r) = p'(r). \tag{3.4}$$

If $p(r) \neq 0$, there are $q-1$ choices for $(B(r), C(r))$. Then any choice of $B'(r)$ determines a unique $C'(r)$. Together $B(r)$ and $B'(r)$ determine B uniquely, and likewise for C, so in this case there are $q(q-1)$ solutions.

If $p(r) = 0$, then $p'(r) \neq 0$ since p has distinct roots, so $B(r)$ and $C(r)$ cannot both be zero. If $B(r) = 0$, then $C(r)$ is nonzero, $B'(r)$ is fixed by 3.4 and $C'(r)$ may by any element of $\mathbb{F}_q$, giving $q(q-1)$ solutions. The case when $C(r) = 0$ is identical.

Thus, across all A with a double root, there are

$$(q-4)q(q-1) + 2q(q-1)$$

solutions.

- *Case 3: $A(x)$ is irreducible.* We consider 1.1 over the extension $\mathbb{F}_{q^2}$ of $\mathbb{F}_q$ with fixed A and p, but temporarily allow B, C, and D to have coefficients in $\mathbb{F}_{q^2}$. Let the distinct roots of $A(x)$ be $r_1, r_2 \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Evaluating at $x = r_i$ gives $B(r_i)C(r_i) = p(r_i)$ with $p(r_i)$ nonzero, so nonzero choices of $B(r_1)$ and $B(r_2)$ determines $C(r_1)$ and $C(r_2)$, and hence B, C, and D. We wish to find the B and C with coefficients in $\mathbb{F}_q$; it follows that D will have coefficients in $\mathbb{F}_q$ too.

Let σ be the generator of $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$, so σ swaps r_1 and r_2. Then B has coefficients in $\mathbb{F}_q$ if and only if $B(r_2) = \sigma(B(r_1))$, so a solution (A, B, C, D) amounts to a choice of $B(r_1) \in \mathbb{F}_{q^2}^\times$. There are $q(q-1)/2$ irreducible A, so this case contributes $q(q-1)(q^2-1)/2$ solutions.

Adding up the number of solutions in each case yields $N_{2,q} = q^4 + 3q^3 + 2q^2$. By omitting those A with 0 as a root, we obtain $M_{2,q}$. □

A similar argument yields the following generalization for $|\Psi^{-1}(A)|$ when the irreducible factorization of A is known.

Proposition 3.5. *Let $A \in \mathbb{F}_q[x]$ be monic of degree n, and suppose A factors into monic irreducibles as*

$$A(x) = \prod_i f_i(x)^{m_i}$$

with the f_i distinct and $\deg f_i = d_i$. Then the number of (B, C, D) satisfying 1.1 is $\prod_i g(f_i)$, where

$$g(f_i) = \begin{cases} 2q-1 & f_i(x) = x-r, k_i = 1, p(r) = 0 \\ 2q^{m_i-1}(q-1) & f_i(x) = x-r, k_i > 1, p(r) = 0 \\ q^{d_i(m_i-1)}(q^{d_i}-1) & \textit{otherwise.} \end{cases}$$

Since the number of irreducible polynomials over $\mathbb{F}_q$ of a given degree is known, Proposition 3.5 gives a way to compute $N_{n,q}$ exactly for a given n. However, despite knowing $|\Psi^{-1}(A)|$ for any given A, as illustrated in the case $n = 2$, the sum over all A is quite complicated. Nevertheless, we make the following conjecture.

Conjecture 3.6. *Let* $c_{n,k} = \binom{n}{k} - \binom{n}{k-1}$*. Then*

$$N_{n,q} = \sum_{k=0}^{n} c_{2n,k} q^{2n-k}$$

and

$$M_{n,q} = \sum_{k=0}^{n} c_{2n-1,k}(q^k + q^{2n-k}).$$

Example 3.7. We give the first few values of $N_{n,q}$ and $M_{n,q}$.

- $N_{1,q} = q^2 + q$
- $N_{2,q} = q^4 + 3q^3 + 2q^2$
- $N_{3,q} = q^6 + 5q^5 + 9q^4 + 5q^3$
- $N_{4,q} = q^8 + 7q^7 + 20q^6 + 28q^5 + 14q^4$
- $N_{5,q} = q^{10} + 9q^9 + 35q^8 + 75q^7 + 90q^6 + 42q^5$
- $M_{1,q} = q^2 + 1$
- $M_{2,q} = q^4 + 2q^3 + 2q + 1$
- $M_{3,q} = q^6 + 4q^5 + 5q^4 + 5q^2 + 4q + 1$
- $M_{4,q} = q^8 + 6q^7 + 14q^6 + 14q^5 + 14q^3 + 14q^2 + 6q + 1$
- $M_{5,q} = q^{10} + 8q^9 + 27q^8 + 48q^7 + 42q^6 + 42q^4 + 48q^3 + 27q^2 + 8q + 1$.

In particular, observe that the coefficients $c_{2n,k}$ are exactly the nonzero Betti numbers of $\mathcal{Y}_n$ and $\mathcal{B}_{n,n}$.

4. Affine Springer fibers and $\mathcal{Y}_n^\times$

4.1. **Affine Springer fibers.** We begin by reviewing the theory of affine Springer fibers, specializing to the case of $G = \mathrm{SL}_{2n}$. We work over the field $k = \mathbb{C}$ of complex numbers.

As a set, the *affine Grassmannian* $\mathrm{Gr}_G(\mathbb{C})$ of G is the group $G(\mathbb{C}(\!(t)\!))/G(\mathbb{C}[[t]])$. In the case of SL_{2n}, we may identify the points of the affine Grassmannian with rank $2n$ $\mathbb{C}[[t]]$-submodules of $\mathbb{C}(\!(t)\!)^{2n}$. We call such a $\mathbb{C}[[t]]$-submodule a *lattice* in $\mathbb{C}(\!(t)\!)^{2n}$. For a regular semisimple element $\gamma \in \mathfrak{sl}_{2n}(\mathbb{C}[[t]])$, the *affine Springer fiber* $\mathcal{X}_\gamma$ of γ corresponds to those lattices $\Lambda \in \mathrm{Gr}_G(\mathbb{C})$ that are stabilized by γ and satisfy the relative length condition $[\Lambda\colon \mathbb{C}[[t]]^{2n}] = 0$. For details and a more general treatment of affine Springer fibers, see [6].

Let $z = t^n$. Then we may view $\mathbb{C}(\!(t)\!)^2$ as a $2n$-dimensional $\mathbb{C}(\!(z)\!)$-vector space, so the affine Grassmannian may be thought of as $\mathbb{C}[[z]]$-lattices in $\mathbb{C}(\!(t)\!)^2$. We will frequently move between these two interpretations. In terms of the latter interpretation, of interest to us is the affine Springer fiber $\mathcal{X}_\gamma$ over the regular semisimple element $\gamma = \mathrm{diag}(t, at)$, where $a \in \mathbb{C}$ satisfies $a^n \neq 1$. The case $n = 1$ is computed in [6]. In this case, $\mathcal{X}_\gamma$ is a countably infinite chain of $\mathbb{P}^1$s with each consecutive pair of $\mathbb{P}^1$s intersecting along a point. We show that there is a similar description in the case $n = 2$ and conjecture a description for the general case.

4.2. **Computing $\mathcal{X}_\gamma$ for $n = 2$.** We compute the points of $\mathcal{X}_\gamma$ explicitly for $n = 2$. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ so that (e_1, te_1, e_2, te_2) is a $\mathbb{C}(\!(t^2)\!)$-basis of $\mathbb{C}(\!(t)\!)^2$. Consider the operator $\gamma = \mathrm{diag}(t, at)$ acting on $\mathbb{C}(\!(t)\!)^2$. In this basis and under the identification $z = t^2$, the matrix of γ is

$$\gamma = \begin{pmatrix} & z & & \\ 1 & & & \\ & & & az \\ & & a & \end{pmatrix}.$$

For a formal Laurent series $p \in \mathbb{C}((z))$, let $\nu(p)$ denote the degree of the lowest degree term of p. Define $\nu(0) = \infty$. If $g \in \mathfrak{gl}_{2n}(\mathbb{C}((z)))$ takes the standard lattice $\mathbb{C}[[z]]^{2n}$ to a lattice Λ, then the relative length $[\Lambda\colon \mathbb{C}[[z]]^{2n}]$ is given by $\nu(\det g)$.

Proposition 4.1. *The elements $\Lambda \in \mathcal{X}_\gamma$ when viewed as $\mathbb{C}[[t^2]]$ lattices in $\mathbb{C}((t))^2$, are precisely those with basis*

$$\begin{pmatrix} t^d \\ 0 \end{pmatrix}, \begin{pmatrix} t^{d+1} \\ 0 \end{pmatrix}, \begin{pmatrix} a\beta t^{d-2} + \alpha t^{d-1} \\ t^{-d} \end{pmatrix}, \begin{pmatrix} \beta t^{d-1} \\ t^{-d+1} \end{pmatrix},$$

where $d \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{C}$.

Proof. The idea of the proof is as follows. Let (v_1, v_2, v_3, v_4) with $v_i \in \mathbb{C}((z))^4$ be a $\mathbb{C}[[z]]$-basis for $\Lambda \in \mathcal{X}_\gamma$, and let A be the matrix with columns v_i. By elementary column operations over $\mathbb{C}[[z]]$, we may assume A is upper-triangular of the form

$$A = \begin{pmatrix} z^{d_1} & p_1 & p_2 & p_4 \\ & z^{d_2} & p_3 & p_5 \\ & & z^{d_3} & p_6 \\ & & & z^{d_4} \end{pmatrix},$$

and $d_1 + d_2 + d_3 + d_4 = 0$ by the relative length condition[1] . Furthermore, by adding multiples of the first column we may assume p_1, p_2, p_4 are polynomials in z, z^{-1} which are either zero or of degree less than d_1, and similarly for the remaining p_i.

The stability condition $\gamma\Lambda \subset \Lambda$ is equivalent to checking that $\gamma v_i \in \Lambda$, which we do via Cramer's rule. Let A^i_j be the matrix obtained by replacing the i-th column of A with γv_j. By Cramer's rule, γv_j is in Λ, i.e. the $\mathbb{C}[[z]]$-span of the v_i, if and only if $\nu(\det A^i_j) \geq 0$ for all i. Thus computing the lattices $\Lambda \in \mathcal{X}_\gamma$ amounts to checking the $\nu(\det A^i_j)$.

First we show that $p_1 = p_6 = 0$. We have

$$\gamma v_1 = \begin{pmatrix} 0 \\ z^{d_1} \\ 0 \\ 0 \end{pmatrix}, \gamma v_2 = \begin{pmatrix} z^{d_2+1} \\ p_1 \\ 0 \\ 0 \end{pmatrix}, \gamma v_3 = \begin{pmatrix} zp_3 \\ p_2 \\ 0 \\ az^{d_3} \end{pmatrix}, \gamma v_1 = \begin{pmatrix} z_5^p \\ p_4 \\ az^{d_4+1} \\ ap_6 \end{pmatrix}.$$

[1] To elaborate slightly, assume without loss of generality that v_4 has the fourth coordinate with the least valuation. This coordinate cannot be zero, otherwise the v_i do not form a basis. We may multiply by a unit in $\mathbb{C}[[z]]$ to assume this coordinate is z^{d_4}, then add multiples of v_4 to the other v_i to assume their fourth coordinate is zero. Then we repeat by choosing v_3 similarly, and so on.

Then we find that

$$\begin{aligned}
\det A_1^1 &= -\det A_2^2 = -p_1 z^{-d_2},\\
\det A_1^2 &= z^{d_1-d_2},\\
\det A_2^1 &= z^{-d_1+d_2+1} - p_1^2 z^{-d_1-d_2},
\end{aligned}$$

and the condition $v(\det A_j^i) \geq 0$ is automatically true for the remaining A_1^i and A_2^i. Since z^{2d+1} is not the square of a polynomial, it follows that $\det A_2^1 \neq 0$. We must have $d_1 \geq d_2$ and $d_2 + 1 \geq d_1$, so $(d_1, d_2) = (d+1, d)$ or (d, d) for some integer d. In either case, the conditions $\deg p_1 \geq d_2 = d$ and $2 \deg p_1 \geq d_1 + d_2$ force $p_1 = 0$, since we recall by assumption that $\deg p_1 < d_1$. By the same argument, the $\det A_j^i$ for $i, j \in \{3, 4\}$ force $p_6 = 0$, and we also conclude that $d_3 \geq d_4$ and $d_4 + 1 \geq d_3$. Together with the relative length condition $d_1 + d_2 + d_3 + d_4 = 0$, this means (d_1, d_2, d_3, d_4) is of the form $(d, d, -d, -d)$ or $(d+1, d, -d, -d-1)$.

Now that we know $p_1 = p_6 = 0$, the remaining four $\det A^{ij}$ are

$$\begin{aligned}
\det A_3^1 &= z^{-d_1+1} p_3 - a z^{d_2+2d_3} p_4,\\
\det A_3^2 &= p_2 z^{-d_2} - a p_5 z^{d_1+2d_3},\\
\det A_4^1 &= z^{-d_4+1} p_5 - a p_2 z^{d_2+2d_4+1},\\
\det A_4^2 &= p_4 z^{-d_2} - p_3 z^{d_1+2d_4+1}.
\end{aligned}$$

Then the conditions $\nu(\det A_j^i) \geq 0$ translate to

$$\begin{aligned}
\deg p_3 \geq d_1 - 1, \deg p_4 \geq -d_2 - 2d_3 \text{ OR } p_3 &= a z^{d_3-d_4+1} p_4,\\
\deg p_2 \geq d_2, \deg p_5 \geq -d_1 - 2d_3 \text{ OR } p_2 &= a z^{d_3-d_4} p_5,\\
\deg p_2 \geq -d_2 - 2d_4 - 1, \deg p_5 \geq d_1 - 1 \text{ OR } p_5 &= a z^{d_4-d_3} p_2,\\
\deg p_3 \geq -d_1 - 2d_4 - 1, \deg p_4 \geq d_2 \text{ OR } p_4 &= a z^{d_4-d_3+1},
\end{aligned}$$

respectively (and the OR's are inclusive).

In the case $(d_1, d_2, d_3, d_4) = (d, d, -d, -d)$, we find that A must be of the form

$$A = \begin{pmatrix} z^d & & a\beta z^{d-1} & \\ & z^d & \alpha z^{d-1} & \beta z^{d-1} \\ & & z^{-d} & \\ & & & z^{-d} \end{pmatrix}$$

for $\alpha, \beta \in \mathbb{C}$. In the case $(d_1, d_2, d_3, d_4) = (d+1, d, -d, -d-1)$, A must be of the form

$$A = \begin{pmatrix} z^{d+1} & & \beta z^d & \alpha z^d \\ & z^d & & a\beta z^{d-1} \\ & & z^{-d} & \\ & & & z^{-d-1} \end{pmatrix}.$$

Letting $z = t^2$ and returning to the $\mathbb{C}[[z]]$-basis (e_1, te_1, e_2, te_2) of $\mathbb{C}((t))^2$, we find that these are exactly the promised bases of $\Lambda \in \mathcal{X}_\gamma$. $\square$

Remark 4.2. It turns out $\mathcal{X}_\gamma = \bigcup_{d \in \mathbb{Z}} \mathcal{C}_d$, where each $\mathcal{C}_d \cong \mathbb{P}^2$ and $\mathcal{C}_i \cap \mathcal{C}_j$ is a copy of $\mathbb{P}^{2-k}$ if $|i - j| = k$ for $k = 1, 2$, and empty otherwise. Explicitly, $\mathcal{C}_d$ consists of the lattices with $\mathbb{C}[[z]]$-basis

$$\begin{pmatrix} t^d \\ 0 \end{pmatrix}, \begin{pmatrix} t^{d+1} \\ 0 \end{pmatrix}, \begin{pmatrix} a\beta t^{d-2} + \alpha t^{d-1} \\ t^{-d} \end{pmatrix}, \begin{pmatrix} \beta t^{d-1} \\ t^{-d+1} \end{pmatrix}$$

or

$$\begin{pmatrix} t^{d-1} \\ 0 \end{pmatrix}, \begin{pmatrix} t^d \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha t^{d-2} \\ t^{-d+1} \end{pmatrix}, \begin{pmatrix} 0 \\ t^{-d+2} \end{pmatrix}$$

for $\alpha, \beta \in \mathbb{C} \cup \{\infty\}$. The lattices with basis of the second type form the copy of $\mathbb{P}^1$ along which $\mathcal{C}_d$ and $\mathcal{C}_{d-1}$ intersect, and $\mathcal{C}_d$ intersects $\mathcal{C}_{d-2}$ at a single point.

This leads to the statement of Conjecture 1.3. A natural next step would be to recover the cohomology of $\mathcal{Y}_n^\times$ from $\mathcal{X}_\gamma$ using Conjecture 1.3.

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