Operators on Bessel Generating Functions With General Coefficients
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Abstract

With operators on formal series in $x_i$, $1 \leq i \leq N$, which are symmetric in $N-1$ of the $x_i$, Bessel generating functions can be studied. These operators are used with the Dunkl transform on Bessel generating functions of sequences of probability measures to get a Law of Large Numbers as the number of variables goes to infinity while $\beta$ is constant. Also, in the results, formal power series are used. However, we consider when the limits of partial derivatives involving any number of distinct indices of the logarithms of Bessel generating functions at 0 can be nonzero. Then, the free cumulant of order $k$ is a linear combination of the limits of order $k$ partial derivatives. Afterwards, we use these results on the $\beta$-Hermite ensemble.

1 Introduction

This paper studies ways Bessel generating functions can be used in probability. We are given a positive real number $\theta$ where $\beta = 2\theta$. Here, $\beta = 1$, 2, and 4 correspond to the GUE, GOE, and GSE, respectively. The multivariate Bessel function $B_{(a_1, \ldots, a_N)}(x_1, \ldots, x_N; \theta)$ for $a_1 < \cdots < a_N$ is based on the $\theta$-corners process, also known as the Gaussian $\beta$-corners process, with a fixed top row $(a_1, \ldots, a_N)$ ([GM18], [BGCG21]), see Definition 2.4. For positive integers $N$, let $\mathcal{M}_N$ denote the set of Borel probability measures on ordered $N$-tuples $(a_1, a_2, \ldots, a_N)$ in $\mathbb{R}^N$ such that $a_1 \leq a_2 \leq \cdots \leq a_N$. With $\theta$, for $\mu \in \mathcal{M}_N$, the Bessel generating function $G_\theta(x_1, \ldots, x_N; \mu)$ is

$$G_\theta(x_1, \ldots, x_N; \mu) = \int_{a_1 \leq \cdots \leq a_N} B_{(a_1, \ldots, a_N)}(x_1, \ldots, x_N; \theta) \mu(da_1, \ldots, da_N),$$

see Definition 2.5.
For a sequence \( \{\mu_N\}_{N \geq 1} \) of probability measures such that \( \mu_N \in \mathcal{M}_N \) for \( N \geq 1 \), for \( k \geq 1 \), we let the random variable \( p_k^N \) be
\[
p_k^N = \frac{1}{N} \sum_{i=1}^{N} \left( \frac{a_i}{N} \right)^k,
\]
(1)
where \( (a_1 \leq \cdots \leq a_N) \) are distributed according to \( \mu_N \). For the main result of the paper, Law of Large Numbers satisfaction is needed, which is in Definition 1.1.

**Definition 1.1.** A sequence \( \{\mu_N\}_{N \geq 1} \) of probability measures such that \( \mu_N \in \mathcal{M}_N \) for \( N \geq 1 \) satisfies a Law of Large Numbers if there exists a sequence \( \{m_k\}_{k \geq 1} \) of real numbers such that
\[
\lim_{N \to \infty} \mathbb{E}_{\mu_N} \left( \prod_{i=1}^{s} p_{k_i}^N \right) = \prod_{i=1}^{s} m_{k_i}
\]
(2)
for any positive integer \( s \) and positive integers \( k_i, 1 \leq i \leq s \).

In the above definition, \( \{m_k\}_{k \geq 1} \) are the moments, where \( m_k = \lim_{N \to \infty} \mathbb{E}_{\mu_N}(p_k^N) \). Later in this paper, we look at \( \mu \in \mathcal{M}_N, N \geq 1 \), which are known as exponentially decaying. The condition for \( \mu \in \mathcal{M}_N \) being exponentially decaying is in Definition 1.2 below.

**Definition 1.2** ([BGCG21, Definition 2.8]). A probability measure \( \mu \) in \( \mathcal{M}_N \) is exponentially decaying with exponent \( R > 0 \) if
\[
\int_{a_1 \leq a_2 \leq \cdots \leq a_N} e^{NR \max_{1 \leq i \leq N} |a_i|} \mu(da_1, \ldots, da_N)
\]
is finite.

**Lemma 1.3** ([BGCG21, Lemma 2.9]). If \( \mu \in \mathcal{M}_N \) is exponentially decaying with exponent \( R > 0 \), then \( G_\theta(x_1, \ldots, x_N; \mu) \) converges for all \( (x_1, \ldots, x_N) \) in the domain
\[
\Omega_R = \{(x_1, \ldots, x_N) \in \mathbb{C}^N | |\text{Re}(x_i)| < R, 1 \leq i \leq N \},
\]
and is holomorphic in the domain.

A reason we consider \( \mu \in \mathcal{M}_N \) which are exponentially decaying is that \( G_\theta(x_1, \ldots, x_N; \mu) \) is convergent and holomorphic in a neighborhood of the origin, see Lemma 1.3. Soon, we will be able to state the main result of the paper, Theorem 1.5.

**Definition 1.4.** For an ordered \( N \)-tuple \( a = (a_1, a_2, \ldots, a_N) \) of nonnegative integers, let the equivalent partition of \( a \), denoted by \( \pi(a) \), be the partition obtained from removing the \( a_i \) which are 0 and ordering the remaining \( a_i \) in nonincreasing order from left to right.

For a partition \( \nu = (a_1 \geq a_2 \geq \cdots \geq a_m) \), let \( P(\nu) \) denote the number of distinct permutations of \( (a_1, a_2, \ldots, a_m) \). Also, for a finite ordered list \( S \) of positive integers with maximum element \( M \), suppose that \( n_i \) of the elements of \( S \) are \( i \) for \( 1 \leq i \leq M \). Then, if \( n = (n_1, n_2, \ldots, n_M) \), let \( \sigma(S) \) be \( \pi(n) \).
Theorem 1.5. Suppose $\theta$ is a positive real number. Let $\{\mu_N\}_{N \geq 1}$ be a sequence of probability measures such that for all $N \geq 1$, $\mu_N$ is in $\mathcal{M}_N$ and is exponentially decaying. Assume that for all partitions $\nu$ with $|\nu| \geq 1$, there exists a real number $c_{\nu}$ such that
\[
\lim_{N \to \infty} \frac{1}{N} \cdot \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_r}} \ln(G_{\theta}(x_1, \ldots, x_N; \mu_N)) \bigg|_{x_i=0, 1 \leq i \leq N} = \frac{\ell(\nu)(|\nu| - 1)!c_{\nu}}{P(\nu)}
\]
for all positive integers $i_1, \ldots, i_r$ such that $\sigma((i_1, \ldots, i_r)) = \nu$. Then, $\{\mu_N\}_{N \geq 1}$ satisfy a LLN, and for positive integers $k$, $m_k = \sum_{\pi \in NC(k)} \prod_{B \in \pi} \theta^{|B|-1} \left( \sum_{\nu \in P, |\nu|=|B|} (-1)^{\ell(\nu)-1} P(\nu) c_{\nu} \right)$.

Note that Theorem 1.5 generalizes [BGCG21, Claim 9.1] and is similar to, but differs in important places with, [BGCG21, Theorem 3.8], where the reverse direction is also shown. Particularly, in [BGCG21, Theorem 3.8], $\theta N \to \gamma$ as $N \to \infty$ and the partial derivatives are not scaled; but, in this paper, $\theta$ is constant and the partial derivatives are scaled by $\frac{1}{N}$. Due to this, different approaches need to be taken. In Theorem 1.5, the free cumulant of order $k$ for $k \geq 1$ would be
\[
c_k = \theta^{k-1} \sum_{\nu \in P, |\nu|=k} (-1)^{\ell(\nu)-1} P(\nu) c_{\nu},
\]
a linear combination of the $c_{\nu}$ for partitions $\nu$ with $|\nu| = k$ ([Spe14]).

Additionally, this paper considers when the limit of any partial derivative can be nonzero. Previously, Bufetov and Gorin have looked at similar results involving limits of partial derivatives of Schur functions in [BG13, BG18, BG19]. If the partial derivatives involve only one integer, the limit can be nonzero, and in [BG18, BG19] where the Central Limit Theorem is being studied, if two distinct indices are involved, the limit can be nonzero. But, if three or more distinct indices are involved, the limit is zero. In this paper, we look at when limits of partial derivatives involving any number of distinct indices can be nonzero.

For exponentially decaying $\mu \in \mathcal{M}_N$, in Proposition 2.7, we see that we can apply Dunkl operators, from [Dun89], to $G_{\theta}(x_1, \ldots, x_N; \mu)$ and set $x_i = 0, 1 \leq i \leq N$, to get the moments of $\mu$. Since $G_{\theta}(x_1, \ldots, x_N; \mu)$ is holomorphic in a neighborhood of the origin, we can express $\ln(G_{\theta}(x_1, \ldots, x_N; \mu))$ as a symmetric polynomial for this. The results for this paper have $\theta$ constant.

To prove Theorem 1.5, we consider results on formal power series. In Section 3, we look at operators on formal power series symmetrical in $x_j$ for $1 \leq j \leq N, j \neq i$. These operators give the leading order terms of the Dunkl operators on $G_{\theta}(x_1, \ldots, x_N; \mu)$ at 0 as $N \to \infty$. Particularly, in Section 3, there are results on the limits of the operators on formal power series as the number of variables $N$ goes to infinity. Afterwards, in Section 4, we look at the result of applying Dunkl operators to formal series. Particularly, in both sections, we consider the constant term, which corresponds to setting $x_i = 0, 1 \leq i \leq N$. 

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Since we can view $\ln(G_\theta(x_1, \ldots, x_N; \mu))$ as a symmetric polynomial, we can use these results on Theorem 1.5, see Subsections 4.1 and 4.6. Also, in Appendix B we give some results of the paper in terms of formal power series.

From [AGZ09, Section 4.5], for all $\beta > 0$, consider the measure, known as the $\beta$-Hermite ensemble, on $\mathcal{M}_N$ with probability density

$$d_{N,\beta}(x_1, \ldots, x_N) = C_{N,\beta} \prod_{1 \leq i < j \leq N} |x_i - x_j|^\beta \prod_{i=1}^{N} e^{-\frac{\beta x_i^2}{4}},$$

where

$$C_{N,\beta} = (2\pi)^{-\frac{N}{2}} \left( \frac{\beta}{2} \right)^{\frac{\beta N(N-1)}{4} + \frac{N}{2}} \Gamma \left( \frac{\beta}{2} \right)^N \left( \prod_{i=1}^{N} \Gamma \left( \frac{i\beta}{2} \right) \right)^{-1}.$$

In the paper, we scale $d_{N,\beta}(x_1, \ldots, x_N)$ by $\sqrt{N}$ to get the probability density

$$d_{N,\beta}^*(x_1, \ldots, x_N) = \frac{C_{N,\beta}}{N^{\frac{\beta N(N-1)}{4} + \frac{N}{2}}} \prod_{1 \leq i < j \leq N} |x_i - x_j|^\beta \prod_{i=1}^{N} e^{-\frac{\beta x_i^2}{4N}},$$

on $\mathcal{M}_N$. For $\beta = 1, 2, 4$, $d_{N,\beta}$ is the probability density of the ordered eigenvalues of the GUE, GOE, and GSE, respectively. Moreover, from [AGZ09, Theorem 4.5.35], there exist random tridiagonal matrices with eigenvalue distribution being the $\beta$-Hermitian ensemble for all $\beta > 0$. In the paper, we see that if $\beta > 0$ is fixed, the main theorem on $\{\mu_N\}_{N \geq 1}$ where $\mu_N \in \mathcal{M}_N$ has density $d_{N,\beta}^*$ gives the moments as well as LLN satisfaction. This is an example of how the results of this paper can be used on probability measures.

The organization of this paper is as follows. In Section 2 we introduce the notation for formal series, Dunkl operators, and Bessel generating functions. Afterwards, in Section 3 we go over results on the limits of operators on formal series with $N$ variables which are symmetric in $N - 1$ variables as $N$ increases to $\infty$. Next, in Section 4 we prove Theorem 1.5, the main theorem, and in Section 5 we apply Theorem 1.5 to $d_{N,\beta}^*(x_1, \ldots, x_N)$. Following this, in Appendix A we state and prove several combinatorial formulas which are used throughout the paper. Finally, in Appendix B we state additional results for formal series.

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2 Operators and Functions

2.1 Formal Power Series

Later on in the paper, we look at formal power series in $x_1, \ldots, x_N$. However, for simplicity, we usually call a formal power series a formal series instead. For $\bar{x} = (x_1, \ldots, x_N)$ and a
partition $\nu$, let
\[
M_\nu(\vec{x}) = \sum_{a=(a_1,\ldots,a_N), \ a_i \in \mathbb{Z}_{\geq 0}, 1 \leq i \leq N, \pi(a)=\nu} \prod_{i=1}^{N} x_i^{a_i}.
\]
Above, $N = \infty$ is possible. Also, suppose that the set of all partitions is $P$.

For a symmetric formal series $F(x_1,\ldots,x_N)$, we let
\[
F(x_1,\ldots,x_N) = \sum_{\nu \in P, \ell(\nu) \leq N} c_F^\nu M_\nu(\vec{x}),
\]
where the $c_F^\nu \in \mathbb{C}$ are constants. Later on, in Section 4, when we view the $c_F^\nu$ as variables, $c_F^\nu$ has degree $|\nu|$ for $\nu \in P$.

For positive integers $N \geq i$, let $\mathcal{F}_i^N$ denote the set of formal series $F(x_1,\ldots,x_N)$ which are symmetrical in $x_j$ for $1 \leq j \leq N, j \neq i$. Suppose that $F(x_1,\ldots,x_N) \in \mathcal{F}_i^N$. Then, where $\vec{x}_i = (x_j)_{1 \leq j \leq N, j \neq i}$ is $\vec{x}$ with $x_i$ removed, we have
\[
F(x_1,\ldots,x_N) = \sum_{d=0}^{\infty} \left( \sum_{\nu \in P, \ell(\nu) \leq N-1} c_F^d,\nu M_\nu(\vec{x}_i) \right) x_i^d.
\]
where the $c_F^d,\nu \in \mathbb{C}$ are constants.

We can let $c_F^\nu = 0$ for $\nu$ such that $\ell(\nu) \geq N+1$, since $\vec{x}$ has $N$ variables, and $c_F^d,\nu = 0$ for $\nu$ such that $\ell(\nu) \geq N$, since $\vec{x}_i$ has $N-1$ variables. Particularly, for $F(x_1,\ldots,x_N) \in \mathcal{F}_i^N$,
\[
F(x_1,\ldots,x_N) = \sum_{d=0}^{\infty} \left( \sum_{\nu \in P, \ell(\nu) \leq N-1} c_F^d,\nu M_\nu(\vec{x}_i) \right) x_i^d.
\]
\[
(7)
\]

**Definition 2.1.** For a sequence of constants $s = \{c_{d,\nu}\}_{d \geq 0, \nu \in P}$ and a positive integer $i$, let the formal series of $s$ in $\mathcal{F}_i^N$ for all $N \geq i$ be
\[
F_i(s)(x_1,\ldots,x_N) = \sum_{d=0}^{\infty} \sum_{\nu \in P, \ell(\nu) \leq N-1} c_{d,\nu} M_\nu(\vec{x}_i) x_i^d.
\]
\[
(8)
\]
Also, where $\vec{x}_i = (x_j)_{j \geq 1, j \neq i}$, let
\[
F_i(s) = \sum_{d=0}^{\infty} \sum_{\nu \in P} c_{d,\nu} M_\nu(\vec{x}_i) x_i^d
\]
\[
(9)
\]
as a formal series in $(x_j)_{j \geq 1}$.

### 2.2 Dunkl Operator

For $N$ variables $x_1,\ldots,x_N$, let $s_{i,j}$ be the operator which switches $x_i$ and $x_j$. 

\[
F_i(s) = \sum_{d=0}^{\infty} \sum_{\nu \in P} c_{d,\nu} M_\nu(\vec{x}_i) x_i^d
\]
\[
(9)
\]
Definition 2.2. The Dunkl operators are, for $1 \leq i \leq N$,

$$D_i := \frac{\partial}{\partial x_i} + \theta \sum_{1 \leq j \leq N, j \neq i} \frac{1}{x_i - x_j} (1 - s_{i,j}).$$ \hspace{1cm} (10)$$

For positive integers $k$, let

$$P_k = \sum_{i=1}^{N} D_i^k.$$ \hspace{1cm} (11)

Also, let $\frac{\partial}{\partial x_i}$ be denoted by $\partial_i$ for $1 \leq i \leq N$. Moreover, for $i \neq j$, let

$$\theta \sum_{1 \leq j \leq N, j \neq i} \frac{1}{x_i - x_j} (1 - s_{i,j})$$

be called the switch from $i$ to $j$. We know that the Dunkl operators are commutative, which is in Proposition 2.3 below. Due to this, the $P_i$ are commutative as well.

Proposition 2.3 ([Dun89, Theorem 1.9]). For positive integers $i,j$, $1 \leq i,j \leq N$, $D_i D_j = D_j D_i$.

2.3 Bessel Generating Functions

A Gelfand-Tsetlin pattern with $N$ rows is a set of real numbers $\{\lambda^j_i\}_{1 \leq i \leq j \leq N}$ such that

$$\lambda^j_{i+1} \leq \lambda^j_i \leq \lambda^j_{i+1}$$

for $1 \leq i \leq j \leq N - 1$. We let $G_N$ denote the set of Gelfand-Tsetlin patterns with $N$ rows.

Definition 2.4. The multivariate Bessel function for $a_1 < \cdots < a_N$ is

$$B(a_1, \ldots, a_N)(x_1, \ldots, x_N; \theta) = \int_{d=\{y^k_i\} \in G_N, \ y_1^N = a_1, \ 1 \leq i \leq N} \left[ \exp \left( \sum_{k=1}^{N} x_k \cdot \left( \sum_{i=1}^{k} y_i^k - \sum_{j=1}^{k-1} y_j^{k-1} \right) \right) \right] \cdot f(a_1, \ldots, a_N)(d) \prod_{1 \leq i \leq k \leq N-1} dy_i^k,$$

where for $d = \{y^k_i\} \in G_N$ such that $y_i^N = a_i$, $1 \leq i \leq N$,

$$f(a_1, \ldots, a_N)(d) = \prod_{k=1}^{N} \frac{\Gamma(k\theta)}{\Gamma(\theta)^k} \prod_{1 \leq i < j \leq N} (a_j - a_i)^{1-2\theta} \cdot \prod_{k=1}^{N-1} \left( \prod_{1 \leq i < j \leq k} (y_j^k - y_i^k)^{2-2\theta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |y_a^k - y_b^{k+1}|^{\theta-1} \right).$$

Definition 2.5. The Bessel generating function of $\mu \in \mathcal{M}_N$ is defined as

$$G_\theta(x_1, \ldots, x_N; \mu) := \int_{a_1 \leq \cdots \leq a_N} B(a_1, \ldots, a_N)(x_1, \ldots, x_N; \theta) \mu(da_1, \ldots, da_N).$$ \hspace{1cm} (12)
Observe that \( f(a_1, \ldots, a_N)(d) \) is the probability density of the \( \theta \)-corners process or \( \beta \)-corners process with top row \( a_1 < \cdots < a_N \) in [GM18], [BGCG21]. Note that this is also referred to as the orbital \( \beta \) process in [Cue19]. A scaled multivariate Bessel function and the Bessel generating function for \( \theta = 1 \) are studied in [Ahn20].

**Theorem 2.6** ([Cue19]). For any symmetric polynomial \( P(x_1, \ldots, x_N) \) and each \( N \)-tuple of reals \( a_1 < a_2 < \cdots < a_N \),

\[
P(D_1, \ldots, D_N)B(a_1, \ldots, a_N) = P(a_1, \ldots, a_N)B(a_1, \ldots, a_N).
\]

(13)

From Lemma [1.3] if \( \mu \in \mathcal{M}_N \) is exponentially decaying, then the BGF of \( \mu \) converges in an open neighborhood of the origin in \( \mathbb{C}^N \). Additionally, from (13),

\[
P_kB(a_1, \ldots, a_N) = \left( \sum_{i=1}^{N} a_i^k \right) B(a_1, \ldots, a_N),
\]

which can be used to prove the following Proposition.

**Proposition 2.7** ([BGCG21, Proposition 2.11]). For a positive integer \( s \), let \( k_1, \ldots, k_s \) be positive integers. Suppose \( \mu \in \mathcal{M}_N \) is exponentially decaying. Then,

\[
\left( \prod_{i=1}^{s} P_{k_i} \right) G_\theta(x_1, \ldots, x_N; \mu) \bigg|_{x_i=0, 1 \leq i \leq N} = \mathbb{E}_\mu \left( \prod_{i=1}^{s} \left( \sum_{j=1}^{N} a_j^{k_i} \right) \right).
\]

(14)

## 3 Limits of Operators On Formal Power Series

In this section, we look at operators on formal power series in \( x_j \), \( 1 \leq j \leq N \) such that there exists \( i, 1 \leq i \leq N, \) for which the formal power series is symmetrical for \( j \neq i \). The main results are Theorem 3.9 and Theorem 3.13 with Theorem 3.9 being a special case of Theorem 3.13.

### 3.1 Basic Results

For a partition \( \nu = (a_1 \geq a_2 \geq \cdots \geq a_m) \), let

\[
S(\nu) = \{(p_1, p_2)|p_i = (b_{i,1}, \ldots, b_{i,m}), 1 \leq i \leq 2, b_{i,j} + b_{2,j} = a_j, 1 \leq j \leq m\}.
\]

If we have partitions \( \nu_1, \nu_2, \ldots, \nu_k \), we let \( \nu = \nu_1 + \cdots + \nu_k \) be the partition formed when \( \nu_1, \ldots, \nu_k \) are joined to each other.

For \( 1 \leq i, j \leq N, i \neq j \), let \( C_{i,j} \) be the operator such that for nonnegative integers \( a_k \), \( 1 \leq k \leq N \),

\[
C_{i,j} \left( \prod_{k=1}^{N} x_k^{a_k} \right) = x_i^{a_j-1} \prod_{1 \leq k \leq N, \ k \neq i, j} x_k^{a_k}.
\]
if \( a_i = 0 \) and \( a_j \geq 1 \). Otherwise, if \( a_i \geq 1 \) or \( a_i = a_j = 0 \),

\[
C_{i,j} \left( \prod_{k=1}^{N} a_k^{q_k} \right) = 0.
\]

With this, for positive integers \( N \geq i \) and a formal series \( f(x_1, \ldots, x_N) \in \mathcal{F}_i^N \), let the operator \( \mathcal{Q}_i^N(f(x_1, \ldots, x_N)) \) be

\[
\mathcal{Q}_i^N(f(x_1, \ldots, x_N)) = \theta \sum_{1 \leq j \leq N, j \neq i} \frac{d_i - C_{i,j}}{N} + f(x_1, \ldots, x_N). \tag{15}
\]

In Proposition 3.4 we show that \( \mathcal{Q}_i^N(f(x_1, \ldots, x_N)) \) is an operator from \( \mathcal{F}_i^N \) to \( \mathcal{F}_i^N \). Sometimes, for \( f(x_1, \ldots, x_N) \in \mathcal{F}_i^N \), we denote \( \mathcal{Q}_i^N(f(x_1, \ldots, x_N)) \) by \( \mathcal{Q}_i^N(f) \), and for \( s = \{c^d \}_{d \geq 0, \nu \in P} \), we denote \( \mathcal{Q}_i^N(F_i(s)(x_1, \ldots, x_N)) \) by \( \mathcal{Q}_i^N(F_i(s)) \). In this paper, for operators \( \mathcal{T}_i, 1 \leq i \leq m \), the product

\[
\prod_{i=1}^{m} \mathcal{T}_i
\]

denotes the \( \mathcal{T}_i \) being applied from \( i = 1 \) to \( m \).

**Definition 3.1.** For a positive integer \( i \), a sequence of formal series \( \{f_N(x_1, \ldots, x_N)\}_{N \geq i} \) is symmetric outside of \( i \) if \( f_N(x_1, \ldots, x_N) \in \mathcal{F}_i^N \) for all \( N \geq i \).

**Definition 3.2.** Suppose that \( i \) is a positive integer and a sequence of formal series \( f = \{f_N(x_1, \ldots, x_N)\}_{N \geq i} \) is symmetrical outside of \( i \). Then, the limit of \( f \) outside of \( i \) exists if \( \lim_{N \to \infty} c^{d, \nu}_f \) exists for all \( d, \nu \). If the limit outside of \( f \) exists, the limiting sequence of \( f \) outside of \( i \) is \( s = \{\lim_{N \to \infty} c^{d, \nu}_f\}_{d \geq 0, \nu \in P} \), and the limit of \( f \) as \( N \to \infty \) outside of \( i \) is

\[
\lim_{N \to \infty} f_N(x_1, \ldots, x_N) = F_i(s) = \sum_{d=0}^{\infty} \left( \sum_{\nu \in P} \left( \lim_{N \to \infty} c^{d, \nu}_f \right) \cdot M_{\nu}(\bar{x}_i) \right) x_i^d.
\]

**Remark 3.3.** For a sequence \( \{f_N(x_1, \ldots, x_N)\}_{N \geq i} \) such that \( f_N(x_1, \ldots, x_N) \) is in \( \mathcal{F}_i^N \) for all \( N \geq i \), the limiting sequence of \( f_N(x_1, \ldots, x_N) \) outside of \( i \) being \( s \) is equivalent to the limit of \( f_N(x_1, \ldots, x_N) \) as \( N \to \infty \) outside of \( i \) being \( F(s) \).

**Proposition 3.4.** For \( f(x_1, \ldots, x_N) \) and \( g(x_1, \ldots, x_N) \) in \( \mathcal{F}_i^N \), \( \mathcal{Q}_i^N(f)g(x_1, \ldots, x_N) \) is in \( \mathcal{F}_i^N \), and is

\[
\sum_{d=0}^{\infty} \sum_{\nu \in P, \ell(\nu) \leq N-1} \left( \theta c^{d+1, \nu}_g - \theta c^{0, \nu+(d+1)}_g + \sum_{a+b=d, \ell(p_1) \in S(\nu)} c^{a, p_1}_f c^{b, p_2}_g \right) M_{\nu}(\bar{x}_i) x_i^d
\]

\[
+ \frac{\theta}{N} \sum_{d=0}^{\infty} \sum_{\nu \in P, \ell(\nu) \leq N-1} (-c^{d+1, \nu}_g + (\ell(\nu) + 1)c^{0, \nu+(d+1)}_g) M_{\nu}(\bar{x}_i) x_i^d.
\]
Proof. We see that
\[ g_1(x_1, \ldots, x_N) = d_i g(x_1, \ldots, x_N) = \sum_{d=1}^{\infty} \sum_{\nu \in P} c_g^{d,\nu} M_{\nu}(\vec{x}_i) x_i^{d-1} \]
\[ = \sum_{d=0}^{\infty} \sum_{\nu \in P} c^{d+1,\nu} M_{\nu}(\vec{x}_i) x_i^d. \]

Also, let
\[ g_2(x_1, \ldots, x_N) = \sum_{1 \leq j \leq N, j \neq i} C_{i,j} g(x_1, \ldots, x_N) = \left( \sum_{1 \leq j \leq N, j \neq i} C_{i,j} \right) \left( \sum_{\nu \in P} c_g^{0,\nu} M_{\nu}(\vec{x}_i) \right). \]

Observe that \( g_2(x_1, \ldots, x_N) \) is a formal power series which is symmetric in \( \vec{x}_i \). We find the coefficient of \( M_{\nu}(\vec{x}_i) x_i^d \). If \( \ell(\nu) \geq N \), the coefficient is 0 in \( g_2(x_1, \ldots, x_N) \). Then, suppose \( \ell(\nu) \leq N - 1 \). For a monomial \( p \) in \( M_{\nu}(\vec{x}_i) \) with variables \( (\vec{x}_i)_j \) with \( 1 \leq j \leq \ell(\nu) \), there are \( N - 1 - \ell(\nu) \) \( j \) such that \( x_j \) is not in \( p \), and these are the \( j \) such that \( C_{i,j} \) applied to a monomial will give \( px_i^d \). For a monomial \( q \), if
\[ C_{i,j} q = px_i^d, \]
then \( q = px_i^{d+1} \) and has coefficient \( c_g^{0,\nu+(d+1)} \) in \( g \). Therefore, the coefficient of \( px_i^d \), and thus \( M_{\nu}(\vec{x}_i) x_i^d \), in \( g_2(x_1, \ldots, x_N) \) is \( (N - \ell(\nu) - 1)c_g^{0,\nu+(d+1)} \). From this,
\[ g_2(x_1, \ldots, x_N) = \sum_{d=0}^{\infty} \sum_{\nu \in P, \ell(\nu) \leq N-1} (N - \ell(\nu) - 1)c_g^{0,\nu+(d+1)} M_{\nu}(\vec{x}_i) x_i^d. \]

Next, let
\[ g_3(x_1, \ldots, x_N) = f(x_1, \ldots, x_N) g(x_1, \ldots, x_N) \]
\[ = \sum_{d_1, d_2=0}^{\infty} \sum_{\nu_1, \nu_2 \in P} c_f^{d_1,\nu_1} c_g^{d_2,\nu_2} M_{\nu_1}(\vec{x}_i) M_{\nu_2}(\vec{x}_i) x_i^{d_1+d_2}. \]

For \( \nu \in P \) where \( \nu \) can be 0, we find the coefficient of \( M_{\nu}(\vec{x}_i) x_i^d \) in \( g_3(x_1, \ldots, x_N) \). If \( \ell(\nu) \geq N \), this coefficient will be 0. Suppose \( \ell(\nu) \leq N - 1 \), and let
\[ q = \prod_{j=1}^{\ell(\nu)} x_{(\vec{x}_i)_j}^{\nu_j}. \]

Note that if \( q_1 \) and \( q_2 \) are monic monomials such that \( q_1 q_2 = q \), then for \( 1 \leq s \leq 2 \), if \( b_{s,j} \) is the degree of \( x_{(\vec{x}_i)_j} \) in \( q_s \) for \( 1 \leq j \leq \ell(\nu) \) and \( p(q_s) = (b_{s,1}, \ldots, b_{s,\ell(\nu)}) \), \( (p(q_1), p(q_2)) \in S(\nu) \). We see that if \( S \) is the set of \( (q_1, q_2) \) such that \( q_1 \) and \( q_2 \) are monic monomials with \( q_1 q_2 = q \),
\[ p : S \to S(\nu), (q_1, q_2) \mapsto (p(q_1), p(q_2)) \]
is injective as well as surjective, and therefore is a bijection. Also, the coefficient of \( q_1 x_1^a \) and \( q_2 x_1^b \) is \( c_f^{a,\pi(p(q_1))} \) in \( f \) and \( c_g^{b,\pi(p(q_2))} \) in \( g \), respectively. Then, the coefficient of \( q x_i^d \) is
\[ \sum_{a+b=d, q_1 q_2=q} c_f^{a,\pi(p(q_1))} c_g^{b,\pi(p(q_2))} = \sum_{a+b=d, (p_1, p_2) \in S(\nu)} c_f^{a,\pi(p_1)} c_g^{b,\pi(p_2)}, \]

9
which is also the coefficient of $M_{\nu}(\vec{x}_i)x_i^d$ in $g_3(x_1, \ldots, x_N)$. From this,

\[ g_3(x_1, \ldots, x_N) = \sum_{d=0}^{\infty} \sum_{\nu \in P, \ell(\nu) \leq N-1} \left( \sum_{a+b=d, (p_1, p_2) \in S(\nu)} c_f^{a, \pi(p_1)} c_g^{b, \pi(p_2)} \right) M_{\nu}(\vec{x}_i)x_i^d. \]

Since $g_1(x_1, \ldots, x_N)$, $g_2(x_1, \ldots, x_N)$, and $g_3(x_1, \ldots, x_N)$ are in $\mathcal{F}_i^N$, $Q_i(f)g(x_1, \ldots, x_N)$ is in $\mathcal{F}_i^N$, with

\[ Q_i(f)g(x_1, \ldots, x_N) = \frac{(N-1)\theta g_1(x_1, \ldots, x_N)}{N} - \frac{\theta g_2(x_1, \ldots, x_N)}{N} + g_3(x_1, \ldots, x_N). \]

With the expressions we obtained, this completes the proof.

**Corollary 3.5.** Let $m \geq 0$ be an integer. Suppose $\{f_{j,N}(x_1, \ldots, x_N)\}_{N \geq i}$ for $1 \leq j \leq m$ and $\{g_N(x_1, \ldots, x_N)\}_{N \geq i}$ are symmetrical outside of $i$. Also, assume that the limit of $f_{j,N}(x_1, \ldots, x_N)$ for $1 \leq j \leq m$ and $g_N(x_1, \ldots, x_N)$ as $N \to \infty$ exist outside of $i$. Then,

\[ \lim_{N \to \infty} \left( \prod_{j=1}^{m} Q_i^N(f_{j,N}) \right) g_N(x_1, \ldots, x_N) \]

exists outside of $i$. Also, the limit of $Q_i^N(f_{1,N})g_N(x_1, \ldots, x_N)$ as $N \to \infty$ outside of $i$ is equal to

\[ \sum_{d=0}^{\infty} \sum_{\nu \in P} \lim_{N \to \infty} \left( \theta c_{g_N}^{d+1, \nu} - \theta c_{g_N}^0,\nu^{(d+1)} + \sum_{a+b=d, (p_1, p_2) \in S(\nu)} c_{f_{1,N}}^{a, \pi(p_1)} c_{g_N}^{b, \pi(p_2)} \right) M_{\nu}(\vec{x}_i)x_i^d. \quad (16) \]

**Proof.** The $m = 0$ case is clear, and for $m = 1$, the limit of $Q_i^N(f_{1,N})g_N(x_1, \ldots, x_N)$ as $N \to \infty$ outside of $i$ existing and being equal to $[16]$ follows from taking the limit as $N \to \infty$ of Proposition 3.4. Then, the rest of Corollary 3.5 follows from $m = 1$ and induction.

**Proposition 3.6.** Let $i \geq 1$ and $m \geq 0$ be integers. Suppose $\{f_{j,N}(x_1, \ldots, x_N)\}_{N \geq 1}$ for $1 \leq j \leq m$ and $\{g_N(x_1, \ldots, x_N)\}_{N \geq 1}$ are symmetrical outside of $i$. Assume that for $1 \leq j \leq m$, $f_{j,N}(x_1, \ldots, x_N)$ has a limit as $N \to \infty$ with limiting sequence $f_j$ outside of $i$. Also, assume that $g_N(x_1, \ldots, x_N)$ has a limit as $N \to \infty$ with limiting sequence $g$ outside of $i$. Then, outside of $i$,

\[ \lim_{N \to \infty} \left( \prod_{j=1}^{m} Q_i^N(f_{j,N}) \right) g_N(x_1, \ldots, x_N) = \lim_{N \to \infty} \left( \prod_{j=1}^{m} Q_i^N(F_i(f_j)) \right) F_i(g)(x_1, \ldots, x_N). \quad (17) \]
Proof. From Corollary [3.3] the limits in both sides exist. Suppose that for $1 \leq j \leq m$, $f_j = \{c_{f_j}^d\}_{d \geq 0, \nu \in P}$, and $g = \{c_g^d\}_{d \geq 0, \nu \in P}$. We use induction on $m$, where the base case $m = 0$ is clear. For $m = 1$, $\lim_{N \to \infty} Q_i^N (f_{1,N}) g_N (x_1, \ldots , x_N)$ is

$$
\sum_{d=0}^{\infty} \sum_{\nu \in P} \left( \theta_d c_{f_1}^{d+1, \nu} - \theta_d c_{g}^{0, \nu+(d+1)} + \sum_{a+b=d, (p_1, p_2) \in S(\nu)} c_{f_1}^{a, \pi(p_1)} c_{g}^{b, \pi(p_2)} \right) M_\nu (\bar{a}_i x_i^d)
$$

which is also $\lim_{N \to \infty} Q_i^N (F(f_1)) F(g)(x_1, \ldots , x_N)$, where [16] is used.

Assume the statement holds for $m$, where $m \geq 1$. We want to show the statement holds for $m + 1$. Let

$$r_N (x_1, \ldots , x_N) = Q_i^N (f_{1,N}) g_N (x_1, \ldots , x_N),$$

and $r$ be the limiting sequence of $Q_i^N (F(f_1)) F(g)(x_1, \ldots , x_N)$ as $N \to \infty$ outside of $i$. From $m = 1$, $\lim_{N \to \infty} r_N (x_1, \ldots , x_N) = \lim_{N \to \infty} Q_i^N (F(f_1)) F(g)(x_1, \ldots , x_N) = F_i(r)$, or $r$ is the limiting sequence of $r_N (x_1, \ldots , x_N)$ as $N \to \infty$ outside of $i$. Then, by the inductive hypothesis,

$$\lim_{N \to \infty} \left( \prod_{j=1}^{m+1} Q_i^N (f_{j,N}) \right) g_N (x_1, \ldots , x_N) = \lim_{N \to \infty} \left( \prod_{j=2}^{m+1} Q_i^N (f_{j,N}) \right) r_N (x_1, \ldots , x_N)
$$

$$= \lim_{N \to \infty} \left( \prod_{j=2}^{m+1} Q_i^N (F_i(f_j)) \right) F_i(r)(x_1, \ldots , x_N)
$$

$$= \lim_{N \to \infty} \left( \prod_{j=2}^{m+1} Q_i^N (F_i(f_j)) \right) Q_i^N (F_i(f_1)) F_i(g)(x_1, \ldots , x_N)
$$

$$= \lim_{N \to \infty} \left( \prod_{j=1}^{m+1} Q_i^N (F_i(f_j)) \right) F_i(g)(x_1, \ldots , x_N).
$$

This completes the proof. \hfill \Box

Proposition 3.7. Suppose that $i, j$ are integers such that $1 \leq i, j \leq N$ and $i \neq j$. Then, for integers $k$ with $1 \leq k \leq N$ and $k \neq i, j$, if $f \in F_k^N$,

$$s_{i,j} Q_k^N (f) = Q_k^N (f) s_{i,j}$$

as operators from $F_k^N$ to $F_k^N$. Also, if $f \in F_i^N$,

$$s_{i,j} Q_i^N (f) = Q_j^N (s_{i,j} f) s_{i,j}$$

as operators from $F_i^N$ to $F_j^N$.

Proof. This is clear from expanding the operators. \hfill \Box
3.2 Constant Term

An important part of Section 3 is to provide results for evaluating (14), which is at \( x_i = 0, 1 \leq i \leq N \). Due to this, the most important part of the formal series following the \( Q_i^N \) operators is the constant term. For a sequence of coefficients \( s = \{c_{d,\nu}\}_{d \geq 0, \nu \in P} \), this corresponds to \( c^{0.0} \). In this subsection, we look at Theorem 3.9 to compute the constant term following \( Q_i^N \) operators where there is only one \( i \). Later on, in Theorem 3.13, we look at the constant term following \( Q_i^N \) operators where there are distinct indices \( i \). These values are computed with free cumulants, which are in Definition 3.8. However, we must go over noncrossing partitions first.

Suppose that we have a partition \( \pi \) of a finite, nonempty set \( S \) of real numbers. We let

\[
\pi = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_m,
\]

where \( B_i, 1 \leq i \leq m \), are the blocks of \( \pi \), such that the smallest element of \( B_{i+1} \) is greater than the smallest element of \( B_i \) for \( 1 \leq i \leq m - 1 \). Also, the length of \( \pi \) is \( \ell(\pi) = m \).

A partition \( \pi \) is noncrossing if for any distinct blocks \( B_1 \) and \( B_2 \) of \( \pi \), there do not exist \( a, b \in B_1 \) and \( c, d \in B_2 \) such that \( a < c < b < d \). Let the set of noncrossing partitions of a finite, nonempty set \( S \) of real numbers be \( NC(S) \), and for \( k \geq 1 \), let \( NC(k) = NC(\{1, \ldots, k\}) \). A way to represent a partition \( \pi \) is the circular representation, where the elements are spaced around a circle in order and the convex hulls of the elements of each block of \( \pi \) are added. If \( \pi \) is noncrossing, the convex hulls will be disjoint.

![Circular representations of a noncrossing and crossing partition of \{1, 2, \ldots, 9\}.](image)

**Figure 1:** Circular representations of a noncrossing and crossing partition of \{1, 2, \ldots, 9\}.

**Definition 3.8.** For a sequence of coefficients \( s = \{c_{d,\nu}\}_{d \geq 0, \nu \in P} \) and a positive integer \( k \), let the free cumulant of order \( k \) for \( s \) be

\[
c_k(s) = \theta^{k-1} \sum_{\nu \in P, d \geq 0, |\nu| + d = k-1} (-1)^{\ell(\nu)} P(\nu) c_{d,\nu}.
\]

A way the free cumulants appear is in (20). Note that these free cumulants will be used in Section 4 to obtain the free cumulants mentioned previously for the moments \( m_k \) in Theorem 1.5.
Theorem 3.9. Let $i$ and $k$ be positive integers. Suppose $\{f_{j,N}(x_1,\ldots,x_N)\}_{N\geq i}$ for $1 \leq j \leq k - 1$ and $\{g_N(x_1,\ldots,x_N)\}_{N\geq i}$ are sequences of formal series which are symmetric outside of $i$. For $1 \leq j \leq k$, assume that $f_{j,N}(x_1,\ldots,x_N)$ has a limit as $N \to \infty$ with limiting sequence $f$ outside of $i$. Also, suppose $g_N(x_1,\ldots,x_N)$ has a limit as $N \to \infty$ with limiting sequence $g$ outside of $i$. Then,

$$
\lim_{N \to \infty} \left( \prod_{j=1}^{k-1} Q_i^N(f_{j,N})g_N(x_1,\ldots,x_N) \right) = \sum_{\pi \in NC(k), \pi = B_1 \cup \cdots \cup B_m} c_{|B_1|}(g) \prod_{i=2}^m c_{|B_i|}(f).
$$

(20)

Proof. From Corollary 3.5, the limit of $\prod_{j=1}^{k-1} Q_i^N(f_{j,N})g_N(x_1,\ldots,x_N)$ as $N \to \infty$ outside of $i$ exists, and the left hand side of (20) is

$$
[1] \lim_{N \to \infty} \prod_{j=1}^{k-1} Q_i^N(f_{j,N})g_N(x_1,\ldots,x_N),
$$

We can prove the Theorem with induction. For the base case $k = 1$,

$$
\lim_{N \to \infty} [1]g_N(x_1,\ldots,x_N) = c_g^{0,0} = c_1(g).
$$

Next, assume that the statement holds for a positive integer $k$. We want to show the statement holds for $k + 1$. From Proposition 3.6 outside of $i$,

$$
\lim_{N \to \infty} \prod_{j=1}^k Q_i^N(f_{j,N})g_N(x_1,\ldots,x_N) = \lim_{N \to \infty} Q_i^N(F_i(f))^k F_i(g)(x_1,\ldots,x_N).
$$

Using Corollary 3.5 let $g'$ be the limiting sequence of $Q_i^N(F_i(f))^k F_i(g)(x_1,\ldots,x_N)$ outside of $i$, with

$$
c_{g'}^{d,\nu} = \theta c_g^{d+1,\nu} - \theta c_g^{0,\nu+(d+1)} + \sum_{a+b=d, (p_1,p_2) \in S(\nu)} c_f^{a,\pi(p_1)} c_g^{b,\pi(p_2)}
$$

for $d \geq 0$ and $\nu \in P$. By Proposition 3.6

$$
\lim_{N \to \infty} Q_i^N(F_i(f))^k F_i(g)(x_1,\ldots,x_N) = \lim_{N \to \infty} Q_i^N(F_i(f))^{k-1} F_i(g')(x_1,\ldots,x_N).
$$

However, by the inductive hypothesis,

$$
\lim_{N \to \infty} Q_i^N(F_i(f))^{k-1} F_i(g')(x_1,\ldots,x_N) = \sum_{\pi \in NC(k), \pi = B_1 \cup \cdots \cup B_m} c_{|B_1|}(g') \prod_{i=2}^m c_{|B_i|}(f)
$$

$$
\sum_{\pi \in NC(k), \pi = B_1 \cup \cdots \cup B_m} \theta^{|B_1|-1} \left( \sum_{\nu \in P, d \geq 0 \atop |\nu| + d = |B_1|-1} (-1)^{f(\nu)} P(\nu) c_{g'}^{d,\nu} \right) \prod_{i=2}^m c_{|B_i|}(f).
$$
Also, from Lemma A.2

\[ \sum_{\pi \in NC(k+1), \pi = B_1 \sqcup \cdots \sqcup B_m} c_{B_1}(g) \prod_{i=2}^{m} c_{B_i}(f) \]

\[ = \sum_{\pi \in NC(k), \pi = B_1 \sqcup \cdots \sqcup B_m} \left( c_{B_1}(g) + \sum_{j=1}^{\left| B_1 \right|} c_j(f) c_{B_1+1-j}(g) \right) \prod_{i=2}^{m} c_{B_i}(f). \]

Then, for \( k+1 \), it suffices to show that

\[ \theta^{\left| B_1 \right|-1} \sum_{\nu \in P, d \geq 0, |\nu| + d = |B_1| - 1} (-1)^{\ell(\nu)} c_g^{d,\nu} P(\nu) = c_{B_1+1}(g) + \sum_{j=1}^{\left| B_1 \right|} c_j(f) c_{B_1+1-j}(g). \]

The left hand side is

\[ \theta^{\left| B_1 \right|} \sum_{\nu \in P, d \geq 0, |\nu| + d = |B_1| - 1} (-1)^{\ell(\nu)} P(\nu) c_g^{d+1,\nu} - \theta^{\left| B_1 \right|} \sum_{\nu \in P, d \geq 0, |\nu| + d = |B_1| - 1} (-1)^{\ell(\nu)} P(\nu) c_g^{0,\nu+(d+1)} \]

\[ + \theta^{\left| B_1 \right|-1} \sum_{\nu \in P, d \geq 0, |\nu| + d = |B_1| - 1} \sum_{a+b=d, (p_1, p_2) \in S(\nu)} (-1)^{\ell(\nu)} P(\nu) c_f^{a,\nu_1} c_g^{b,\nu_2}. \]

However, the right hand side is

\[ \theta^{\left| B_1 \right|} \sum_{\nu \in P, d \geq 0, |\nu| + d = |B_1|} (-1)^{\ell(\nu)} P(\nu) c_g^{d,\nu} \]

\[ + \theta^{\left| B_1 \right|-1} \sum_{j=1}^{\left| B_1 \right|} \left( \sum_{\nu_1 \in P, d_1 \geq 0, |\nu_1| + d_1 = j-1} (-1)^{\ell(\nu_1)} P(\nu_1) c_f^{d_1,\nu_1} \right) \left( \sum_{\nu_2 \in P, d_2 \geq 0, |\nu_2| + d_2 = |B_1| - j} (-1)^{\ell(\nu_2)} P(\nu_2) c_g^{d_2,\nu_2} \right). \]

Observe that

\[ \sum_{\nu \in P, d \geq 0, |\nu| + d = |B_1|} (-1)^{\ell(\nu)} P(\nu) c_g^{d,\nu} = \sum_{\nu \in P, d \geq 0, |\nu| + d = |B_1| - 1} (-1)^{\ell(\nu)} P(\nu) c_g^{d+1,\nu} + \sum_{\nu \in P, |\nu| = |B_1|} (-1)^{\ell(\nu)} P(\nu) c_g^{0,\nu}. \]

Here, we want to show that

\[ \sum_{\nu \in P, |\nu| = |B_1|} (-1)^{\ell(\nu)} P(\nu) c_g^{0,\nu} = \sum_{\nu \in P, d \geq 0, |\nu| + d = |B_1| - 1} (-1)^{\ell(\nu)+1} P(\nu) c_g^{0,\nu+(d+1)}. \]

We look at the coefficient of \( c_g^{0,\nu} \) with \( |\nu| = |B_1| \geq 1 \). If \( \nu = (a_1 \geq \cdots \geq a_m) \) and \( R(\nu) \) is the set \( \{ a_i | 1 \leq i \leq m \} \), the coefficient on the right hand side is

\[ (-1)^{\ell(\nu)} \sum_{i \in R(\nu)} P(\nu_i), \]
where \( \nu_i \) is \( \nu \) with \( i \) removed. However, in a permutation of \( \nu \), the first integer must be an element \( i \) of \( R(\nu) \). Then, there are \( P(\nu_i) \) permutations of the remaining components of \( \nu \), and therefore, for \( \nu \) with \(|\nu| \geq 1\),

\[
P(\nu) = \sum_{i \in R(\nu)} P(\nu_i).
\]

(21)

With this, the coefficient on the left and right hand side are equal.

In order to complete the proof, it suffices to show that

\[
\sum_{\nu \in P, d \geq 0, |\nu| + d = |B_1| - 1} \sum_{(p_1, p_2) \in S(\nu)} (-1)^{\ell(\nu)} P(\nu) c_f^{a, \pi(p_1)} c_g^{b, \pi(p_2)}
\]

\[
= \sum_{j=1}^{|B_1|} \left( \sum_{\nu_1 \in P, d_1 \geq 0, |\nu_1| + d_1 = j - 1} (-1)^{\ell(\nu_1)} P(\nu_1) c_f^{d_1, \nu_1} \right) \left( \sum_{\nu_2 \in P, d_2 \geq 0, |\nu_2| + d_2 = |B_1| - j} (-1)^{\ell(\nu_2)} P(\nu_2) c_g^{d_2, \nu_2} \right).
\]

(22)

We find the coefficient of \( c_f^{d_1, \nu_1} c_g^{d_2, \nu_2} \) on both sides, where \( d_1 + d_2 + |\nu_1| + |\nu_2| = |B_1| - 1 \). On the right hand side, the coefficient is \( (-1)^{\ell(\nu_1) + \ell(\nu_2)} P(\nu_1) P(\nu_2) \). For \( \nu \) with \(|\nu| = |B_1| - d - 1 \), let \( T(\nu) = \{(p_1, p_2) \in S(\nu)|\pi(p_1) = \nu_1, \pi(p_2) = \nu_2\} \). Then, on the left hand side, the coefficient is, where \( d = d_1 + d_2 \),

\[
\sum_{\nu \in P, |\nu| = |B_1| - d - 1} \sum_{\nu_1 \in P, \pi(p_1) = \nu_1, \pi(p_2) = \nu_2} (-1)^{\ell(\nu)} P(\nu) = \sum_{\nu \in P, |\nu| = |B_1| - d - 1} (-1)^{\ell(\nu)|T(\nu)|} \cdot P(\nu).
\]

Suppose \(|T(\nu)| \neq 0\). Then, if \( \ell(\nu) = \ell(\nu_1) + \ell(\nu_2) - k, 0 \leq k \leq \min(\ell(\nu_1), \ell(\nu_2)) \).

**Claim 3.9.1.** For \( 0 \leq k \leq \min(\ell(\nu_1), \ell(\nu_2)) \),

\[
\sum_{\nu \in P, |\nu| = |B_1| - d - 1, \ell(\nu) = \ell(\nu_1) + \ell(\nu_2) - k} |T(\nu)| \cdot P(\nu) = \frac{\binom{\ell(\nu_1)}{k} \binom{\ell(\nu_2)}{k}}{\binom{\ell(\nu_1) + \ell(\nu_2)}{k}} \cdot \frac{(\ell(\nu_1) + \ell(\nu_2))!}{\ell(\nu_1)! \ell(\nu_2)!} P(\nu_1) P(\nu_2).
\]

Proof. Let \( l = \ell(\nu_1) + \ell(\nu_2) - k \) and \( \bar{x} = (x_1, \ldots, x_l) \). Suppose partition \( \nu \) has \(|\nu| = |B_1| - d - 1 \) and \( \ell(\nu) = l \). We see that the coefficient of any term of \( M_\nu(\bar{x}) \) in \( M_{\nu_1}(\bar{x}) M_{\nu_2}(\bar{x}) \) is \(|T(\nu)|\). Therefore, the sum of the coefficients in \( M_{\nu_1}(\bar{x}) M_{\nu_2}(\bar{x}) \) of terms of \( M_\nu(\bar{x}) \) is \(|T(\nu)| \cdot P(\nu)\), where there are \( P(\nu) \) terms of \( M_\nu(\bar{x}) \).

Suppose that \( S \) is the sum of the terms of \( M_{\nu_1}(\bar{x}) M_{\nu_2}(\bar{x}) \) which contain all of the \( x_i \), \( i = 1, \ldots, l \). If a term \( p \) of \( S \) contains all of the \( x_i \), \( p \) without its coefficient must be a term of \( M_\nu(\bar{x}) \) for some \( \nu \in P \) with \(|\nu| = |B_1| - d - 1 \) and \( \ell(\nu) = l \). But, the sum of the coefficients in \( M_{\nu_1}(\bar{x}) M_{\nu_2}(\bar{x}) \) of terms of \( M_\nu(\bar{x}) \) is \(|T(\nu)| \cdot P(\nu)\). Therefore, the sum of the coefficients of \( S \) is

\[
C = \sum_{\nu \in P, |\nu| = |B_1| - d - 1, \ell(\nu) = \ell(\nu_1) + \ell(\nu_2) - k} |T(\nu)| \cdot P(\nu).
\]
However, for terms $p_i$ in $M_{\nu}(\bar{x})$, $1 \leq i \leq 2$, if $p_1p_2$ contains all of the $x_i$, then exactly $k$ of the $x_i$ must be in both $p_1$ and $p_2$. Note that $S$ is the sum of $p_1p_2$ for such $(p_1, p_2)$, and $C$ is the number of $(p_1, p_2)$; we count $C$. There are
\[
\frac{(\ell(\nu_1) + \ell(\nu_2) - k)!}{k!(\ell(\nu_1) - k)!(\ell(\nu_2) - k)!} = \frac{(\ell(\nu_1) k)}{(\ell(\nu_1) + \ell(\nu_2)) k} \cdot \frac{(\ell(\nu_1) + \ell(\nu_2))!}{\ell(\nu_1)!(\ell(\nu_2))!}
\]
ways to choose the $x_i$ for $p_1$ and $p_2$. For each choice of the $x_i$, there are $P(\nu_1)$ and $P(\nu_2)$ possibilities for $p_1$ and $p_2$, respectively. Thus,
\[
\sum_{\nu \in P, \nu = |B_1| - d - 1, \ell(\nu) = \ell(\nu_1) + \ell(\nu_2) - k} |T(\nu)| \cdot P(\nu) = C = \frac{(\ell(\nu_1) k)}{(\ell(\nu_1) + \ell(\nu_2)) k} \cdot \frac{(\ell(\nu_1) + \ell(\nu_2))!}{\ell(\nu_1)!(\ell(\nu_2))!} \cdot P(\nu_1)P(\nu_2).
\]

Afterwards, the coefficient of $c_f^{d_1, \nu_1}c_f^{d_2, \nu_2}$ on the left hand side of (22) is, with Claim 3.9.1 and Lemma A.1,
\[
\sum_{\nu \in P, \nu = |B_1| - d - 1} (-1)^{\ell(\nu)} |T(\nu)| \cdot P(\nu) = \sum_{k=0}^{\min(\ell(\nu_1), \ell(\nu_2))} (-1)^{\ell(\nu_1) + \ell(\nu_2) - k} \sum_{\nu \in P, \nu = |B_1| - d - 1, \ell(\nu) = \ell(\nu_1) + \ell(\nu_2) - k} |T(\nu)| \cdot P(\nu) = (-1)^{\ell(\nu_1) + \ell(\nu_2)} P(\nu_1)P(\nu_2).
\]
We are done.

### 3.3 Distinct Indices

Also, we look at the constant term following combinations of $Q_f^i$ with distinct indices $i$ in Theorem 3.13. This is important for the proof of Theorem 1.5, as will be seen in Section 4, and for this, we use the operator in (23). But, it is necessary to consider the free cumulants of $s = \{c_f^{d, \nu}\}_{d \geq 0, \nu \in P}$ which are symmetrical in the $x_i$ beforehand.

**Proposition 3.10.** Suppose that $i$ is a positive integer, and $s = \{c_f^{d, \nu}\}_{d \geq 0, \nu \in P}$ is a sequence such that $F_i(s)$ is symmetrical with respect to $(x_j)_{j \geq 1}$. Then, $c_k(s) = 0$ for all integers $k \geq 2$.

**Proof.** Since $F_i(s)$ is symmetrical, we have that for all $\lambda \in P$, there exists a constant $c_\lambda$ such that $c_f^{d, \nu} = c_\lambda$ for all $d, \nu$ such that $\nu + (d) = \lambda$. The free cumulant of $s$ of order $k$ is
\[
c_k(s) = \theta^{k-1} \sum_{\nu \in P, d \geq 0, |\nu| + d = k - 1} (-1)^{\ell(\nu)} P(\nu)c_f^{d, \nu} = \theta^{k-1} \sum_{\lambda \in P, |\lambda| = k - 1} c_\lambda \sum_{\nu \in P, d \geq 0, \nu + (d) = \lambda} (-1)^{\ell(\nu)} P(\nu),
\]
and $c_k(s) = 0$ for $k \geq 2$ follows from (21).
Suppose we have positive integers \( i \leq j \), and \( k \). For \( F(x_1, \ldots, x_j), G(x_1, \ldots, x_j) \in \mathcal{F}_i \), let

\[
R^k_{ij}(F(x_1, \ldots, x_j))G(x_1, \ldots, x_j) = Q^j_i(F(x_1, \ldots, x_j))^k G(x_1, \ldots, x_j)|_{x_i=0}.
\] (23)

**Lemma 3.11.** Suppose \( \lambda \) is a partition of length \( m \geq 1 \), \( N \geq m \) is a positive integer, \( f_i \in \mathcal{F}_{N-i+1}^{N-i+1} \) for \( 1 \leq i \leq m \), and \( g \in \mathcal{F}_N^N \). Then,

\[
\left( \prod_{i=1}^m R^{\lambda_i}_{N-i+1,N-i+1}(f_i) \right) (g)
\]

is a symmetric formal series in \( x_1, \ldots, x_{N-m} \).

**Proof.** This follows from induction on \( m \) from \( m = 1 \) to \( N \). \( \square \)

**Lemma 3.12.** Suppose \( \lambda \) is a partition of length \( m \geq 1 \). Let \( \{f_{i,N}\}_{N \geq m} \) for \( 1 \leq i \leq m \) and \( \{g_{i,N}(x_1, \ldots, x_N)\}_{N \geq m} \) be sequences of formal series such that \( f_{i,N} \in \mathcal{F}_{N-i+1}^{N-i+1} \) and \( g_{i,N} \in \mathcal{F}_N^N \) for \( N \geq m \). Also, suppose \( \{s_{i,N-i+1}f_{i,N}\}_{N \geq m} \) for \( 1 \leq i \leq m \) and \( \{s_{i,N}g_{i,N}\}_{N \geq m} \), which are symmetrical outside of \( 1 \), have limits as \( N \to \infty \) outside of \( 1 \). Then, for \( i \geq 1 \), the limit from \( N = i + m \) to \( \infty \) of the formal series

\[
\left( \prod_{i=1}^m R^{\lambda_i}_{N-i+1,N-i+1}(f_{i,N}) \right) (g_N)
\]

in \( x_1, \ldots, x_{N-m} \) exists outside of \( i \).

**Proof.** We show this with induction on \( m \). For the base case \( m = 1 \),

\[
\lim_{N \to \infty} Q^1_N(s_{1,N}f_{1,N})^{\lambda_1}(s_{1,N}g_N)
\]

exists outside of \( 1 \) from Corollary 3.5 because \( s_{1,N}f_{1,N} \) and \( s_{1,N}g_N \) have limits outside of \( 1 \). For some \( i \geq 1 \), consider \( N \geq i + 1 \). From Lemma 3.11, \( R^{\lambda_i}_{N,N}(f_{i,1},N) (g_N) \in \mathcal{F}_i^{N-i+1} \).

From Proposition 3.7, \( s_{1,N}Q^N_N(f_{1,N})^{\lambda_1}(g_N) = Q^N_N(s_{1,N}f_{1,N})^{\lambda_1}(s_{1,N}g_N) \). With this, the coefficient \( c^{d,\nu} \) of \( Q^N_N(f_{1,N})^{\lambda_1}(g_N)|_{x_i=0} \) as a formal series in \( x_1, \ldots, x_{N-1} \) outside of \( i \) will be the coefficient \( c^{d,\nu+(d)} \) of \( Q^N_N(f_{1,N})^{\lambda_1}(g_N) \) outside of \( N \), and thus the coefficient \( c^{d,\nu+(d)} \) of \( Q^N_N(s_{1,N}f_{1,N})^{\lambda_1}(s_{1,N}g_N) \) exists, the limit of this coefficient exists from \( N = i + 1 \) to \( \infty \). As this holds for all \( d, \nu \), the base case is complete.

Next, assume that the result holds for \( m \geq 1 \). We want to show that the result holds for \( m + 1 \). We have that

\[
h_{N-m} = \left( \prod_{i=1}^m R^{\lambda_i}_{N-i+1,N-i+1}(f_{i,N}) \right) (g_N).
\]

By the inductive hypothesis and because \( h_{N-m} \) is symmetric, the limit of \( s_{1,N-m}h_{N-m} = h_{N-m} \) from \( N - m = 1 \) to \( \infty \) exists outside of \( 1 \). Then,

\[
\left( \prod_{i=1}^{m+1} R^{\lambda_i}_{N-i+1,N-i+1}(f_{i,N}) \right) (g_N) = R^{\lambda_{m+1}}_{N-m,N-m}(f_{m+1,N})(h_{N-m}).
\]
is the base case $m = 1$ with $f_{m+1,N}$ as $f_{1,N-m}$, $h_{N-m}$ as $g_{N-m}$, and $N - m$ as $N$, where the conditions are satisfied. Therefore, from $m = 1$, the limit of the above expression outside of $i$ for $i \geq 1$ exists from $N - m = i + 1$ to $\infty$, or from $N = i + m + 1$ to $\infty$, completing the proof.

**Theorem 3.13.** Suppose $\lambda$ is a partition of length $m \geq 1$. Let $\{f_{i,N}(x_1, \ldots, x_{N-i+1})\}_{N \geq m}$ for $1 \leq i \leq m$ and $\{g_N(x_1, \ldots, x_N)\}_{N \geq m}$ be sequences of formal series such that $f_{i,N} \in \mathcal{F}_{N-i+1}$ and $g_N \in \mathcal{F}_N$ for $N \geq m$. Moreover, for $1 \leq i \leq m$, assume that $\{s_{i,N-i+1}f_{i,N}\}_{N \geq m}$, which is symmetrical outside of 1, has a limit as $N \to \infty$ with limiting sequence $f_i$ outside of 1. Also, assume that $\{s_{1,N}g_N\}_{N \geq m}$, which is symmetrical outside of 1, has a limit as $N \to \infty$ with limiting sequence $g$ outside of 1. Then,

$$
\lim_{N \to \infty} \left[ 1 \right] \prod_{i=1}^{m} \mathcal{R}_{N-i+1,N-i+1}^{\lambda_i}(f_{i,N})(g_N)
$$

is symmetric in $x_1, \ldots, x_{N-m}$ for all $N \geq m$. We denote $h_N(x_1, \ldots, x_N)$ by $h_N$. Then,

$$s_{1,N-m}Q_{N-m}(f_{m+1,N})^{\lambda_{m+1}}h_{N-m} = Q_{1,m}^{N-m}(s_{1,N-m}f_{m+1,N})^{\lambda_{m+1}}h_{N-m}.$$

from Proposition [3.7] Since the switch does not change the constant,

$$[1]\mathcal{R}_{N-m,N-m}^{\lambda_{m+1}}(f_{m+1,N}) \left( \prod_{i=1}^{m} \mathcal{R}_{N-i+1,N-i+1}^{\lambda_i}(f_{i,N}) \right)(g_N) = [1]Q_{1,m}^{N-m}(f_{m+1,N})^{\lambda_{m+1}}h_{N-m}$$

$$= [1]Q_{1,m}^{N-m}(s_{1,N-m}f_{m+1,N})^{\lambda_{m+1}}h_{N-m}.$$

We want to compute the limit as $N \to \infty$ of this. With $f_{N-m} = s_{1,N-m}f_{m+1,N}$ for $N \geq m + 1$, outside of 1, as $N \to \infty$, $f_N$ has limit $f_{m+1}$, and from Lemma 3.12, $h_N$ has a limit; let $h$ be the limiting sequence of $h_N$ as $N \to \infty$. Then, from Theorem 3.9 with $N - m$ for $N$,

$$\lim_{N \to \infty} [1]Q_{1,m}^{N-m}(f_{N-m})^{\lambda_{m+1}}h_{N-m} = \sum_{\pi \in NC(\lambda_{m+1}+1), \pi = B_1 \sqcup \cdots \sqcup B_{\ell(\pi)}} c_{[B_1]}(h) \prod_{i=2}^{\ell(\pi)} c_{[B_i]}(f_{m+1}).$$

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But, as $h_N$ is symmetric in $x_i$, $1 \leq i \leq N$, for $N \geq 1$, $h$ is symmetric. Then, by Proposition 3.10 $c_k(h) = 0$, $k \geq 2$, and $c_1(h) = c_h^{0,0}$. From this,

$$
\lim_{N \to \infty} [1]Q_N^{N-m}(f_{N-m})^\lambda h_{N-m} = c_h^{0,0} \left( \sum_{\pi \in NC(\lambda_{m+1}), \pi = B_1 \cup \cdots \cup B_{\ell(\pi)}} \prod_{i=1}^{\ell(\pi)} c_{|B_i|}(f_{m+1}) \right)
$$

$$
= \left( \sum_{\pi \in NC(\lambda_1+1), \pi = B_1 \cup \cdots \cup B_{\ell(\pi)}} c_{|B_1|}(g) \prod_{i=2}^{m+1} c_{|B_i|}(f_i) \right) \prod_{i=2}^{m+1} \left( \sum_{\pi \in NC(\lambda_i), \pi = B_1 \cup \cdots \cup B_{\ell(\pi)}} \prod_{i=1}^{\ell(\pi)} c_{|B_i|}(f_i) \right)
$$

using the inductive hypothesis, as needed.

\section{Proof of Main Theorem}

\subsection{Setup}

First, we go over some setup to the proof of Theorem 1.5. Note that Theorem 1.5 is a generalization of Claim 9.1 of [BGCG21], and some of techniques of that paper are used to prove the Theorem. Particularly, Lemma 5.3, Corollary 5.4, and Claim A of [BGCG21] correspond to Corollary 4.3, Proposition 4.9, and Lemma 4.10 as well as Lemma 4.12, respectively. However, we use important new techniques since partial derivatives with two or more indices can have a nonzero limit.

In Theorem 1.5, let $F_N(x_1, \ldots, x_N) = \ln(G_N(x_1, \ldots, x_N; \theta))$ for $N \geq 1$. Note that for $\nu \in P$ with $|\nu| \geq 1$, if $i_1, \ldots, i_r$ are positive integers such that $\sigma((i_1, \ldots, i_r)) = \nu$, then

$$
\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_r}} \ln(G_N^\theta) \bigg|_{x_{i_1}=0, \ldots, i_r=0} = \frac{|\nu|!}{P(\nu)} \cdot c_{\nu}^\nu.
$$

After this, the condition of Theorem 1.5 gives

$$
\lim_{N \to \infty} \frac{|\nu|}{\ell(\nu)} \cdot \frac{c_{\nu}^\nu}{N} = c_\nu.
$$

(25)

Also, from Proposition 2.7 for exponentially decaying $\mu \in \mathcal{M}_N$, for a partition $\lambda$ with $|\lambda| \geq 1$,

$$
\mathbb{E}_\mu \left( \prod_{i=1}^{\ell(\lambda)} \mathcal{P}_{\lambda_i} \right) = \mathbb{E}_\mu \left( \prod_{i=1}^{\ell(\lambda)} \left( \frac{1}{N} \sum_{j=1}^{N} \left( \frac{a_j}{N} \right)^\lambda_j \right) \right)
$$

$$
= \frac{1}{N^{\ell(\lambda)+|\lambda|}} \left( \prod_{i=1}^{\ell(\lambda)} \mathcal{P}_{\lambda_i} \right) G_\theta(x_1, \ldots, x_N; \mu) \bigg|_{x_{i_r}=0, 1 \leq i_r \leq N}.
$$

(26)

In the proof of Theorem 1.5 we use (26) with $\mu = \mu_N$ and take the $N \to \infty$ limit to get that $\{\mu_N\}_{N \geq 1}$ satisfy a LLN, see Subsection 4.6.
Proposition 4.1 ([BGCG21 Lemma 5.2]). Suppose that \( F \) is a \((k+1)\)-times continuously differentiable function on a neighborhood of \((0, \ldots, 0) \in \mathbb{C}^N\), with Taylor series expansion

\[
F(x_1, \ldots, x_N) = \sum_{\nu, |\nu| \leq k, \ell(\nu) \leq N} c_\nu^F M_\nu(\bar{x}) + O(\|x\|^{k+1}).
\]

Then, for any \( \lambda = (\lambda_1, \ldots, \lambda_m) \) with \(|\lambda| = k\),

\[
\left( \prod_{i=1}^m P_{\lambda_i} \right) \exp(F(x_1, \ldots, x_N)) \bigg|_{x_i=0, 1 \leq i \leq N} = \left( \prod_{i=1}^m P_{\lambda_i} \right) \exp(\tilde{F}(x_1, \ldots, x_N)) \bigg|_{x_i=0, 1 \leq i \leq N},
\]

where

\[
\tilde{F}(x_1, \ldots, x_N) = \sum_{\nu, |\nu| \leq k, \ell(\nu) \leq N} c_\nu^F M_\nu(\bar{x}).
\]

Suppose \( \mu \in \mathcal{M}_N \) is exponentially decaying. Then, from Lemma 1.3, \( G_\theta(x_1, \ldots, x_N; \mu) \) is symmetric and holomorphic in a neighborhood of the origin. Due to this, we see that \( F(x_1, \ldots, x_N) = \ln(G_\theta(x_1, \ldots, x_N; \mu)) \) is symmetric and holomorphic in a neighborhood of the origin as well, with \( F(0, \ldots, 0) = \ln(G_\theta(0, \ldots, 0; \mu)) = 0 \). Then, with Proposition 4.1, we can replace \( F(x_1, \ldots, x_N) \) by \( \tilde{F}(x_1, \ldots, x_N) \) for the proof of Theorem 1.5, where \( \tilde{F}(x_1, \ldots, x_N) = F_N(x_1, \ldots, x_N) \) for \( N \geq 1 \). Observe that \( \tilde{F}(x_1, \ldots, x_N) \) is a symmetric polynomial in \( x_i, 1 \leq i \leq N \), which is a symmetric formal series. Later on, we prove the results for \( F(x_1, \ldots, x_N) \) as a symmetric formal series.

From now on, assume that \( F(x_1, \ldots, x_N) \) is a symmetric formal series with \( F(0, \ldots, 0) = 0 \), and is not necessarily \( \ln(G_\theta(x_1, \ldots, x_N; \mu)) \) for some exponentially decaying \( \mu \in \mathcal{M}_N \). Later on, we look at operators on formal series rather than functions as well. Also, view the coefficients \( c_\nu^F \) of \( F(x_1, \ldots, x_N) \) as variables rather than constants. We view \( c_\nu^F \) as having degree \(|\nu|\), and from [25], \( c_\nu^F \) has order \( N \). Additionally, since \( F(0, \ldots, 0) = 0, c_0^F = 0 \), and when we refer to a coefficient \( c_\nu^F \) with \( \nu \in P \), assume \(|\nu| \geq 1 \).

### 4.2 Sequences

Suppose indices \( \mathfrak{r} = \{i_j\}_{1 \leq j \leq k} \) and a positive integer \( N \geq \max(\mathfrak{r}) \) are given. For a symmetric formal series \( F(x_1, \ldots, x_N) \) with \( F(0, \ldots, 0) = 0 \), let

\[
\mathcal{D}_\mathfrak{r}(F(x_1, \ldots, x_N)) = \left( \prod_{j=1}^k \mathcal{D}_{i_j} + \frac{\partial}{\partial x_{i_j}} F(x_1, \ldots, x_N) \right) (1).
\]

Above, the operators are for formal series.

Suppose we have a symmetric polynomial \( \tilde{F}(x_1, \ldots, x_N) \) with \( \tilde{F}(0, \ldots, 0) = 0 \). Observe that

\[
\frac{\partial}{\partial x_i} R(x_1, \ldots, x_N) \exp(\tilde{F}(x_1, \ldots, x_N))
= \left( \frac{\partial}{\partial x_i} R(x_1, \ldots, x_N) + R(x_1, \ldots, x_N) \frac{\partial}{\partial x_i} \tilde{F}(x_1, \ldots, x_N) \right) \exp(\tilde{F}(x_1, \ldots, x_N)).
\]
Moreover, because $\tilde{F}(x_1, \ldots, x_N)$ is symmetric,

$$s_{i,j}(R(x_1, \ldots, x_N) \exp(\tilde{F}(x_1, \ldots, x_N))) = (s_{i,j} R(x_1, \ldots, x_N)) \exp(\tilde{F}(x_1, \ldots, x_N)).$$

Also, since $\tilde{F}(0, \ldots, 0) = 0$, $R(x_1, \ldots, x_N) \exp(\tilde{F}(x_1, \ldots, x_N))$ at $x_i = 0$, $1 \leq i \leq N$ is $R(0, \ldots, 0)$. With the above expressions, we see that

$$\left( \prod_{j=1}^{k} D_{ij} \right) \exp(\tilde{F}(x_1, \ldots, x_N))$$

$$= \left( \prod_{j=1}^{k} \left( D_{ij} + \frac{\partial}{\partial x_{ij}} \tilde{F}(x_1, \ldots, x_N) \right) \right) (1) \cdot \exp(\tilde{F}(x_1, \ldots, x_N))$$

$$= \mathcal{D}_t(\tilde{F}(x_1, \ldots, x_N)) \cdot \exp(\tilde{F}(x_1, \ldots, x_N)),$$

and

$$\left( \prod_{j=1}^{k} D_{ij} \right) \exp(\tilde{F}(x_1, \ldots, x_N)) \bigg|_{x_i=0, 1 \leq i \leq N} = [1] \mathcal{D}_t(\tilde{F}(x_1, \ldots, x_N)).$$

Note that since $\tilde{F}(x_1, \ldots, x_N)$ is a polynomial, $\mathcal{D}_t(\tilde{F}(x_1, \ldots, x_N))$ is also a polynomial. Also, for a partition $\lambda$ with $\ell(\lambda) = m$, we have that

$$\left( \prod_{i=1}^{m} \mathcal{P}_{\lambda_i} \right) \exp(\tilde{F}(x_1, \ldots, x_N)) \bigg|_{x_i=0, 1 \leq i \leq N} = \sum_{\ell \in I_N(\lambda)} [1] \mathcal{D}_t(\tilde{F}(x_1, \ldots, x_N)), \quad (28)$$

where $I_N(\lambda)$ consists of indices $l$ of length $|\lambda|$ such that the first $\lambda_1$ indices are $i_1$, the next $\lambda_2$ indices are $i_2$, and so forth, until the last $\lambda_m$ indices are $i_m$, where $1 \leq i_j \leq N$ for $1 \leq j \leq m$.

With (28), we can express (26) in terms of formal series, which is then used in the proof of Theorem 1.5 where $F_N(x_1, \ldots, x_N)$ is reduced to a symmetric polynomial. Note that in $\mathcal{D}_t(F(x_1, \ldots, x_N))$, there are sequences of $k$ operators. We discuss this next.

We consider sequences $s$ of $k$ operators over $x_1, \ldots, x_N$, where $s_j$ is the operator at step $j$, $1 \leq j \leq k$. Each $s_j$ has an associated index $i_j$, where $1 \leq i_j \leq N$. We say that $r = \{i_j\}_{1 \leq j \leq k}$ are the indices of $s$. Here are the possibilities for $s_j$:

1. (Derivative) $\frac{\partial}{\partial x_{i_j}}$, denoted by $\partial_{i_j}$.
2. (Switch) $\frac{\theta}{x_{i_j} - x_{i}} (1 - s_{i,j,i})$, where $1 \leq i \leq N$, $i \neq i_j$. This is the switch from $i_j$ to $i$.
3. (Term Multiplication) Multiplication by $c^\nu_{\ell} \partial_{i_j}(x_1^{a_1^i} \cdots x_N^{a_N^i})$ for $a_i$, $1 \leq i \leq N$, such that $\nu \in \mathcal{P}$ with $|\nu| \geq 1$ and $\pi((a_1, \ldots, a_N)) = \nu$. We say that $c^\nu_{\ell}$ is the constant of the term multiplication.
4. (Change) $\theta(d_{i_j} - C_{i,j,i})$, where $1 \leq i \leq N$, $i \neq i_j$. This is the change from $i_j$ to $i$.

Observe that the term multiplications are equivalent to multiplying by $\partial_{i_j} F(x_1, \ldots, x_N)$. Also, if $s_j$ is a term multiplication by

$$s_{i,j}(\partial_{i_j} F(x_1, \ldots, x_N)) = (s_{i,j} \partial_{i_j}) F(x_1, \ldots, x_N).$$

$$p = c^\nu_{\ell} \frac{\partial}{\partial x_{i_j}} x_1^{a_1^i} \cdots x_N^{a_N^i},$$

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we say that \( x_i \) is in \( s_j \) or \( s_j \) contains \( x_i \) if \( p \neq 0 \) and the degree of \( x_i \) in \( p \) is at least 1.

Next, for \( 0 \leq j \leq k \), where \( r(s_0) = 1 \), let

\[
r(s)_j = s_j \circ s_{j-1} \circ \cdots \circ s_1(1).
\]

Also, suppose we have indices \( \mathbf{r} = \{i_j\}_{1 \leq j \leq k} \) and \( N \geq \max(\mathbf{r}) \). For sequences \( s \) with indices \( \mathbf{r} \) over \( x_1, \ldots, x_N \) such that \( r(s)_k \) is nonzero and has degree 0 in the \( x_i \), let \( D_{N,\mathbf{r}}(s) \) be the set of \( x_i \) such that \( i \notin \mathbf{r} \) which are in term multiplications in \( s \).

**Lemma 4.2.** Suppose \( s \) is a sequence of length \( k \). For \( 0 \leq j \leq k \),

\[
r(s)_j = \prod_{i=1}^{m} c_{F_i}^{\nu_i} \cdot P(x_1, \ldots, x_N),
\]

where \( s_j \) is term multiplication for \( j = j_i, 1 \leq i \leq m \), with \( j_i < j_{i+1} \) for \( 1 \leq i \leq m-1 \), the term multiplication \( s_j \) has constant \( c_{F_i}^{\nu_i} \), and \( P(x_1, \ldots, x_N) \) is some homogeneous integer polynomial which is 0 or has degree \( \sum_{i=1}^{m} |\nu_i| - j \).

**Proof.** We can prove this with induction on \( j \). The base case \( j = 0 \) is clear. Assume the result holds for \( j \), \( 0 \leq j \leq k-1 \). We want to show the result holds for \( j + 1 \). Suppose \( P(x_1, \ldots, x_N) \) and \( P'(x_1, \ldots, x_N) \) are the polynomials for \( r(s)_j \) and \( r(s)_{j+1} \), respectively.

If \( r(s)_j = 0 \), then \( r(s)_{j+1} = 0 \), and the statement holds. Assume \( r(s)_j \neq 0 \). If \( s_{j+1} \) is a derivative, a switch, or a change, we see that in \( r(s)_{j+1}, P'(x_1, \ldots, x_N) = 0 \) or \( P'(x_1, \ldots, x_N) \) is a homogeneous integer polynomial with degree \( \sum_{i=1}^{m} |\nu_i| - j - 1 \), and the \( c_{F_i}^{\nu_i} \) remain the same. However, if \( s_{j+1} \) is term multiplication by \( c_{F_{m+1}}^{\nu_{m+1}} \partial_{i_1} (a_1^{\nu_1} \cdots a_N^{\nu_N}) \),

\[
r(s)_{j+1} = \prod_{i=1}^{m+1} c_{F_i}^{\nu_i} \cdot \left( \frac{\partial}{\partial x_{i_1}^{\nu_1} \cdots x_N^{\nu_N}} \right) P(x_1, \ldots, x_N).
\]

There are now \( m+1 \) term multiplications, where \( j_{m+1} = j + 1 \) and \( s_{j_{m+1}} = s_{j+1} \) has constant \( c_{F_{m+1}}^{\nu_{m+1}} \). Moreover, because \( \pi((a_1, \ldots, a_N)) = \nu_{m+1} \), we have that \( P'(x_1, \ldots, x_N) = (\partial_{i_1} x_1^{\nu_1} \cdots x_N^{\nu_N}) P(x_1, \ldots, x_N) \) is 0 or is a homogeneous integer polynomial with degree \( \sum_{i=1}^{m+1} |\nu_i| - j - 1 \). The induction is complete. \( \blacksquare \)

**Corollary 4.3.** For indices \( \mathbf{r} = \{i_j\}_{1 \leq j \leq k} \) and \( N \geq \max(\mathbf{r}) \), \([1]D_{\mathbf{r}}(F(x_1, \ldots, x_N))\) is a polynomial in the \( c_{F_i}^{\nu} \) for \( \nu \in P \) with \( |\nu| \geq 1 \) which is homogeneous of degree \( k \), where \( c_{F_i}^{\nu} \) has degree \( |\nu| \) for \( \nu \in P \) with \( |\nu| \geq 1 \).

**Proof.** This follows from Lemma 4.2. \( \blacksquare \)

From Lemma 4.2, if a sequence \( s \) of length \( k \) has \( r(s)_k \) have degree 0 in the \( x_i \), we know that \( r(s)_k = d \cdot \prod_{i=1}^{m} c_{F_i}^{\nu_i} \) for nonzero \( d \) with the sum of the \( |\nu_i| \) equal to \( k \), where the \( c_{F_i}^{\nu_i} \) are from Lemma 4.2. Then, for such \( s \), where \( k \) can be any positive integer, let

\[
C(s) = d.
\]
Definition 4.4. For a positive integer \( k \), suppose we have a sequence of indices \( \mathbf{r} = \{i_j\}_{1 \leq j \leq k} \). For \( N \geq \max(\mathbf{r}) \), let \( T_N(\mathbf{r}) \), \( T_N^1(\mathbf{r}) \), and \( T_N^2(\mathbf{r}) \) denote set of sequences \( s \) over \( x_1, \ldots, x_N \) with indices \( \mathbf{r} \) such that \( r(s)_k \) is nonzero and has degree 0 in the \( x_i \) such that:

- The operators of \( s \) in \( T_N(\mathbf{r}) \) can be any operator (derivatives, term multiplications, switches, and changes).
- The operators of \( s \) in \( T_N^1(\mathbf{r}) \) are only derivatives, term multiplications, and switches.
- The operators of \( s \) in \( T_N^2(\mathbf{r}) \) are only term multiplications and changes.

Throughout Section 4, \( T_N(\mathbf{r}), T_N^1(\mathbf{r}) \), and \( T_N^2(\mathbf{r}) \) are referred to. Particularly, we have that

\[
[1] \mathcal{D}_t(F(x_1, \ldots, x_N)) = [1] \left( \prod_{j=1}^{k} \left( \mathcal{D}_{ij} + \frac{\partial}{\partial x_{ij}} F(x_1, \ldots, x_N) \right) \right) (1) = \sum_{s \in T_N^1(\mathbf{r})} r(s)_k.
\]

4.3 Orders

Definition 4.5. Suppose \( Q(c^\nu; N) \) is a polynomial in the \( c^\nu \), \( \nu \in \mathbb{P}, |\nu| \geq 1 \), with finite degree and finitely many terms, and coefficients which are functions of \( N \). Then, \( Q(c^\nu; N) \) has an order of \( N^k \) if for a term

\[
\prod_{i=1}^{m} c^{\nu_i},
\]

the coefficient \( f(N) \) in \( Q(c^\nu; N) \) has \( |f(N)| = O(N^{k-m}) \) for positive integers \( N \).

If \( Q(c^\nu; N) \) has an order of \( N^k \), we say that \( Q(c^\nu; N) \) is \( O(N^k) \). Later, we use \( c^\nu_F \) for \( c^\nu \), and generally, for \( k \) indices \( \mathbf{r} \), if we have sets \( S_N \) in \( T_N(\mathbf{r}) \) for \( N \geq 1 \), we let

\[
Q(c^\nu_F; N) = \sum_{s \in S_N} r(s)_k.
\]

We get \( Q(c^\nu_F; N) = \mathcal{D}_t(F(x_1, \ldots, x_N))|_{x_i=0, 1 \leq i \leq N} \) with \( S_N = T_N^1(\mathbf{r}) \) for \( N \geq 1 \). Later on, Definition 4.5 and (31) are used with \( \mathcal{D}_t \) as well as other operators.

Proposition 4.6. Suppose \( \mathbf{r} = \{i_j\}_{1 \leq j \leq k} \). There exists a constant \( C > 0 \) such that for all positive integers \( N \), for any sequence \( s \in T_N(\mathbf{r}), |C(s)| \leq C \).

Proof. For such a sequence \( s \), for \( 0 \leq j \leq k \), suppose \( P_j(x_1, \ldots, x_N) \) is the polynomial for \( r(s)_j \) in (29). Suppose \( s \) has \( m \) term multiplications. Since each other operator decreases the degree in the \( x_i \) by 1, the degree in the \( x_i \) of \( P_j(x_1, \ldots, x_N) \) for \( 0 \leq j \leq k \) is at most \( k - m \), which is less than \( k \). Using this, Proposition 4.6 can be seen by, for \( 0 \leq j \leq k - 1 \), bounding the factor \( s_{j+1} \) changes the sum of the absolute values of the coefficients from \( P_j(x_1, \ldots, x_N) \) to \( P_{j+1}(x_1, \ldots, x_N) \). Note that \( P_k(x_1, \ldots, x_N) = C(s) \).

Proposition 4.7. For a positive integer \( k \) and indices \( \mathbf{r} = \{i_j\}_{1 \leq j \leq k} \), for \( N \geq \max(\mathbf{r}) \), \([1] \mathcal{D}_t(F(x_1, \ldots, x_N)) \) has order \( N^k \).
Proof. We know from Corollary 4.3 that the expression will be a polynomial in the $c_F^{\nu}$ which is homogeneous of order $k$. Suppose that, for a positive integer $m$, for $1 \leq i \leq m$, $\nu_i \in P$ such that $|\nu_i| \geq 1$, and the sum of the $|\nu_i|$ is $k$. Note that the number of possible $\nu_1, \ldots, \nu_m$, $1 \leq m \leq k$, is finite. Then, if we show that for $N \geq \max(\tau)$, the coefficient of $p = \prod_{i=1}^{m} c_F^{\nu_i}$ is of order $N^{k-m}$, we will be done by Definition 4.5. For $N \geq \max(\tau)$, suppose the coefficient of $p$ is $r$; we want to show that $|r| = O(N^{k-m})$. Let $T$ be the set of sequences $s$ such that $r(s)_k = dp$, where $d \neq 0$ is a real number. From Proposition 4.6, there exists $C > 0$ such that for all $N$, for all $s \in T$, $|C(s)| \leq C$. Then, $r = \sum_{s \in T} C(s), |r| = \left| \sum_{s \in T} C(s) \right| \leq \sum_{s \in T} |C(s)| \leq C \cdot |T|$, with the triangle inequality, and it suffices to show that $|T| = O(N^{k-m})$.

Suppose $s \in T$. We know that exactly $m$ of the $s_i$ must be term multiplications, and as seen in the proof of Proposition 4.6, the degree in the $x_i$ of $r(s)_j$ for $0 \leq j \leq k$ is at most $k - m$.

**Claim 4.7.1.** Suppose $s \in T$. If $x_i \in D_{N,\tau}(s)$, suppose the first term multiplication $x_i$ is in is $s^j$. Then, $s_j$ must be a switch from $i_j$ to $i$ for some $j, j' < j \leq k$.

**Proof.** Observe that, in $r_j(s)$, $x_i$ is in all of the terms, and must be removed because $r_k(s)$ cannot contain $x_i$. For the sake of contradiction, assume the statement does not hold. If $j' = k$, then $r_k(s) = r_{j'}(s)$ contains $x_i$, a contradiction since $r_k(s) = dp$ for a nonzero real number $d$. Therefore, $j' < k$. Below, we view $r_j(s)$ for $1 \leq j \leq k$ as a polynomial in the $x_i$, and a term in $r_j(s)$ is a product of the $x_i$ with its coefficient.

Consider $s_j, j' < j \leq k$. We see that if $s_j$ is a derivative, term multiplication, or switch, all nonzero terms of $r_j(s)$ will have $x_i$. Then, in $r_k(s) \neq 0$, the terms will contain $x_i$, a contradiction.

Let $S = \{x_i|1 \leq j \leq k\}$. If $|D_{N,\tau}(s)| = d$, from Claim 4.7.1, we see that we can find one switch from $i_j$ to $i$ for each $x_i \in D_{N,\tau}(s)$, giving $d$ switches in total, where $0 \leq d \leq k - m$. Also, the number possibilities for $D_{N,\tau}(s)$, the number subsets of $\{x_1, \ldots, x_N\}\_S$ with size $d$, is at most $N^d$. Next, suppose $X \subset \{x_1, \ldots, x_N\}\_\backslash S$, $|X| = d$. Because the variables outside of $r$ are symmetrical, the number of $s \in T$ such that $D_{N,\tau}(s) = X$ is the same for all such $X$; let this number be $T_d$.

In $s$, there are $m$ term multiplications, known as $\alpha(s)$ operators. Also, there are $d \leq k - m$ operators which are switches from $i_j$ to $i$, where $x_i \in D_{N,\tau}(s)$ and the $i$ are distinct, known as $\beta(s)$ operators. The other $k - m - d$ operators, known as $\gamma(s)$ operators, can be derivatives or any switch. In this proof, there can be overcounting of $N_d$, with $s \in T$ counted multiple times.
For \( d, 0 \leq d \leq k - m \),
\[
\frac{k!}{m!d!(k-m-d)!}
\]
possible groupings of the \( s_j \) into \( \alpha(s) \), \( \beta(s) \), and \( \gamma(s) \) operators. For a term multiplication, or a \( \alpha(s) \) operator, suppose the term before the derivative is \( q \). We know that all \( x_i \) in \( q \) must be in \( S \cup D_{N,x}(s) \). Also, the total degree of \( q \) after the derivative is at most \( k-m \). We look at degrees of the \( x_i \) in \( q \). Note that for \( x_i \), where \( \partial_i q \neq 0 \), the degree must be at least 1 and at most \( k-m+1 \), giving \( k-m+1 \) possibilities. For the other \( x_i \), the degree must be at least 0 and at most \( k-m \), also giving \( k-m+1 \) possibilities. With this, the number of possibilities for the \( \alpha(s) \) operators is at most
\[
(k - m + 1)^{(k+|D_{N,x}(s)|)m} = (k - m + 1)^{(k+d)m}.
\]
On the other hand, for the \( \beta(s) \) operators, there being \( d \) in total, there are \( d! \) possible orderings. Finally, each of the other \( \gamma(s) \) operators, there being \( k-m-d \) in total, have \( N \) possibilities, giving a total of \( N^{k-m-d} \). Then, for a constant \( C_d \) not depending on \( N \),
\[
T_d \leq \frac{k!}{m!d!(k-m-d)!} \cdot (k - m + 1)^{(k+d)m} \cdot d! \cdot N^{k-m-d} = C_d \cdot N^{k-m-d}.
\]
From this, for \( 0 \leq d \leq k - m \), the number of \( s \in T \) such that \( |D_{N,x}(s)| = d \) is at most \( N^d \cdot T_d \leq C_d \cdot N^{k-m} \), where there are at most \( N^d \) choices for \( D_{N,x}(s) \). Therefore,
\[
|T| \leq \left( \sum_{d=0}^{k-m} C_d \right) N^{k-m}
\]
where the \( C_d \) do not depend on \( N \), and \( |T| = O(N^{k-m}) \), as desired. 

\[\blacksquare\]

### 4.4 Remainders

**Proposition 4.8.** Suppose \( s = \{a_\nu(N)\}_{\nu \in P} \) is a sequence of variables over positive integers \( N \) such that for each \( \nu \in P \), there exists a constant \( C > 0 \) such that \( |a_\nu(N)| \leq CN \) for all positive integers \( N \). Then, if \( Q(c'^\nu, N) \) is order \( N^k \),
\[
\lim_{N \to \infty} \frac{Q(a_\nu(N); N)}{N^{k+1}} = 0,
\]
where we let \( c'^\nu = a_\nu(N) \) for all \( \nu \in P \).

**Proof.** This is clear from considering each term of \( Q(a_\nu(N); N) \), with there being a finite number of terms. 

\[\blacksquare\]

Suppose we have a polynomial \( R \) in the \( c'_P \) which is homogeneous of degree \( k \) with order \( N^k \). For \( N \geq 1 \), we can let \( c'_P = c'_{F_N} \), and from the conditions of Theorem 1.5, we see that for each \( \nu \in P \), if \( a_\nu(N) = c'_{F_N} \), there exists a constant \( C > 0 \) such that
\[|a_\nu(N)| \leq CN \text{ for all } N \geq 1. \] Afterwards, we use Proposition 4.8 on \( R \) with \( c'_F = a_\nu(N) \) for \( N \geq 1 \) to get that
\[
\lim_{N \to \infty} \frac{R(a_\nu(N); N)}{N^{k+1}} = \lim_{N \to \infty} \frac{R(c'_F; N)}{N^{k+1}} = 0. \tag{32}
\]
We use (32) in the proof of Theorem 1.5 in Subsection 4.6 to show that as \( N \to \infty \), such remainders \( R \) go to 0. Particularly, (32) can be used with various results from Subsections 4.4 and 4.5.

**Proposition 4.9.** Suppose that \( \lambda \) is a partition with \( \ell(\lambda) = m \) and \( |\lambda| = k \). Also, suppose \( r = \{i_j\}_{1 \leq j \leq k}, \) where the first \( \lambda_1 \) indices are 1, the next \( \lambda_2 \) indices are 2, and so forth, until the last \( \lambda_m \) indices are \( m \). Then, for \( N \geq m \),
\[
\frac{1}{N^m} \sum_{l \in I_N(\lambda)} [1]D_l(F(x_1, \ldots, x_N)) = [1]D_1(F(x_1, \ldots, x_N)) + R,
\]
where \( R \) is a homogeneous polynomial in the \( c'_F \) with degree \( k \) and order \( N^{k-1} \).

**Proof.** Suppose that in \( l \in I_N(\lambda) \), the first \( \lambda_1 \) indices are \( i_1 \), the next \( \lambda_2 \) indices are \( i_2 \), and so forth, until the last \( \lambda_m \) indices are \( i_m \), where \( 1 \leq i_j \leq N \) for \( 1 \leq j \leq m \). From Proposition 4.7, for all \( l \in I_N(\lambda) \), \( D_l(F(x_1, \ldots, x_N)) \) has order \( N^k \). Note that the \([1]D_l(F(x_1, \ldots, x_N))\) with \( l \) having all \( i_j \) distinct are equal, by symmetry, to \([1]D_1(F(x_1, \ldots, x_N))\), a polynomial in the \( c'_F \) which is homogeneous of degree \( k \). Consider the set \( S \) of other \( l \), where some of the \( i_j \) are equal. Let
\[
R' = \frac{1}{N^m} \sum_{l \in S} [1]D_l(F(x_1, \ldots, x_N)).
\]
From Corollary 4.3, \( R' \) is a polynomial in the \( c'_F \), which is homogeneous of degree \( k \).

Note that there are \( N^m - O(N^{m-1}) \) \( l \) where all of the \( i_j \) are distinct, and dividing by \( N^m \) will give
\[
\left(1 - O\left(\frac{1}{N}\right)\right) [1]D_1(F(x_1, \ldots, x_N)).
\]
Also, \(|S| = O(N^{m-1})\) and each \( D_1(F(x_1, \ldots, x_N)) \) is order \( N^k \), so \( R' \) is \( O(N^{m-1}) \cdot \frac{1}{N^m} N^k = O(N^{k-1}) \). Then, we get
\[
R = R' - O\left(\frac{1}{N}\right) D_1(F(x_1, \ldots, x_N))|_{x_i=0,1 \leq i \leq N}
\]
is a homogeneous polynomial in the \( c'_F \) with degree \( k \) and order \( N^{k-1} \), as desired. \( \blacksquare \)

**Lemma 4.10.** Suppose that \( r = \{i_j\}_{1 \leq j \leq k} \) are indices. For \( N \geq \max(r) \), let \( H \) be the set of sequences \( s \) in \( T_N^1(\tau) \) which satisfy the following conditions:

- For \( 1 \leq j \leq k \), if \( s_j \) is the switch from \( i_j \) to \( i \), \( i \not\in r \).
• There do not exist integers $i$, $j_1$, and $j_2$, $1 \leq i \leq N$, $1 \leq j_1, j_2 \leq k$, $j_1 \neq j_2$, such that $s_{j_1}$ is the switch from $i_{j_1}$ to $i$ and $s_{j_2}$ is the switch from $i_{j_2}$ to $i$.

Then,

$$[1] D_c(F(x_1, \ldots, x_N)) = \sum_{s \in H} r(s)_k + R,$$

where $R$ is a homogeneous polynomial in the $c_F^\nu$ with degree $k$ and order $N^{k-1}$.

**Proof.** Suppose that $Q$ is the set $s \in T_N^1(\mathcal{r})$ such that at least one of the conditions is not followed. Then,

$$R = \sum_{s \in Q} r(s)_k.$$

Since $Q \subset T_N^1(\mathcal{r})$, $R$ is a polynomial in the $c_F^\nu$ which is homogeneous of degree $k$. It suffices to show that the coefficient in $R$ of each term

$$p = \prod_{i=1}^{m} c_F^\nu_i$$

is order $N^{k-m-1}$ from Definition [4.5]. We consider the set $T$ of $s \in Q$ which end with $p$. Following the proof of Proposition [4.7], let the coefficient of $p$ be $r$. We want to show that $r$ is $O(N^{k-m-1})$, and we know that $|C(s)| \leq C$ for a $C > 0$ independent of $N$. Therefore, it suffices to show that $|T| = O(N^{k-m-1})$.

In $s \in T$, there are $m$ term multiplications, or $\alpha(s)$ operators. Also, there are $d$, $d \leq k-m$, $\beta(s)$ operators which are switches from $i_j$ to $i$ where $x_i \in D_{N,i}(s)$, such that each $x_i \in D_{N,i}(s)$ is in exactly one $\beta(s)$ operator. The other $k-m-d$ operators are $\gamma(s)$ operators, and are derivatives or switches.

Consider when one of the switches is from $i_j$ to $i \in \mathcal{r}$. Since $\mathcal{r}$ and $D_{N,i}(s)$ are disjoint, the switch must be a $\gamma(s)$ operator, and the number of possibilities for the $\gamma(s)$ operators is $O(N^{k-m-d-1})$. With the number of possibilities for $D_{N,i}(s)$ being at most $N^d$ and for the other choices being bounded, the total number of possible $s$ is then $O(N^{k-m-1})$.

Also, suppose there are two switches with the same $i$. Then, we could have a $\beta(s)$ switch with the same $i$ as a $\gamma(s)$ switch, or two $\gamma(s)$ switches with the same $i$. For the first case, there are at most $N$ ways to pick $i$. Afterwards, the number of possibilities for $D_{N,i}(s)$ is at most $N^{d-1}$ and the $\gamma(s)$ switches is $O(N^{k-m-d-1})$. Where there are $O(1)$ possibilities for the other choices, this gives $O(N^{k-m-1})$ possibilities for the first case. Next, for the second case, there are at most $N$ ways to choose $i$. Afterwards, the number of possibilities for $D_{N,i}(s)$ is at most $N^d$, and the $\gamma(s)$ switches is $O(N^{k-m-d-2})$. Where there are $O(1)$ possibilities for the other choices, this gives $O(N^{k-m-1})$ possibilities for the second case. Therefore, $|T| = O(N^{k-m-1})$, as desired.

### 4.5 Sequences For Different Operators

Next, we consider the operators $Q_i^N$ from Section 3. Particularly,

$$N Q_i^N \left( \frac{\partial F}{\partial N} \right) = \theta \sum_{1 \leq j \leq N, \ j \neq i} (d_i - C_{i,j}) + \frac{\partial}{\partial x_i} F(x_1, \ldots, x_N)$$
is used. In Theorem 4.13, we show that we can replace each \( D_i \) with \( N Q_{i}^N \left( \frac{\partial_{x_j} F}{N} \right) \). Afterwards, in Theorem 4.14, we use the \( R_{ij}^k \) operators to evaluate the resulting expression. Similarly to before, for \( \mathbf{r} = \{ i_j \}_{1 \leq j \leq k} \) and \( N \geq \max(\mathbf{r}) \), we let

\[
Q_{\mathbf{r}}^N(F(x_1, \ldots, x_N)) = N^k \left( \prod_{j=1}^{k} Q_{i_j}^N \left( \frac{\partial_{x_j} F}{N} \right) \right) (1) = \sum_{s \in T_{2}^k(\mathbf{r})} r(s)_k. \tag{33}
\]

**Proposition 4.11.** Let \( \mathbf{r} = \{ i_j \}_{1 \leq j \leq k} \). For \( N \geq \max(\mathbf{r}) \), \( [1]Q_{\mathbf{r}}^N(F(x_1, \ldots, x_N)) \) has order \( N^k \).

**Proof.** We can use the same proof as Proposition 4.7, but with no derivatives and each switch replaced by a change, with the switch from \( i \) to \( j \) replaced by the change from \( i \) to \( j \).

**Lemma 4.12.** Suppose that \( \mathbf{r} = \{ i_j \}_{1 \leq j \leq k} \). For \( N \geq \max(\mathbf{r}) \), suppose \( H \) is the set of sequences \( s \) in \( T_{2}^N(\mathbf{r}) \) which satisfy the following conditions:

- For \( 1 \leq j \leq k \), if \( s_j \) is the change from \( i_j \) to \( i \), \( i \notin \mathbf{r} \).
- There do not exist integers \( i, j_1, j_2 \), \( 1 \leq i \leq N \), \( 1 \leq j_1, j_2 \leq k \), \( j_1 \neq j_2 \), such that \( s_{j_1} \) is the change from \( i_{j_1} \) to \( i \) and \( s_{j_2} \) is the change from \( i_{j_2} \) to \( i \).

Then,

\[
[1]Q_{\mathbf{r}}^N(F(x_1, \ldots, x_N)) = \sum_{s \in H} r(s)_k + R,
\]

where \( R \) is a homogeneous polynomial in the \( c_F^\nu \) with degree \( k \) and order \( N^{k-1} \).

**Proof.** We can use the same proof as Lemma 4.10, but with no derivatives and each switch replaced by a change, with the switch from \( i \) to \( j \) replaced by the change from \( i \) to \( j \).

**Theorem 4.13.** Suppose that \( k \) is a positive integer and \( \mathbf{r} = \{ i_j \}_{1 \leq j \leq k} \). For \( N \geq \max(\mathbf{r}) \),

\[
[1]D_{\mathbf{r}}(F(x_1, \ldots, x_N)) = [1]Q_{\mathbf{r}}^N(F(x_1, \ldots, x_N)) + R, \tag{34}
\]

where \( R \) is a homogeneous polynomial in the \( c_F^\nu \) with degree \( k \) and order \( N^{k-1} \).

**Proof.** We note that in \( T_{N}^1(\mathbf{r}) \), the sequences can have derivatives, but cannot in \( T_{N}^2(\mathbf{r}) \), where \( T_{N}^1(\mathbf{r}) \) and \( T_{N}^2(\mathbf{r}) \) correspond to the left and right hand side of (34), respectively. Then, let \( T_{N}^3(\mathbf{r}) \) be the set of \( s \in T_{N}^1(\mathbf{r}) \) which do not contain derivatives.

Suppose that \( H \) is the set of \( s \) in \( T_{N}^3(\mathbf{r}) \) satisfying the Lemma 4.10 conditions. From the Lemma,

\[
[1]D_{\mathbf{r}}(F(x_1, \ldots, x_N)) = \sum_{s \in H} r(s)_k + R_1,
\]

where \( R_1 \) is a homogeneous polynomial in the \( c_F^\nu \) with degree \( k \) and order \( N^{k-1} \). Consider \( H_1 = H \cap T_{N}^3(\mathbf{r}) \). We see that \( H_1 \) is the set of \( s \in T_{N}^1(\mathbf{r}) \) satisfying the conditions of Lemma 4.10 with only switches and term multiplications, and

\[
\sum_{s \in H} r(s)_k = \sum_{s \in H_1} r(s)_k + R_2, R_2 = \sum_{s \in H \cap (T_{N}^1(\mathbf{r}) \setminus T_{N}^3(\mathbf{r}))} r(s)_k.
\]
Also, let $H_2$ be the set of $s$ in $T_N^3(r)$ satisfying the Lemma 4.12 conditions. By the Lemma,

$$[1] Q^N_c(F(x_1, \ldots, x_N)) = \sum_{s \in H_2} r(s)_k + R_3,$$

where $R_3$ is a homogeneous polynomial in the $c'_k$ with degree $k$ and order $N^{k-1}$.

**Claim 4.13.1.** $R_2$ is a homogeneous polynomial in the $c'_k$ with degree $k$ and order $N^{k-1}$

**Proof.** We bound the number of $s \in T_N^1(r) \setminus T_N^3(r)$, the $s \in T_N^1(r)$ which have a derivative, such that $r(s)_k = dp$, where

$$p = \prod_{i=1}^{m} c'_{p_i}.$$

Observe that, looking at the proof of Proposition 4.7, the derivatives in $s \in T_N^1(r)$ only appear in $\gamma(s)$ operators. Where $|D_{N, k}(s)| = d$, the number of ways to choose the $\gamma(s)$ operators such that there is at least 1 derivative is $O(N^{k-d-1})$. Then, we see that the number of $s$ in $T_N^1(r) \setminus T_N^3(r)$, and therefore $H \cap (T_N^1(r) \setminus T_N^3(r))$, ending with $p$ is $O(N^{k-m-1})$. So, $R_2$ is also a homogeneous polynomial in the $c'_k$ with degree $k$ and order $N^{k-1}$. 

For a sequence $s$ in $H_1$, let $c(s)$ be the sequence in $H_2$ where if $s_j$ is the switch from $i_j$ to $i$, then $c(s)_j$ is the change from $i_j$ to $i$, and all other operators are the same. We have that $s \in H_1$ have switches and term multiplications, while $s \in H_2$ have changes and term multiplications. Clearly, $c$ is a bijection from $H_1$ to $H_2$.

**Claim 4.13.2.** For all $s \in H_1$, $r(s)_k = r(c(s))_k$.

**Proof.** Consider a sequence $s \in T_N(r)$ with term multiplications, switches, and changes which satisfy the conditions of Lemma 4.10 and Lemma 4.12, where $s_j, 1 \leq j \leq k$, is a switch from $i_j$ to $i$ and $s_j', j < j' \leq k$, are not changes. It suffices to show that if we make $s_j$ the change from $i_j$ to $i$, then $r(s)_k$ is unchanged. Then, for $s \in H_1$, if the switches of $s$ are $s_{j_1}, s_{j_2}, \ldots, s_{j_k}, 1 \leq j_1 < \cdots < j_k \leq k$, we could make $s_{j_u}$ into a change from $u = 1$ to $r$ and get $c(s)$. This would give that $r(c(s))_k = r(s)_k$.

Suppose $q$ is a term of $r(s)_{j-1}$. We look at $s_j(q)$, which is a sum of terms or 0. Suppose $q'$ is a term of $s_j(q)$ and the degree of $x_i$ in $q'$ is at least 1. Afterwards, where $j < j' \leq k$, $s_j'$ is the switch from $i_{j'}$ to $i'$ where $i' \neq i$, or a term multiplication. In $r(s)_k$, the degree of $x_i$ must be 0, but to decrease the degree of $x_i$, $s_{j'}, j' > j$, will make all terms 0. Therefore, $q'$ will not contribute to $r(s)_k$. So, the only term of $s_j(q)$ which will contribute to $r(s)_k$ will be $q'$ with degree of $x_i$ equal to 0, and since $s_j$ is the switch from $i_{j}$ to $i$, in $s_j(q)$, there will be at most one such $q'$. With this, we can set

$$s_j(q) = q'$$

if such $q'$ exists, and $s_j(q) = 0$ if not.

Assume the degree of $x_i$ and $x_{i_j}$ are both at least 1 in $q$. Then, in $s_j(q)$, the degree in $x_i$ of all terms will be at least 1, so $q$ does not contribute to $r(s)_k$. Now, replace $s_j$ with the change from $i_j$ to $i$. Note that in $\theta(d_{i_j} - C_{i,j,i})(q) = \theta d_{i_j}(q)$, the degree of $x_i$ is at
least 1, and because of this, using the logic from above, after replacing \( s_j \) there are also no contributions to \( r(s)_k \). Therefore, in this case, \( r(s)_k \) is unchanged.

Next, suppose the degree of \( x_i \) is 0 and \( x_{ij} \) is at least 1 in \( q \). In this case, \( q' \) exists and will be \( \theta d_{ij}(q) \). But, since the degree of \( x_{ij} \) is at least 1,
\[
\theta(d_{ij} - C_{ij,i})(q) = \theta d_{ij}(q) = q' = s_j(q).
\]
Therefore, we can replace \( s_j \) with \( \theta(d_{ij} - C_{ij,i}) \) and \( r(s)_k \) will be the same.

Consider when the degree of \( x_{ij} \) in \( q \) is 0. If the degree of \( x_i \) is at least 1 in \( q \), then \( q' \) will exist, and be \(-\theta C_{ij,i}(q)\). Otherwise, if the degree of \( x_i \) is 0, \( s_j(q) = 0 = -\theta C_{ij,i}(q) \).

However, since the degree of \( x_{ij} \) in \( q \) is 0,
\[
\theta(d_{ij} - C_{ij,i})(q) = -\theta C_{ij,i}(q) = s_j(q).
\]
Therefore, after replacing \( s_j \) with \( \theta(d_{ij} - C_{ij,i}) \), \( r(s)_k \) is unchanged here as well. ■

Since \( c : H_1 \rightarrow H_2 \) is a bijection, with Claim \ref{claim:4.13.2}
\[
\sum_{s \in H_1} r(s)_k = \sum_{s \in H_2} r(s)_k
\]
and we are done. ■

**Theorem 4.14.** Suppose that \( \lambda \) is a partition with \( \ell(\lambda) = m \) and \( |\lambda| = k \). Then, for \( N \geq m \), if, for \( 1 \leq i \leq N \), \( \overline{x}_i = (x_1, \ldots, x_i, 0, \ldots, 0) \) is \( \overline{x} \) with \( x_j = 0 \) for \( i + 1 \leq j \leq N \),
\[
[1]N^k \prod_{i=1}^{m} Q_i^N \left( \frac{\partial F}{N} \right)^{\lambda_i} (1)
= [1] \prod_{i=1}^{m} \left( (N - i + 1)^{\lambda_i} \mathcal{R}_{N-i+1,N-i+1} \left( \frac{\partial_{N-i+1} F(\overline{x}_{N-i+1})}{N - i + 1} \right) \right) (1) + R,
\]
where \( R \) is a homogeneous polynomial in the \( c_F' \) with degree \( k \) and order \( N^{k-1} \).

**Proof.** By symmetry,
\[
[1]N^k \prod_{i=1}^{m} Q_i^N \left( \frac{\partial F}{N} \right)^{\lambda_i} (1) = [1]N^k \prod_{i=1}^{m} Q_i^N_{N-i+1} \left( \frac{\partial_{N-i+1} F}{N} \right)^{\lambda_i} (1).
\]
Here, \( \tau = \{i_j\}_{1 \leq j \leq k} \) consists of \( k \) indices, the first \( \lambda_1 \) being \( N \), the next \( \lambda_2 \) being \( N - 1 \), and so forth, until the last \( \lambda_m \) are \( N - m + 1 \). Let \( T \) be the set of sequences \( s \) with indices \( \tau \) over \( x_1, \ldots, x_N \) consisting of term multiplications and changes such that:

- For \( 1 \leq j \leq m - 1 \), after \( \lambda_1 + \cdots + \lambda_j \) operators, the next \( \lambda_{j+1} \) operators cannot have changes from \( N - j \) to \( i \) and term multiplications which contain \( x_i \), \( N - j + 1 \leq i \leq N \).

- \( r(s)_k \) has degree 0 in the \( x_i \).
Note that it is possible for $s \in T$ to have $r(s)_k = 0$.

Suppose $0 \leq d \leq m$. For a sequence $s$ of term multiplications and changes, let $r_d(s)_i$, $0 \leq i \leq k$, be as follows. Set $r_d(s)_0 = 1$, and for $1 \leq i \leq k$, $i \neq \lambda_1 + \cdots + \lambda_j$, $1 \leq j \leq d$, let $r_d(s)_i = s_i(r_d(s)_{i-1})$. Also, for $1 \leq j \leq d$, let

$$r_d(s)_{\lambda_1 + \cdots + \lambda_j} = (r_d(s)_{\lambda_1 + \cdots + \lambda_j-1})|_{x_{N-j+1}=0}.$$ 

Note that $r_0(s)_i = r(s)_i$, $0 \leq i \leq k$.

**Claim 4.14.1.** For $0 \leq i \leq k$ and any sequence $s$ of term multiplications and changes, there exists a positive integer $n$ such that for $0 \leq d \leq m$, $r_d(s)_i$ is homogeneous in $x_1, \ldots, x_N$ with degree $n$ or $r_d(s)_i = 0$.

**Proof.** This is clear by induction from $i = 0$ to $k$. —

**Claim 4.14.2.** For all $s \in T$, $r_d(s)_k$ is the same for $0 \leq d \leq m$.

**Proof.** To show this, we prove that $r_d(s)_k = r_{d-1}(s)_k$, $1 \leq d \leq m$. Observe that $r_d(s)_i = r_{d-1}(s)_i$ for $1 \leq i \leq \lambda_1 + \cdots + \lambda_d - 1$, and

$$r_{d-1}(s)_k - r_d(s)_k = s_k \cdots s_{\lambda_1+\cdots+\lambda_d-1}(r_{d-1}(s)_{\lambda_1+\cdots+\lambda_d} - r_d(s)_{\lambda_1+\cdots+\lambda_d}|_{x_{N-d+1}=0}),$$

where each term of $r_{d-1}(s)_{\lambda_1+\cdots+\lambda_d} - r_d(s)_{\lambda_1+\cdots+\lambda_d}|_{x_{N-d+1}=0}$ contains $x_{N-d+1}$. But, if $\lambda_1 + \cdots + \lambda_d + 1 \leq i \leq k$, $s_i$ does not have a $x_{N-d+1}$ since $s \in T$, and $s_i$ cannot remove $x_{N-d+1}$, unless all terms become 0. Then, all nonzero terms of $r_d(s)_k - r_{d-1}(s)_k$ contain $x_{N-d+1}$. Since $s \in T$, $r_0(s)_k = r(s)_k$ has degree 0 in the $x_i$, so $r_{d-1}(s)_k$ and $r_d(s)_k$, and hence $r_d(s)_k - r_{d-1}(s)_k$, have degree 0 in the $x_i$ from Claim 4.14.1. But, all nonzero terms of $r_d(s)_k - r_{d-1}(s)_k$ contain $x_i$, and therefore, $r_d(s)_k - r_{d-1}(s)_k = 0$. —

**Claim 4.14.3.**

$$[1] \prod_{i=1}^{m} (N - i + 1)^{\lambda_i} R_{N-i+1,N-i+1}^{\lambda_i} \left( \frac{\partial_{N-i+1} F(\bar{X}_{N-i+1})}{N-i+1} \right)(1) = \sum_{s \in T \cap T_2^3(\mathfrak{r})} r(s)_k.$$ 

**Proof.** The left hand side of the above expression is

$$\sum_{s \in T} r_m(s)_k = \sum_{s \in T} r(s)_k = \sum_{s \in T \cap T_2^3(\mathfrak{r})} r(s)_k,$$

since, by Claim 4.14.2, $r_m(s)_k = r(s)_k$ for $s \in T$, and $r(s)_k \neq 0$ if and only if $s \in T \cap T_2^3(\mathfrak{r})$. —

Suppose that $T'$ is the set of $s \in T_2^3(\mathfrak{r})$ satisfying the Lemma 4.12 conditions. From

$$[1] N^k \prod_{i=1}^{m} Q_{N-i+1}^N \left( \frac{\partial_{N-i+1} F}{N} \right)^{\lambda_i}(1) = \sum_{s \in T'} r(s)_k + R_1,$$

where $R_1$ is a homogeneous polynomial in the $c'_{\mathfrak{r}}$ with degree $k$ and order $N^{k-1}$. 

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Claim 4.14.4. The set $T'$ is a subset of $T$.

Proof. It suffices to show that if $s \in T'$, for $d > \lambda_1 + \cdots + \lambda_j$, $s_d$ does not contain $x_i$, $N - j + 1 \leq i \leq N$. For the sake of contradiction, assume $s \in T'$ and $s_d$ contains $x_i$, $N - j + 1 \leq i \leq N$, for some $d > \lambda_1 + \cdots + \lambda_j$. From the conditions of Lemma 4.12, $x_i$ must be in a term multiplication. After, $x_i$ will be in all terms of $r(s)_d$. But, $x_i$ cannot be removed by an operator unless all the terms become 0, a contradiction to $s \in T'$.

Since $T' \subset T_N^2(r)$ as well, $T' \subset T \cap T_N^2(r)$. Again, we consider the coefficient of

$$p = \prod_{i=1}^{m} c_{\mu_i}^p.$$ 

Let $U$ and $U'$ be the set of $s$ in $T_N^2(r)$ and $T'$, respectively, such that $r(s)_k = d p$, $d \neq 0$. Then, $U \cap T$ will be the set of $s$ in $T \cap T_N^2(r)$ such that $r(s)_k = dp$, $d \neq 0$. We show that $|U \setminus U'|$ is $O(N^{k-m-1})$; where $U' \subset U \cap T$, $|U \cap T \setminus U'|$ will be $O(N^{k-m-1})$ as well. But, as in Lemma 4.12, this follows from the proof of Lemma 4.10 without derivates and using changes instead of switches. After, with Proposition 4.6,

$$\sum_{s \in T \cap T_N^2(r)} r(s)_k = \sum_{s \in T'} r(s)_k + R_2,$$

where $R_2$ is a homogeneous polynomial in the $c_{\mu_i}^p$ with degree $k$ and order $N^{k-1}$, completing the proof. ■

4.6 Proof of Theorem 1.5

Let $\lambda$ be a partition of length $m$ with $|\lambda| \geq 1$. Suppose $r = \{i_j\}_{1 \leq j \leq |\lambda|}$, where the first $\lambda_1$ indices are 1, the next $\lambda_2$ are 2, and so forth, until the last $\lambda_m$ are $m$. For Theorem 1.5, we look at the limit of (26) for $\mu = \mu_N$ as $N \to \infty$. From Proposition 4.1 (26), and (28),

$$E_{\mu_N} \left( \prod_{i=1}^{m} \mathcal{P}_{\lambda_i}^N \right) = \frac{1}{N^{m+|\lambda|}} \left( \prod_{i=1}^{m} \mathcal{P}_{\lambda_i} \right) \exp(F_N(x_1, \ldots, x_N)) \bigg|_{x_i=0, 1 \leq i \leq N}$$

$$= \frac{1}{N^{m+|\lambda|}} \left( \prod_{i=1}^{m} \mathcal{P}_{\lambda_i} \right) \exp(\tilde{F}_N(x_1, \ldots, x_N)) \bigg|_{x_i=0, 1 \leq i \leq N}$$

$$= \frac{1}{N^{m+|\lambda|}} \sum_{l \in I_N(\lambda)} [1] \mathcal{D}_l(\tilde{F}_N(x_1, \ldots, x_N)).$$

From now on in the proof, we set $F_N(x_1, \ldots, x_N)$ as $\tilde{F}_N(x_1, \ldots, x_N)$.

Note that the results hold for $F_N(x_1, \ldots, x_N)$ because $F_N(x_1, \ldots, x_N)$, a symmetric polynomial, is a symmetric formal series. By Proposition 4.9 as $N \to \infty$,

$$\lim_{N \to \infty} \left( \frac{1}{N^{m+|\lambda|}} \sum_{l \in I_N(\lambda)} [1] \mathcal{D}_l(F_N(x_1, \ldots, x_N)) \right) = \lim_{N \to \infty} \frac{1}{N^{|\lambda|}} [1] \mathcal{D}_l(x_1, \ldots, x_N),$$

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using \((32)\) with \(k = |\lambda| - 1\). Afterwards, from Theorem 4.13 and Theorem 4.14
\[
\lim_{N \to \infty} \frac{1}{N^{|\lambda|}} \mathbb{1}_{\mathcal{D}_i(x_1, \ldots, x_N)} = \lim_{N \to \infty} \frac{1}{N^{|\lambda|}} \left( \prod_{i=1}^{m} \left( \mathcal{R}^{\lambda_i}_{N-i+1,N-i+1} \left( \frac{\partial_{N-i+1} F_N(X_{N-i+1})}{N-i+1} \right) \right) \right),
\]
where \((32)\) is used again.

To evaluate this, note that
\[
s_{1,N-i+1} \frac{\partial_{N-i+1} F_N(X_{N-i+1})}{N-i+1} = \frac{\partial_{1} F_N(X_{N-i+1})}{N-i+1},
\]
and the coefficient of \(x_1^d M_{\nu}(x_2, \ldots, x_{N-i+1})\) in \((35)\) is \(\frac{(d+1)c_{\nu}^{\ell+1}+1}{N-i+1}\) for \(\nu \in P\) with \(\ell(\nu) \leq N-i\). Here, we can have that \(\nu = 0\). But, from \((25)\),
\[
\lim_{N \to \infty} \frac{(d+1)c_{\nu}^{\ell+1}+1}{N-i+1} = \frac{(d+1)(\ell(\nu) + 1)c_{\nu}^{\ell+1}+1}{|\nu| + d + 1},
\]
and the limiting sequence of \((35)\) outside of 1 for \(1 \leq i \leq m\) is \(f = \{c_{\nu}^{d,\nu}\}_{d \geq 0, \nu \in P}\), where \(c_{d,\nu} = (d+1)(\ell(\nu) + 1)c_{\nu}^{\ell+1}+1\). From Theorem 3.13,
\[
\lim_{N \to \infty} \frac{1}{N^{|\lambda|}} \left( \prod_{i=1}^{m} \left( \mathcal{R}^{\lambda_i}_{N-i+1,N-i+1} \left( \frac{\partial_{N-i+1} F_N(X_{N-i+1})}{N-i+1} \right) \right) \right) = \prod_{\pi \in \mathcal{NC}(\lambda)} \prod_{B \in \pi} c_{|B|}(f),
\]
From this, \(\{\mu_N\}_{N \geq 1}\) satisfy a LLN, where
\[
\lim_{N \to \infty} \mathbb{E}_{\mu_N} \left( \prod_{i=1}^{m} p_{\pi_i}^{N} \right) = \prod_{i=1}^{m} \mu_{\lambda_i}, m_k = \sum_{\pi \in \mathcal{NC}(k)} \prod_{B \in \pi} c_{|B|}(f).
\]
for \(k \geq 1\). Suppose that for \(\nu \in P\), for \(d \geq 1\), \(N_d(\nu)\) of the components of \(\nu\) are \(d\). Then, if \(\nu' + (d) = \nu\), \(\ell(\nu) P(\nu') = N_d(\nu) P(\nu)\), and for \(l \geq 1\),
\[
c_l(f) = \theta^{l-1} \sum_{\nu' \in P, \nu' + (d) = \nu} \left( -1 \right)^{\ell(\nu')} P(\nu') \frac{(d+1)(\ell(\nu') + 1)c_{\nu' + (d+1)}}{|\nu'| + d + 1},
\]
which gives \((4)\), as required.
5 Eigenvalue Distribution of Random Matrices

In [AGZ09, Section 4.5], random tridiagonal matrices are given with the eigenvalue distribution of the \( \beta \)-Hermite ensemble and density \( d_{N,\beta}(x_1, \ldots, x_N) \) for all \( N \geq 1 \). We can apply Theorem 1.5 to the \( \beta \)-Hermite ensemble scaled by \( \sqrt{N} \), with density \( d_{N,\beta}(x_1, \ldots, x_N) \).

**Proposition 5.1** ([Cue19, Corollary 3.7]). Let \( a_1, \ldots, a_N, y_1, \ldots, y_N, c \) be \( 2N+1 \) arbitrary complex numbers. Then,

\[
B_{(ca_1, \ldots, ca_N)}(y_1, \ldots, y_N) = B_{(a_1, \ldots, a_N)}(cy_1, \ldots, cy_N).
\]

**Proposition 5.2.** Suppose \( \mu \in \mathcal{M}_N \) is exponentially decaying. For \( c > 0 \), let \( c\mu \) be \( \mu \) scaled by \( c \). Then,

\[
G_\theta(x_1, \ldots, x_N; c\mu) = G_\theta(cx_1, \ldots, cx_N; \mu).
\]

**Proof.** The probability density of \( c\mu \) is \( c\mu(a_1, \ldots, a_N) = \frac{1}{c^N} \mu\left(\frac{a_1}{c}, \ldots, \frac{a_N}{c}\right) \). With Proposition 5.1 where \( a' = (a'_1 \leq \cdots \leq a'_N) \), \( a'_i = \frac{a_i}{c} \), \( 1 \leq i \leq N \),

\[
G_\theta(x_1, \ldots, x_N; c\mu) = \int_{a=(a_1 \leq \cdots \leq a_N)} \ldots \int_{a=(a_1 \leq \cdots \leq a_N)} \ldots \int_{a=(a_1 \leq \cdots \leq a_N)} \ldots \int_{a=(a_1 \leq \cdots \leq a_N)} B_a(x_1, \ldots, x_N; \theta) c\mu(a_1, \ldots, a_N) da_1 \cdots da_N = G_\theta(cx_1, \ldots, cx_N; \mu).
\]

**Proposition 5.3.** Suppose \( \theta > 0 \) is fixed. Consider the sequence \( \{\mu_N\}_{N \geq 1} \) of probability measures such that for all positive integers \( N \), \( \mu_N \in \mathcal{M}_N \) and \( \mu_N \) has probability density \( d_{N,2\theta}^\kappa \) from (6). Then, \( \{\mu_N\}_{N \geq 1} \) satisfy a LLN, with \( m_{2k-1} = 0 \) and

\[
m_{2k} = \frac{1}{k+1} \binom{2k}{k}
\]

for \( k \geq 1 \).

**Proof.** For \( N \geq 1 \), consider \( \frac{\mu_N}{\sqrt{N}} \), which has density \( d_{N,2\theta}^{\sqrt{N}} \) in (5). It is easy to see that \( \mu_N \) and \( \frac{\mu_N}{\sqrt{N}} \) are exponentially decaying. From [Cue19, Proposition 4.2],

\[
G_\theta \left( x_1, \ldots, x_N; \frac{\mu_N}{\sqrt{N}} \right) = \exp \left( \frac{1}{2\theta} \sum_{i=1}^N x_i^2 \right). \tag{36}
\]

Then, from Proposition 5.2 and (36),

\[
G_\theta(x_1, \ldots, x_N; \mu_N) = G_\theta \left( \sqrt{N}x_1, \ldots, \sqrt{N}x_N; \frac{\mu_N}{\sqrt{N}} \right) = \exp \left( \frac{N}{2\theta} \sum_{i=1}^N x_i^2 \right).
\]
After this, from Theorem 1.5, we know that \( \{\mu_N\}_{N \geq 1} \) satisfy a LLN with \( c_{(2)} = \frac{1}{7} \) and \( c_\nu = 0 \) for other \( \nu \in P, |\nu| \geq 1 \). For \( k \geq 1, m_{2k-1} = 0 \) and \( m_{2k} = |T(k)| \), where \( T(k) \) is the set of \( \pi \in NC(2k) \) such that \( \pi \) has all blocks of size 2. But, we see that
\[
|T(k)| = C_k = \frac{1}{k+1} \binom{2k}{k},
\]
where \( C_k \) is the \( k \)th Catalan number.

From Proposition 5.3, if \( (a_1 \leq \cdots \leq a_N) \) are distributed with density \( d_{N,\beta}(x_1, \ldots, x_N) \), for positive integer \( s \) and positive integers \( k_i, 1 \leq i \leq s \),
\[
\lim_{N \to \infty} \mathbb{E}_{\mu_N} \left( \prod_{i=1}^{s} \left( \frac{1}{N} \sum_{i=1}^{N} \left( \frac{a_i}{\sqrt{N}} \right)^{k_i} \right) \right) = \prod_{i=1}^{s} m_{k_i},
\]
with the moments \( m_k \) given in Proposition 5.3. Sometimes, as seen above, for sequences \( \{\mu_N\}_{N \geq 1} \) of exponentially decaying probability measures, for \( N \geq 1 \), we can scale \( \mu_N \) by an appropriate power of \( N \) and use Proposition 5.2 to satisfy the conditions of Theorem 1.5.

### A Combinatorial Formulas

**Lemma A.1.** For all nonnegative integers \( a, b, \) and \( m \) such that \( m \leq a \),
\[
\sum_{k=m}^{a} \frac{\binom{a}{k} \binom{b}{k-m}}{\binom{a+b}{k}} (-1)^k = \frac{(-1)^m a! b!}{(a+b)!}. \tag{37}
\]

**Proof.** We use induction on \( m \), from \( m = a \) to 0. The \( m = a \) base case is clear. Suppose the statement holds for \( m + 1, 0 \leq m < a \). We show the statement holds for \( m \) with induction on \( b \), where the base case \( b = 0 \) is clear. For the inductive step, assume (37) for \( m \) holds for \( b \geq 0 \). We want to show (37) for \( m \) holds for \( b + 1 \). Note that
\[
\begin{align*}
\sum_{k=m}^{a} \frac{\binom{a}{k} \binom{b+1}{k-m}}{(a+b+1)} (-1)^k &= \sum_{k=m}^{a} \frac{\binom{a}{k} \binom{b}{k-m}}{(a+b)} \cdot \frac{b+1}{a+b+1} \cdot \frac{(a+b+1) - k}{b+1 - k} \cdot (-1)^k \\
&= \frac{(-1)^m a! (b+1)!}{(a+b+1)!} + \sum_{k=m}^{a} \frac{\binom{a}{k} \binom{b}{k-m}}{(a+b)} \cdot \frac{a-m}{b+1-k+m} (-1)^k \\
&= \frac{(-1)^m a! (b+1)!}{(a+b+1)!} + \frac{a-m}{b+1} \cdot \sum_{k=m}^{a} \frac{\binom{a}{k} \binom{b+1}{k-m}}{(a+b)} (-1)^k \\
&= \frac{(-1)^m a! (b+1)!}{(a+b+1)!} + \frac{a-m}{b+1} \cdot \left( \sum_{k=m}^{a} \frac{\binom{a}{k} \binom{b}{k-m}}{(a+b)} (-1)^k + \sum_{k=m+1}^{a} \frac{\binom{a}{k} \binom{b}{k-m}}{(a+b)} (-1)^k \right) \\
&= \frac{(-1)^m a! (b+1)!}{(a+b+1)!}
\end{align*}
\]
with the inductive hypotheses, which completes the proof.

Lemma A.2. For a positive integer $k$,

$$\sum_{\pi \in NC(k+1), \pi = B_1 \sqcup \cdots \sqcup B_m} \prod_{i=2}^{m} b_{|B_i|} \cdot \prod_{i=2}^{m} a_{|B_i|} = \sum_{\pi \in NC(k), \pi = B_1 \sqcup \cdots \sqcup B_m} \left( b_{|B_1|+1} + \sum_{j=1}^{|B_1|} a_j b_{|B_1|+1-j} \right) \cdot \prod_{i=2}^{m} a_{|B_i|}. $$

Proof. Suppose that for $k \geq 1$, $NC'(k) = NC(\{1,3,4,\ldots,k+1\})$. For $\pi \in NC'(k)$, define the set $S(\pi)$ of partitions $\lambda$ in $NC(k+1)$ as follows. The set $S(\pi)$ consists of partitions $\lambda$ such that $\lambda$ is $\pi$ with 2 added to $B_1$, or $\lambda$ is $\pi$ with $B'_1$ and $B''_1$ instead of $B_1$, where $B'_1$ is the last $j-1$ elements of $B_1$ with 1, and $B''_1$ is the remaining elements of $B_1$ with 2 added, where $1 \leq j \leq |B_1|$. Clearly, $S(\pi) \subset NC(k+1)$.

Claim A.2.1. The union of the $S(\pi)$ for $\pi \in NC'(k)$ is $NC(k+1)$ and the $S(\pi)$ are disjoint for $\pi \in NC'(k)$.

Proof. For $\lambda \in NC(k+1)$, suppose the decomposition into blocks is

$$C_1 \sqcup C_2 \sqcup \cdots \sqcup C_m,$$

following the same conditions as the decomposition in (18). Note that $2 \in C_1$ or $2 \in C_2$.

If $2 \in C_1$, let $\pi = B_1 \sqcup \cdots \sqcup B_m \in NC'(k)$ have $B_i = C_i$ for $2 \leq i \leq m$ and $B_1 = C_1 \setminus \{2\}$. We see that $\pi$ with 2 added to $B_1$ is $\lambda$, so $\lambda \in S(\pi)$. Suppose $\lambda \in S(\pi)$, $\pi = B_1 \sqcup \cdots \sqcup B_m \in NC'(k)$. However, $C_1 \subset NC'(k)$, or $\lambda$ is noncrossing. Thus, $\pi$ is unique, and $\lambda$ is not in $S(\pi)$ for more than one $\pi \in NC'(k)$.

Suppose that $2 \in C_2$. Then, let $\pi = B_1 \sqcup \cdots \sqcup B_{m-1} \in NC'(k)$ have $B_{i-1} = C_i$ for $3 \leq i \leq m$. Since $\lambda$ is noncrossing, $\pi$ is also noncrossing. Note that $1 \in C_1$ and $2 \in C_2$. Suppose $j_1 \in C_1$ and $j_2 \in C_2$, where $j_1, j_2 > 2$. Because $\lambda$ is noncrossing, we must have that $j_1 > j_2$. Therefore, the elements of $C_1$ which are not 1 are greater than all of the elements of $C_2$. We see that, where $1 \leq j = |C_1| \leq |B_1|$, $C_1$ is the last $j-1$ elements of $B_1$ with 1 added, and $C_2$ is the remaining elements of $B_1$ with 2 added. Therefore, $\lambda \in S(\pi)$.

Assume $\lambda \in S(\pi)$, where $\pi = B_1 \sqcup \cdots \sqcup B_m \in NC'(k)$. Since $\lambda$ does not have 2 in the first block, $\lambda$ is $\pi$ with $B'_1$ and $B''_1$ instead of $B_1$, where $B'_1$ is the last $j-1$ elements of $B_1$ with 1 added, and $B''_1$ is the remaining elements of $B_1$ with 2 added, for some $1 \leq j \leq |B_1|$. Then, the first and second block of $\lambda$ would be $C_1 = B'_1$ and $C_2 = B''_1$, respectively. But, in $\pi$,

$$B_1 = (B'_1 \sqcup B''_1)/\{2\} = (C_1 \cup C_2)/\{2\},$$

and $B_i = C_{i+1}$ for $i \geq 2$. With this, $\pi$ is unique and $\lambda$ is not in $S(\pi)$ for more than one $\pi \in NC'(k)$.

\[36\]
For a finite, nonempty set $S$ of real numbers and partition $\pi = B_1 \sqcup \cdots \sqcup B_m \in NC(S)$, let
\[ W_S(\pi) = b_{|B_1|} \cdot \prod_{i=2}^{m} a_{|B_i|}. \]

With Claim A.2.1,
\[ \sum_{\pi \in NC(k+1)} W_{L_{k+1}}(\pi) = \sum_{\pi \in NC'(k)} \sum_{\lambda \in S(\pi)} W_{L_{k+1}}(\lambda) \]
\[ = \sum_{\pi \in NC'(k), \pi = B_1 \sqcup \cdots \sqcup B_m} \left( b_{|B_1|+1} + \sum_{j=1}^{|B_1|} a_j b_{|B_1|+1-j} \right) \cdot \prod_{i=2}^{m} a_{|B_i|}, \]
giving Lemma A.2 since $|\{1, 3, 4, \ldots, k+1\}| = |\{1, 2, 3, \ldots, k\}| = k$. We are done. ■

Lemma A.3. For all positive integers $n$,
\[ \sum_{\pi \in NC(n)} \prod_{B \in \pi} r_{|B|} = \frac{1}{n+1} \cdot \left[ z^{-1} \right] \left( \frac{1}{z} + \sum_{j=1}^{\infty} r_j z^{j-1} \right)^{n+1}. \quad (38) \]

Proof. For integers $m_1, m_2, \ldots, m_n$ such that $m_k \geq 0$ for $1 \leq k \leq n$ and $\sum_{k=1}^{n} k \cdot m_k = n$, let $N(m_1, m_2, \ldots, m_n)$ denote the number of $\pi \in NC(n)$ with $m_k$ blocks of size $k$, $1 \leq k \leq n$. From [Kre72],
\[ N(m_1, m_2, \ldots, m_n) = \frac{n!}{(n-b+1)! \prod_{k=1}^{n} m_k!}, \quad (39) \]
where $b = \sum_{k=1}^{n} m_k$. Then,
\[ \sum_{\pi \in NC(n)} \prod_{B \in \pi} r_{|B|} = \sum_{d, \sum_{k=1}^{n} k \cdot m_k = n} \frac{n!}{d!} \cdot \prod_{k=1}^{n} \frac{r_{m_k}^{m_k}}{m_k!} \quad (40) \]
We have that the right hand side of (38) is
\[ \frac{1}{n+1} \cdot \sum_{j_i \geq 0, 1 \leq i \leq n+1, \sum_{i=1}^{n+1} j_i = n} n+1 \prod_{i=1}^{n+1} r_{j_i} = \sum_{\sum_{k=0}^{n} m_k = n+1, \sum_{k=1}^{n} k \cdot m_k = n} \frac{n!}{m_0!} \cdot \prod_{k=1}^{n} \frac{r_{m_k}^{m_k}}{m_k!}, \]
which is (40) with $m_0$ as $d$. We are done. ■
B  Polynomial Coefficients

Lemma B.1. Suppose the indices are \( r = \{i_j\}_{1 \leq j \leq k} \). For a term

\[
p = \prod_{i=1}^{m} c_{F_{i_{j}}}^{\nu_{i_{j}}},
\]

there exists a polynomial \( f(x) \) such that the coefficient of \( p \) in \([1]D_{x_{i_{j}}}(F(x_{1}, \ldots, x_{N}))\) is \( f(N) \) for \( N \geq \max(r) \).

Proof. Suppose \( N \geq \max(r) \). If the sum of the \( |\nu_{i_{j}}| \) is not \( k \), from Corollary 4.3, the coefficient is \( f(N) = 0 \). Now, suppose that the sum of the \( |\nu_{i_{j}}| \) is \( k \). Let \( S \) be the set of \( x_{i_{j}} \) and \( T \) be the set of \( s \in T_{N}^{1}(r) \) such that \( r(s)_{k} = dp \) for a nonzero \( d \). Then, the ending coefficient of \( p \) will be

\[
\sum_{s \in T} C(s).
\]

Also, for \( s \in T \), let \( X(s) \) be the set \( x_{i_{j}} \) such that \( i_{j} \notin r \) and \( x_{i_{j}} \) appears in switches or term multiplications of \( s \). Since each \( x_{i_{j}} \in X(s) \) from term multiplications must be eliminated by at least one switch by Claim 4.7.1 and each other \( x_{i_{j}} \in X(s) \) appears in at least one switch, because there are at most \( k \) switches, \( |X(s)| \leq k \).

For each \( A \subset \{x_{1}, \ldots, x_{N}\} \setminus S \), where the number of \( s \in T \) such that \( X(s) = A \) is finite, let

\[
g(A) = \sum_{s \in T, X(s) = A} C(s).
\]

Suppose \( D \) is a positive integer, \( 0 \leq D \leq k \). By symmetry, \( g(A) \) is the same for all \( A \subset \{x_{1}, \ldots, x_{N}\} \setminus S \), such that \( |A| = D \). Let this value be \( r_{D} \). Note that there are \( \binom{N-|S|}{D} \) possible \( A \), while \( r_{D} \) is independent of \( N \). Since \( 0 \leq |X(s)| \leq k \), we see that the coefficient of \( p \) is \( f(N) \) for

\[
f(x) = \sum_{0 \leq D \leq k} r_{D} \cdot \binom{x - |S|}{D},
\]

where \( \binom{N-|S|}{D} \) is the number of \( A \) such that \( |A| = D \). We are done. \( \blacksquare \)

Theorem B.2. Suppose \( \{F_{N}(x_{1}, \ldots, x_{N})\}_{N \geq 1} \) is a sequence of symmetric formal series. Assume that for all partitions \( \nu \) with \( |\nu| \geq 1 \), there exists a real number \( c_{\nu} \) such that

\[
\lim_{N \to \infty} \frac{1}{N} \cdot [1] \frac{\partial}{\partial x_{i_{1}}} \cdots \frac{\partial}{\partial x_{i_{r}}} F_{N}(x_{1}, \ldots, x_{N}) = \frac{\ell(\nu)(|\nu| - 1)!c_{\nu}}{P(\nu)}
\]

for all positive integers \( i_{1}, \ldots, i_{r} \) such that \( \sigma((i_{1}, \ldots, i_{r})) = \nu \). Then, for a partition \( \lambda \) with \( |\lambda| \geq 1 \),
\[
\lim_{N \to \infty} \left( \frac{1}{N^{\ell(\lambda)+|\lambda|}} \sum_{i \in I_N(\lambda)} [1]D_i(F_N(x_1, \ldots, x_N)) \right) = \prod_{i=1}^{\ell(\lambda)} \left( \sum_{\pi \in NC(\lambda)_i} \prod_{B \in \pi} \theta^{|B|-1} \left( \sum_{\nu \in P, |\nu|=|B|} (-1)^{\ell(\nu)-1} P(\nu) c^{(\nu)}_{p,\nu} \right) \right).
\]

**Proof.** The same proof as the proof of Theorem 1.5 can be used. \hfill\Box

Suppose \( k \) is a positive integer. Let \( S_k \) be the set of all multisets \( q \) of partitions with size at least 1 such that the sum of \( |\nu| \) for \( \nu \in q \) is \( k \). Additionally, let

\[
\mathcal{P}_k = \left\{ \prod_{\nu \in q} c^{(\nu)}_F \mid q \in S_k \right\},
\]

where if

\[
p = \prod_{\nu \in q} c^{(\nu)}_F,
\]

the length of \( p \) is \( \ell(p) = |q| \).

For indices \( r \) of length \( k \), for \( p \) in \( \mathcal{P}_k \), let the coefficient of \( p \) in \([1]D_r(F(x_1, \ldots, x_N))\) be \( f_{p,r}(N) \) for \( N \geq \max(r) \). Note that for \( N \geq \max(r) \),

\[
[1]D_r(F(x_1, \ldots, x_N)) = \sum_{p \in \mathcal{P}_k} f_{p,r}(N)p
\]

from Lemma \[B.1\]. Also, from Proposition \[4.7\], \( f_{p,r}(x) \) has degree at most \( k - \ell(p) \). Let the coefficient of \( x^{k-\ell(p)} \) in \( f_{p,r}(x) \) be \( s_r(p) \). Note that \( s_r(p) \), which can be 0, is the leading order coefficient of \( f_{p,r}(x) \).

**Proposition B.3.** For \( k \geq 1 \), consider indices \( r = \{i_j\}_{1 \leq j \leq k} \). Where \( \lambda = \sigma(r) \) and \( m = \ell(\lambda) \),

\[
\sum_{p \in \mathcal{P}_k} s_r(p)p = \prod_{i=1}^{m} \left( \sum_{\pi \in NC(\lambda)_i} \prod_{B \in \pi} \theta^{|B|-1} \left( \sum_{\nu \in P, |\nu|=|B|} (-1)^{\ell(\nu)-1} |\nu| P(\nu) c^{(\nu)}_F \right) \right).
\]

**Proof.** Consider the indices \( r' \) of length \( k \), where the first \( \lambda_1 \) indices are 1, the next \( \lambda_2 \) are 2, and so forth, until the last \( \lambda_m \) indices are \( m \). From commutativity of the \( D_i \) and symmetry in the \( x_i, D_i(F(x_1, \ldots, x_N))|_{x_i=0,1 \leq i \leq N} = D_{i'}(F(x_1, \ldots, x_N))|_{x_i=0,1 \leq i \leq N} \) for all \( N \geq \max(r) \). Therefore, \( f_{p,r}(x) = f_{p,r'}(x) \) and \( s_r(p) = s_{r'}(p) \) for all \( p \in \mathcal{P}_k \).

Suppose \( c^{(\nu)}_F \) is a real for all \( \nu \in P, |\nu| \geq 1 \) and \( F_N(x_1, \ldots, x_N) \) has \( c^{(\nu)}_{F_N} = Nc^{(\nu)}_F \) for \( \nu \in P \) with \( 1 \leq \ell(\nu) \leq N \) and \( c^{(\nu)}_{F_N} = 0 \) for \( \nu \in P \) with \( \ell(\nu) \geq N+1 \). Then, with
\( a_\nu(N) = c_{\nu,F}^p \), from Proposition 4.8, Proposition 4.9, and Theorem B.2 for \( \lambda \),

\[
\lim_{N \to \infty} \frac{1}{N^k} [1] \mathcal{D}^p(F_N(x_1, \ldots, x_N)) = \lim_{N \to \infty} \left( \frac{1}{N^{k+m}} \sum_{\ell \in I_N(\lambda)} [1] \mathcal{D}_\ell(F_N(x_1, \ldots, x_N)) \right)
\]

\[
= \prod_{i=1}^m \left( \sum_{\pi \in \mathcal{NC}(\lambda_i)} \prod_{B \in \pi} \theta^{\text{card}(B)} \left( \sum_{\nu \in \mathcal{P}, |\nu| = |B|} \left( -1 \right)^{\ell(\nu)-1} \frac{|\nu| P(\nu)}{\ell(\nu)} c_{\nu,F}^p \right) \right),
\]

where \( c_{\nu} = \frac{|\nu|}{\ell(\nu)} c_{\nu,F}^p \). However, substituting \( c_{\nu,F}^p = N c_{\nu,F}^p \) for \( c_{\nu,F}^p \) in (41) gives

\[
\frac{1}{N^k} [1] \mathcal{D}^p(F_N(x_1, \ldots, x_N)) = \sum_{p \in \mathcal{P}_k} \frac{f_{p,e}(N)p}{N^{k-\ell(p)}},
\]

where the \( c_{\nu,F}^p \) and thus \( p \in \mathcal{P}_k \) are constants. Since \( f_{p,e}(x) \) has degree at most \( k - \ell(p) \) from Proposition 4.7, \( \lim_{N \to \infty} \frac{f_{p,e}(N)}{N^{k-\ell(p)}} = s_{e,p} = s_t(p) \). Taking the limit \( N \to \infty \) gives

\[
\lim_{N \to \infty} \frac{1}{N^k} [1] \mathcal{D}^p(F_N(x_1, \ldots, x_N)) = \sum_{p \in \mathcal{P}_k} s_{e,p} = \sum_{p \in \mathcal{P}_k} s_t(p).\]

Since the above expression is a finite polynomial in the \( c_{\nu,F}^p \), varying the \( c_{\nu,F}^p \) to be any reals gives the result. \( \square \)
References


