HOMOTOPY GROUPS OF THE FIBERS OF THE CHROMATIC RESOLUTION OF DETERMINANT TWISTS OF $Q(\ell)$

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CHRIS XU
MENTOR: ROBERT BURKLUND
PROJECT SUGGESTED BY: WEI ZHANG

Abstract. The $E(2)$-local ring spectrum $Q(\ell)$ was introduced by Behrens and used in [Beh08] to associate generators $\beta_{i/j,k}$ of the 2-line of the Adams-Novikov spectral sequence with certain p-adic modular forms $f_{i/j,k}$ of weight $t+j(p-1)$. Associated to $Q(\ell)$ are *twists* $Q_d(\ell)$ parametrized by $d \in \mathbb{Z}$, which can be though of as invertible $Q(\ell)$-module spectra. In this paper, we compute parts of the 0-, 1- and 2-line of the chromatic resolution of $Q_d(\ell)$, of which computations of $Q(\ell) = Q_0(\ell)$ in [Beh08] were instrumental in producing the association between β elements and modular forms. In the future, we hope to obtain analogous results and shed more light on the $E(2)$-local sphere.

1. Introduction

Fix a prime p. The Adams-Novikov spectral sequence

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*) \Rightarrow \pi_{t-s}(S)_{(p)}$$

yields information about the p-components of the stable homotopy groups of spheres.

It is well-known that the 1-line is generated by elements

$$\alpha_{i/j} \in \mathrm{Ext}^{1,2(p-1)i-1}_{BP_*BP}(BP_*, BP_*)$$

of order p^j, where $i \geq 1$ and $j = v_p(i) + 1$. The following theorem characterizes $\alpha_{i/j}$.

Theorem 1.1. *Let $t := (p-1)i$. Then there is a bijection between generators $\alpha_{i/j}$ and Bernoulli numbers B_t. In particular, the order of $\alpha_{i/j}$ equals the p-factor of the denominator of B_t/t, where B_t denotes the tth Bernoulli number.*

A classical computation by Miller, Ravenel and Wilson in [MRW77] shows that for $p \geq 5$, the 2-line is generated by elements

$$\beta_{i/j,k} \in \mathrm{Ext}^{2,2i(p^2-1)-2j(p-1)}_{BP_*BP}(BP_*, BP_*)$$

of order p^k, where $i = sp^n$, $p \nmid s$, and:

- $p^{k-1} \mid j$.
- $j \leq p^{n-k+1} + p^{n-k} - 1$.
- If $p^k \mid j$ then $j > p^{n-k} + p^{n-k-1} - 1$.

Work by Behrens in [Beh08] shows that, much in the same vein as the 1-line, the generators $\beta_{i/j,k}$ are in a similar correspondence with p-adic modular forms:

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Theorem 1.2. *Each generator $\beta_{i/j,k}$ can be associated with a weight t modular form $f_{i/j,k} \in M_t$, where $t = i(p^2-1)$ and $f_{i/j,k}$ satisfies:*

- *We have $f_{i/j,k}(q) \not\equiv 0 \bmod p$.*
- *We have $ord_q f_{i/j,k}(q) > \frac{t-j(p-1)}{12}$ or equal to $\frac{t-j(p-1)-2}{12}$.*
- *The form $f_{i/j,k}(q)$ has the lowest possible weight out of all modular forms reducing to the reduction of $f_{i/j,k}$ modulo p^k.*
- *For any prime $\ell \neq p$, there exists a form $g_\ell \in M_{t-j(p-1)}(\Gamma_0(\ell))$ such that*

$$f_{i/j,k}(q^\ell) - f_{i/j,k}(q) \equiv g_\ell \bmod p^k.$$

The main machinery used to prove theorem 1.2 is the introduction of a certain ring spectrum denoted $Q(\ell)$, which is $E(2)$-local; then, parts of the 0- through 2-line of the chromatic resolution of $Q(\ell)$ (denoted $M_0Q(\ell)$, $M_1Q(\ell)$ and $M_2Q(\ell)$) are computed.

The $E(2)$-locality of $Q(\ell)$ also induces the natural map

$$S_{E(2)} \to Q(\ell)$$

from the unit map $S \to Q(\ell)$. Work by Behrens in the same paper yields the following:

Theorem 1.3. *In the natural map*

$$\pi_* S_{E(2)} \to \pi_* Q(\ell)$$

the images of all $\alpha_{i/j}$ and $\beta_{i/j,k}$ are nontrivial. Therefore, the spectrum $Q(\ell)$ detects Greek letter phenomena from the 1- and 2-lines of the Adams-Novikov spectral sequence.

The goal of this paper is to perform similar computations to [Beh08] for *twists* $Q_d(\ell)$ of $Q(\ell)$, which can be thought of as a class of $Q(\ell)$-module spectra invertible under smash product. In particular, we have

$$Q_d(\ell) \wedge_{Q(\ell)} Q_{-d}(\ell) \cong Q(\ell).$$

In this paper, we compute parts of $\pi_t M_0 Q_d(\ell), \pi_t M_1 Q_d(\ell)$ and $\pi_t M_2 Q_d(\ell)$. Namely, we compute the following:

Theorem 1.4. *Let ℓ, p be primes such that $p \geq 5$ and ℓ is a topological generator of $\mathbb{Z}_p^\times$. Further let $t, d \in \mathbb{Z}$. Then the following statements hold for $\pi_t M_i Q_d(\ell)$:*

(1) *We have*

$$\pi_t M_0 Q_d(\ell) = \begin{cases} 0 & t \neq 4d, 4d-1, 4d-2 \\ \mathbb{Q}_p & t = 4d. \end{cases}$$

(2) *We have*

$$\pi_t M_1 Q_d(\ell) = \begin{cases} 0 & t \not\equiv 0, -1, -2 \bmod 2(p-1) \text{ or } p-1 \nmid d \\ \mathbb{Z}/p^k\mathbb{Z} & \begin{matrix} t =: 4d + 2(p-1)p^{i-1}s \\ d =: (p-1)p^{i'-1}s' \end{matrix} \quad k := \min(i', i). \end{cases}$$

(3) *Let t be even. Then there is a bijection between the additive order p^k generators of $\pi_{2t} M_2 Q_d(\ell)$ and $f \in (M_{t+j(p-1)})^0_{\mathbb{Z}_p}$ for $p^{k-1} \mid j$ such that*
   (a) *$(p-1)p^{k-1}$ divides $t - 2d$.*
   (b) *$ord_q f(q) \in \left\{\frac{t-2}{12}\right\} \cup \left(\frac{t}{12}, \infty\right)$.*

(c) *p does not divide $f(q)$.*
(d) *The weight of f is the smallest possible weight for its reduction modulo p^k.*
(e) *There is $g \in M_t(\Gamma_0(\ell))^0_{\mathbb{Z}_p}$ such that $\ell^d f(q^\ell) - f(q) \equiv g(q) \bmod p^k$.*

Our hope is that these computations will shed more light on $\pi_* S_{E(2)}$ in the future, and in particular the 1- and 2-lines of the Adams-Novikov spectral sequence.

1.1. **Layout of the paper.** The layout of this paper is as follows:

Section 2 introduces the ring spectrum $Q(\ell)$ and its twists $Q_d(\ell)$ in terms of the totalization of a three-term semicosimplicial set $Q_d(\ell)^\bullet$. Work from [Beh08, Sec. 5-7] is then adapted to compute $\pi_t M_n Q(\ell)$ for $0 \le n \le 2$ in terms of the cohomologies of the three-term chain complex $C_d(\ell)^\bullet$ arising from applying homotopy to $Q_d(\ell)^\bullet$. In particular, the differentials in $C_d(\ell)^\bullet$ are computed.

Sections 3, 4 and 5 compute $\pi_t M_n Q(\ell)$ for certain values of t and for $0 \le n \le 2$, respectively. In the future, we hope to compute $\pi_t M_n Q(\ell)$ for all values of t, which would require looking at the first and second cohomologies of $C_d(\ell)$.

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2. The setup

2.1. **Topological modular forms.** Fix primes p, ℓ such that $p \ge 5$ and ℓ is a topological generator of $\mathbb{Z}_p^\times$. Define

$$\hat{\mathbb{Z}}^S := \prod_{p \notin S} \mathbb{Z}_p$$
$$\mathbb{A}^{S,\infty} := \hat{\mathbb{Z}}^S \otimes \mathbb{Q}$$

We briefly recall the setup of TMF in [Beh08].

Definition 2.1. Let $\mathcal{C}$ be the category of compact open subgroups of $\mathrm{GL}_2(\mathbb{A}^{p,\infty})$. Then we define TMF (standing for *topological modular form*) to be a certain functor

$$\mathrm{TMF}\colon \mathcal{C} \to \mathbf{RingSpectra}.$$

For more information on TMF, see [Beh08, Sec. 3-4].

Letting

$$K_0(\ell) := \left\{ A \in \mathrm{GL}_2(\mathbb{Z}_\ell)\colon A \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \bmod \ell \right\}$$
$$K_0^p := \mathrm{GL}_2(\hat{\mathbb{Z}}^p)$$
$$K_0^p(\ell) := \mathrm{GL}_2(\hat{\mathbb{Z}}^{p,\ell}) K_0(\ell)$$

we define

$$\mathrm{TMF}_p := \mathrm{TMF}(K_0^p)$$
$$\mathrm{TMF}_0(\ell)_p := \mathrm{TMF}(K_0^p(\ell))$$

The relevant fact here is the following statement of the homotopy groups of TMF_p and $\mathrm{TMF}_0(\ell)_p$:

Theorem 2.2. *The graded homotopy rings of* TMF_p *and* $\mathrm{TMF}_0(\ell)_p$ *are precisely the graded rings of modular forms over* $\mathbb{Z}_p$ *for* $\Gamma_0(1)$ *and* $\Gamma_0(\ell)$*, respectively, with weight the* k *modular forms in each ring concentrated in dimension* $2k$ *for all* k*. In particular:*

$$\begin{aligned} \pi_{2k}(\mathrm{TMF}_p) &= (M_k)^0_{\mathbb{Z}_p} \\ \pi_{2k}(\mathrm{TMF}_0(\ell)_p) &= (M_k(\Gamma_0(\ell))^0_{\mathbb{Z}_p}. \end{aligned}$$

2.2. **The ring spectrum $Q(\ell)$.** In [Beh05, Part 1], the spectrum $Q(\ell)$ is introduced. Much of the exposition here is adapted from there as well as [Beh08, Sec. 4-5].

Definition 2.3. The spectrum $Q(\ell)$ is defined to be the totalization $\mathrm{Tot}\ Q(\ell)^\bullet$ of the semicosimplicial set

$$Q(\ell)^\bullet := \left(\begin{array}{ccccc} & & \mathrm{TMF}_p & \longrightarrow & \\ \mathrm{TMF}_p & \rightrightarrows & \times & \longrightarrow & \mathrm{TMF}_0(\ell)_p. \\ & & \mathrm{TMF}_0(\ell)_p & \longrightarrow & \end{array} \right)$$

with certain face maps d, of which $\pi_*(d)$ will be later addressed in proposition 2.9 and corollary 2.12. Additionally, there are natural multiplication and unit maps which make $Q(\ell)$ a ring spectrum.

To compute $\pi_* Q(\ell)$, we may take the homotopy of $Q(\ell)^\bullet$ and then apply the Bousfield-Kan spectral sequence as per [BK72]. More specifically, taking homotopy of $Q(\ell)^\bullet$ yields

$$C(\ell)^\bullet_{2k} := \left(\begin{array}{ccccc} & & (M_k)^0_{\mathbb{Z}_p} & \longrightarrow & \\ (M_k)^0_{\mathbb{Z}_p} & \rightrightarrows & \times & \longrightarrow & (M_k(\Gamma_0(\ell))^0_{\mathbb{Z}_p}. \\ & & (M_k(\Gamma_0(\ell))^0_{\mathbb{Z}_p} & \longrightarrow & \end{array} \right)$$

Remark 2.4. We can interpret $C(\ell)^{\cdot}_k$ as a three-term cohomological chain complex whose differentials are the alternating sum of the corresponding maps. In particular, $H^s(C(\ell)_\bullet) = 0$ for $s \neq 0, 1, 2$.

Remark 2.5. Since M_k and $M_k(\Gamma_0(\ell))$ are 0 for odd k (this follows from $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \Gamma_0(\ell)$), we have $C(\ell)^\bullet_t = 0$ for $t \neq 0$ mod 4.

Now [BK72] states that there is an associated convergent spectral sequence given by

$$E_1^{s,t} = C(\ell)^s_t \Rightarrow \pi_{t-s} Q(\ell).$$

In our case, we note that in the second page of E, the differentials given by $d_2 \colon E_2^{s,t} \to E_2^{s+2,t-1}$ are all 0 by remark 2.4. Furthermore, all higher page differentials vanish by remark 2.5. Therefore, the spectral sequence collapses to the second page. Thus:

$$\begin{aligned} \pi_n Q(\ell) &= \bigoplus_{t-s=n} E_2^{s,t} \\ &= H^0(C(\ell)_n) \oplus H^1(C(\ell)_{n+1}) \oplus H^2(C(\ell)_{n+2}) \end{aligned}$$

By remark 2.5 we obtain the following:

Proposition 2.6. *We have $\pi_n Q(\ell) = H^s(C(\ell)_{4k})$ where $0 \leq s < 4$ is such that $4 \mid n+s$, and $k := (n+s)/4$.*

2.3. **Face maps.** In [Beh08, Sec. 6] and [Beh05], Behrens describes another way to obtain $C(\ell)$, by pulling back from the semisimplicial set of moduli stacks of elliptic curves

$$\mathcal{M}_\bullet := \left((\mathcal{M}_{ell})_p \leftleftarrows \begin{matrix} \mathcal{M}(\Gamma_0(\ell))_p \\ \sqcup \\ (\mathcal{M}_{ell})_p \end{matrix} \begin{matrix} \longleftarrow \\ \longleftarrow \\ \longleftarrow \end{matrix} \mathcal{M}(\Gamma_0(\ell))_p. \right)$$

where $(\mathcal{M}_{ell})_p$ is the moduli stack of elliptic curves C over $\mathbb{Z}_{(p)}$, and $\mathcal{M}(\Gamma_0(\ell))_p$ is the moduli stack over $\mathbb{Z}_{(p)}$ of elliptic curves (C, H) with *level ℓ structure*.

Remark 2.7. Informally, a level ℓ structure is a pair (C, H) where C is an elliptic curve and H is an order ℓ subgroup.

The face maps are interpreted on R-points as follows. An R-point of $\mathcal{M}(\Gamma_0(\ell))_p \sqcup \mathcal{M}_{ell})_p$ is a morphism $C \to C/H$ (for $(C, H) \in \mathcal{M}(\Gamma_0(\ell))_p(R)$) or $C \to C/C[\ell]$ (for $C \in \mathcal{M}_{ell})_p(R)$). The top map $d_0 \colon \mathcal{M}_1 \to \mathcal{M}_0$ takes a morphism to its target, while the bottom map d_1 takes a morphism to its source.

Remark 2.8. Similarly, the R-points of $\mathcal{M}_2$ may be interpreted as the chain $C \to C/H \to C/C[\ell]$, and then d_0, d_1 and $d_2 \colon \mathcal{M}_2 \to \mathcal{M}_1$ omit the first, middle and last objects of the chain, respectively.

From our determination of the face maps, we may then obtain $C(\ell)^\bullet$ by pulling back on the ring of modular forms for each $\mathcal{M}_i$. We note the following.

Proposition 2.9. *The face maps $d_0, d_1 \colon \mathcal{M}_1 \to \mathcal{M}_0$ pull back to coface maps*

$$(M_k)^0_{\mathbb{Z}_p} \rightrightarrows \begin{matrix} (M_k)^0_{\mathbb{Z}_p} \\ \times \\ (M_k(\Gamma_0(\ell))^0_{\mathbb{Z}_p} \end{matrix}$$

given by

$$f \mapsto (\ell^k f(q^\ell), \ell^k f(q))$$
$$f \mapsto (f(q), f(q))$$

respectively. Hence the differential is given by

$$f \mapsto (\ell^k f(q^\ell) - f(q), \ell^k f(q) - f(q)).$$

Proof. See [Beh08, Prop. 6.2]. □

2.4. $Q_d(\ell)$**: determinant twists of** $Q(\ell)$**.** Since $Q(\ell)$ is a ring spectrum, there is a natural notion of invertible $Q(\ell)$-module spectra: precisely the $Q(\ell)$-module spectra M for which there exists M' such that $M \wedge M' = Q(\ell)$.

One important class of invertible spectra are *suspension spectra*, which gives rise to the usual graded homotopy ring $\pi_* Q(\ell)$. However, it turns out there is another class known as the *determinant twists* of $Q(\ell)$, parametrized by $d \in \mathbb{Z}$. We will denote these twists by $Q_d(\ell)$, and the corresponding semicosimplicial homotopy groups of $Q_d(\ell)^\bullet$ as $C_d(\ell)^\bullet$.

Definition 2.10. The *homotopy graded Picard group* of a ring spectrum A, denoted $\operatorname{Pic} A$, is the abelian group of invertible A-module spectra.

There is a natural injection

$$H := \mathbb{Z} \oplus \mathbb{Z} \to \operatorname{Pic} Q(\ell)$$

where $(t, d) \in H$ is such that t is the suspension parameter and d is the determinant twist parameter.

The spectrum $Q_d(\ell)$ is the totalization of $Q_d(\ell)^\bullet$, which has the same objects as $Q(\ell)^\bullet$ but whose maps are "twisted" by some parameter depending on d. It, too, arises from the pullback of a semisimplicial object $M(d)_\bullet$ whose objects are the same as $M_\bullet$ but whose maps are different. The effect of this twisting on $C_d(\ell)_\bullet$ is described in the following proposition.

Proposition 2.11. *Let $d_{i,j} \colon \mathcal{M}_i \to \mathcal{M}_{i-1}$ be a coface map in $\mathcal{M}_\bullet$, and let r be the rank of the corresponding map of elliptic curves. (For example, if $d_{i,j}$ takes (C, H) to C/H, then $r = \ell$, the rank of H.) Then, in $C_d(\ell)^\bullet$, the map induced by pulling back on $d_{i,j}$ is precisely the corresponding map in $C(\ell)$ multiplied by r^{-d}.*

Therefore:

Corollary 2.12. *The map $M_k \to M_k \times M_k(\Gamma_0(\ell))$ in $C_d(\ell)^\bullet$ is precisely the alternating sum given by $f \mapsto (\ell^{k-2d} f(q) - f(q), \ell^{k-d} f(q^\ell) - f(q))$.*

Proof. Follows evidently from proposition 2.11 and proposition 2.9. □

2.5. **The chromatic resolution of $Q_d(\ell)$.** This section is largely adapted from [Beh08, Sec. 7], as many of the facts of $Q(\ell)$ remain true for $Q_d(\ell)$. We note the following fact:

Proposition 2.13. *The spectra TMF_p and $\mathrm{TMF}_0(\ell)_p$ are $E(2)$-local.*

Since $Q_d(\ell)^\bullet$ is a semicosimplicial set from TMF_p and $\mathrm{TMF}_0(\ell)_p$, we obtain the following corollary:

Corollary 2.14. *The spectrum $Q_d(\ell)$ is $E(2)$-local.*

Therefore, the chromatic resolution of $Q_d(\ell)$ stops at $E(2)$:

$$\begin{tikzcd}
M_0 Q_d(\ell) \arrow[d, equal] & M_1 Q_d(\ell) \arrow[d] & M_2 Q_d(\ell) \arrow[d] & \\
Q_d(\ell)_{E(0)} & Q_d(\ell)_{E(1)} \arrow[l] & Q_d(\ell)_{E(2)} \arrow[l] \arrow[r, equal] & Q_d(\ell)
\end{tikzcd}$$

(For more information on the chromatic resolution of a general spectrum, see [Beh08, p. 2.1].) We note the following fact about homotopy groups of $M_i Q_d(\ell)$, adapting the arguments from the proof of [Beh08, Cor. 7.7]:

Proposition 2.15. *The following isomorphisms hold for $Q_d(\ell)$:*

$$\pi_t M_0 Q_d(\ell) \cong H^s(C_d(\ell)^\bullet[p^{-1}])_{4k}$$
$$\pi_t M_1 Q_d(\ell) \cong H^s(C_d(\ell)^\bullet/(p^\infty)[v_1^{-1}])_{4k}$$
$$\pi_t M_2 Q_d(\ell) \cong H^s(C_d(\ell)^\bullet/(p^\infty, v_1^\infty))_{4k}$$

where $k \in \mathbb{Z}$ is such that $4k$ is the smallest multiple of 4 greater than t, and $s := 4k - t$.

3. Computing $\pi_t M_0 Q_d(\ell)$

Here we compute

$$\pi_t M_0 Q_d(\ell) \cong H^s(C_d(\ell)^\bullet[p^{-1}])_{4k}.$$

Analogous to [Beh08, Prop 8.1], we have the following:

Proposition 3.1. *The cohomology $H^s(C_d(\ell)_\bullet)_{2t}$ is p^j-torsion if $t \equiv 2d \bmod (p-1)p^{j-1}$ and in particular 0 if $t \not\equiv 2d \bmod p-1$.*

Proof. Since the objects in $Q(\ell)^\bullet$ remain the same in $Q_d(\ell)^\bullet$, the central element $[\ell] \in \mathrm{GL}_2(\mathbb{Q}_\ell)$ still acts as the identity on $Q_d(\ell) \cong \mathcal{V}_d^{(K_0^{p,\ell})_+}$. In fact, everything in the proof remains the same, except that the action of $[\ell]$ on $\pi_*(\mathrm{TMF}(K^p))$ is now given by

$$\begin{aligned} [\ell] \colon \pi_{2k}\,\mathrm{TMF}(\Gamma_0(N)) &\to \pi_{2k}\,\mathrm{TMF}(\Gamma_0(N)) \\ f &\mapsto \ell^{k-2d} f \end{aligned}$$

since the induced map of elliptic curves is the ℓth power map, which has rank ℓ^2 (thus we divide by ℓ^{-2d}). We deduce that multiplication by $\ell^{k-2d}-1$ is the zero homomorphism on $H^s(C_d(\ell)_\bullet)_{2k}$. Since ℓ topologically generates $\mathbb{Z}_p^\times$, it follows that $k \equiv 2d \bmod (p-1)p^{j-1}$ implies that $\ell^{k-2d}-1 \in p^j\mathbb{Z}_p^\times$, so $p^j = 0$ in H^s. This yields the desired result. □

Since tensoring with $\mathbb{Z}[1/p]$ kills p^∞-torsion, we have the following corollary:

Corollary 3.2. *The homotopy groups of $M_0 Q_d(\ell)$ are*

$$\pi_t M_0 Q_d(\ell) = \begin{cases} 0 & t \neq 4d, 4d-1, 4d-2 \\ H^s(C_d(\ell)^\bullet[p^{-1}])_{4d} & t = 4d-s,\ 0 \le s < 3. \end{cases}$$

In particular, when $s = 0$ in the second case, we obtain

Proposition 3.3. *We have the isomorphism $\pi_{4d} M_0 Q_d(\ell) \cong \mathbb{Q}_p$.*

Proof. We compute H^0, the kernel of the coface map for $k = 4d$. In this case, the coface map is given by

$$\begin{aligned} (M_{2d})^0_{\mathbb{Q}_p} &\to (M_{2d})^0_{\mathbb{Q}_p} \times M_{2d}(\Gamma_0(\ell))^0_{\mathbb{Q}_p} \\ f &\mapsto (\ell^{2d-2d} f(q) - f(q), \ell^{2d-d} f(q^\ell) - f(q)) = (0, \ell^d f(q^\ell) - f(q)). \end{aligned}$$

The condition for f to lie in H^0 therefore reduces down to

$$\ell^d f(q^\ell) - f(q) = 0$$

or in other words, writing $f := \sum a_n q^n$:

$$a_n = \begin{cases} \ell^d a_{n/\ell} & \ell \mid n \\ 0 & \text{o.w.} \end{cases}$$

This forces a_n for positive n to equal 0, while a_0 can vary over $\mathbb{Q}_p$. Hence the 0th cohomology group $H^s(C_d(\ell)^\bullet)_{4d}$ is isomorphic to $\mathbb{Q}_p$. □

Putting this all together yields theorem 1.4(1):

Theorem 3.4. *We have*

$$\pi_t M_0 Q_d(\ell) = \begin{cases} 0 & t \neq 4d, 4d-1, 4d-2 \\ \mathbb{Q}_p & t = 4d \\ H^1(C_d(\ell)^\bullet[p^{-1}])_{4d} & t = 4d-1 \\ H^2(C_d(\ell)^\bullet[p^{-1}])_{4d} & t = 4d-2 \end{cases}$$

4. Computing $\pi_t M_1 Q_d(\ell)$

Here we compute

$$\pi_t M_1 Q_d(\ell) \cong H^s(C_d(\ell)^\bullet/(p^\infty)[v_1^{-1}])_{4k} \qquad (t = 4k-s).$$

Define

$$\mathcal{A}_{t/j} := H^0(C_d(\ell)/(p^j))_{2t}$$
$$\mathcal{A}_{t/\infty} := \varinjlim_j \mathcal{A}_{t/j}$$

Then

$$\mathcal{A}_{t/j} = \left\{ f \in (M_t)^0_{\mathbb{Z}/p^j\mathbb{Z}} \colon \begin{array}{c} \ell^{t-d} f(q^\ell) \equiv f(q) \bmod p^j \\ (\ell^{t-2d}-1) \equiv 0 \bmod p^j \end{array} \right\}$$

Writing $f := \sum a_n q^n$, we see that the first condition is equivalent to

$$(\ell^d - 1)a_0 \equiv 0 \bmod p^j$$
$$a_n \equiv \begin{cases} 0 & \ell \nmid n \\ \ell^d a_{n/\ell} & \ell \mid n \end{cases}$$

forcing all a_i to 0 for i positive. Combining this information with $(\ell^d-1)a_0 \equiv 0$ and the second condition, we obtain

$$\mathcal{A}_{t/j} = \left\{ f \in (M_t)^0_{\mathbb{Z}/p^j\mathbb{Z}} \colon \begin{array}{c} f(q) \equiv a \bmod p^j \text{ and } v_p(a) + \min(i, v_p(\ell^d-1) \geq j \\ \text{for } i \text{ } s.t. \text{ } t = 2d + (p-1)p^{i-1}s \end{array} \right\}$$

and in particular $\mathcal{A}_{t/j} = 0$ if $t \neq 2d \bmod p-1$ or $p-1 \nmid d$. Note that this forces $p-1$ to divide both d and t for $\mathcal{A}_{t/j}$ to be nonzero. From here on out, assume $p-1 \mid d, t$.

For $t \in \{4, 6, 8, \dots\}$, let $E_t \in (M_t)^0_{\mathbb{Q}}$ denote the weight t Eisenstein series

$$E_t(q) = 1 - \frac{2t}{B_t} \sum_{i \geq 1} \sigma_{k-1}(i) q^i.$$

We recall a classical lemma in p-adic modular forms.

Lemma 4.1. *If $p-1 \mid t$, then E_t is p-integral, so "reduction mod p^j" makes sense. Moreover, if we can write t as $t =: (p-1)p^{j-1}s$, then we have $E_t \equiv 1 \bmod p^j$.*

Proof. See [Kat72]. □

Note that $k := \min(i, v_p(\ell^{d-1}))$ is the smallest value of j for which $\mathcal{A}_{t/j}$ can possibly be all of $\mathbb{Z}/p^j\mathbb{Z}$; here, we are allowed to have $v_p(a) \geq 0$, i.e. a can range along anything in $\mathbb{Z}/p^k\mathbb{Z}$. It is now a matter of showing that all possible $a \in \mathbb{Z}/p^k\mathbb{Z}$ can be hit by the image of M_t. Let $i' := v_p(\ell^{d-1} - 1)$; then we may write

$$\begin{aligned} t &= 2d + (p-1)p^{i-1}s \\ d &= (p-1)p^{i'-1}s'. \end{aligned}$$

Then $v_p(t) \geq \min(v_p(d), v_p(t)) = k - 1$; so we may write t as $(p-1)p^{k-1}s''$. The lemma above implies that $E_t \equiv 1 \bmod p^k$, so $\mathcal{A}_{t/k} \cong \mathbb{Z}/p^k\mathbb{Z}$. Condition (1) means that for no $j > k$ will it be possible for 1 to lie in $\mathcal{A}_{t/j}$; hence $\mathcal{A}_{t/\infty} \cong \mathbb{Z}/p^k\mathbb{Z}$. Adjoining v_1^{-1} doesn't do anything:

$$H^0(C_d(\ell)/(p^\infty)[v_1^{-1}])_{2t} \cong \begin{cases} \mathbb{Z}/p^k\mathbb{Z} & k = \min(i', i) \\ 0 & p-1 \nmid d \text{ or } p-1 \nmid t. \end{cases}$$

We arrive at theorem 1.4(2):

Theorem 4.2. *We have*

$$\pi_t M_1 Q_d(\ell) = \begin{cases} 0 & t \not\equiv 0, -1, -2 \bmod 2(p-1) \text{ or } p-1 \nmid d \\ \mathbb{Z}/p^k\mathbb{Z} & \begin{matrix} t =: 4d + 2(p-1)p^{i-1}s \\ d =: (p-1)p^{i'-1}s' \end{matrix} \quad k := \min(i', i). \end{cases}$$

5. Computing $\pi_t M_2 Q_d(\ell)$

We have

$$\begin{aligned} H^0(C_d(\ell)^\bullet/(p^\infty, v_1^\infty))_{2t} &= \varinjlim_k \varinjlim_{j = sp^{k-1}} H^0(C_d(\ell)^\bullet/(p^k, v_1^j))_{2t+2j(p-1)} \\ &=: \varinjlim_k \varinjlim_{j=sp^{k-1}} \mathcal{B}_{t/j,k}. \end{aligned}$$

Then a series of computations shows that

$$\mathcal{B}_{t/j,k} = \left\{ f \in \frac{(M_{t+j(p-1)})^0_{\mathbb{Z}/p^k\mathbb{Z}}}{(M_t)^0_{\mathbb{Z}/p^k\mathbb{Z}}} : \begin{matrix} (\ell^{t+j(p-1)-2d} - 1)f(q) = g_1(q) \text{ for } g_1 \in (M_t)^0_{\mathbb{Z}/p^k\mathbb{Z}} \\ \ell^{t+j(p-1)-d} f(q^\ell) - f(q) = g_2(q) \text{ for } g_2 \in M_t(\Gamma_0(\ell))^0_{\mathbb{Z}/p^k\mathbb{Z}} \end{matrix} \right\}$$

where the embedding $M_t \to M_{t+j(p-1)}$ is given by multiplication by $E_{j(p-1)}$, since $E_{j(p-1)} \equiv 1 \bmod p^k$ from the above lemma.

Proposition 5.1. *We have a split short exact sequence*

$$0 \longrightarrow (M_t)^0_{\mathbb{Z}/p^k\mathbb{Z}} \xrightarrow{\cdot E^j_{p-1}} (M_{t+j(p-1)})^0_{\mathbb{Z}/p^k\mathbb{Z}} \longrightarrow \frac{(M_{t+j(p-1)})^0_{\mathbb{Z}/p^k\mathbb{Z}}}{(M_t)^0_{\mathbb{Z}/p^k\mathbb{Z}}} \longrightarrow 0$$

Proof. See [Beh08, Lem. 11.4]. □

Let $r_{j,k}$ and $\iota_{j,k}$ denote the retractions and sections

$$(M_{t+j(p-1)})^0_{\mathbb{Z}/p^k\mathbb{Z}} \xrightarrow{r_{j,k}} (M_t)^0_{\mathbb{Z}/p^k\mathbb{Z}}$$

$$\frac{(M_{t+j(p-1)})^0_{\mathbb{Z}/p^k\mathbb{Z}}}{(M_t)^0_{\mathbb{Z}/p^k\mathbb{Z}}} \xrightarrow{\iota_{j,k}} (M_{t+j(p-1)})^0_{\mathbb{Z}/p^k\mathbb{Z}}$$

of the split short exact sequence in proposition 5.1.

Proposition 5.2. *The image of $\iota_{j,k}$ is given by*

$$\left\{ f \in (M_{t+j(p-1)})^0_{\mathbb{Z}/p^k\mathbb{Z}} \colon \mathit{ord}_q f(q) > \frac{t}{12} \text{ or equal to } \frac{t-2}{12} \right\}.$$

Proof. See [Beh08, Lem. 11.6]. □

We now state and prove theorem 1.4(3), a statement on the structure of $\pi_t M_2 Q_d(\ell)$, analogous to [Beh08, Thm. 11.3].

Theorem 5.3. *There is a bijection between the additive order p^k generators of $H^0(C_d(\ell)^\bullet/(p^\infty, v_1^\infty))_{2t}$ and $f \in (M_{t+j(p-1)})^0_{\mathbb{Z}_p}$ for $p^{k-1} \mid j$ such that*

1. *$(p-1)p^{k-1}$ divides $t-2d$.*
2. *$\mathit{ord}_q f(q) \in \left\{\frac{t-2}{12}\right\} \cup \left(\frac{t}{12}, \infty\right)$.*
3. *p does not divide $f(q)$.*
4. *The weight of f is the smallest possible weight for its reduction modulo p^k.*
5. *There is $g \in M_t(\Gamma_0(\ell))^0_{\mathbb{Z}_p}$ such that $\ell^d f(q^\ell) - f(q) \equiv g(q) \bmod p^k$.*

Proof. We have a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & (M_t)^0_{\mathbb{Z}/p^k\mathbb{Z}} \arrow[r] & (M_{t+j(p-1)})^0_{\mathbb{Z}/p^k\mathbb{Z}} \arrow[r] \arrow[l, bend right, dotted, "r_{j,k}"'] & \frac{(M_{t+j(p-1)})^0_{\mathbb{Z}/p^k\mathbb{Z}}}{(M_t)^0_{\mathbb{Z}/p^k\mathbb{Z}}} \arrow[r] \arrow[l, bend right, dotted, "\iota_{j,k}"'] & 0 \end{tikzcd}$$

Let $b \in \mathcal{B}_{t/j,k}$ have order $p^{k'}$ for $k' < k$. Then $\iota_{j,k}(b)$ has order $p^{k'}$, so $\iota_{j,k}(b) = p^{k-k'} f$. Taking the image of f mod M_t yields an element b' of order p^k in $\mathcal{B}_{t/j,k}$. Hence, every additive generator of order p^k lies in $\mathcal{B}_{t/j,k}$.

A generator $b \in \mathcal{B}_{t/j,k}$ of order p^k lifts to $f \in M_{t+j(p-1)}$ such that

1. $(\ell^{t+j(p-1)-2d} - 1) f(q) \equiv g_1(q) \bmod p^k$ for $g_1(q) \in (M_t)^0_{\mathbb{Z}/p^k\mathbb{Z}}$.
2. $\ell^{t+j(p-1)-d} f(q^\ell) - f(q) \equiv g_2(q) \bmod p^k$ for $g_2(q) \in M_t(\Gamma_0(\ell))^0_{\mathbb{Z}/p^k\mathbb{Z}}$.

Since $p^{k-1} \mid j$, conditions (i) and (ii) implies

$$(\ell^{t-2d} - 1) f(q) \equiv g_1(q) \bmod p^k$$
$$\ell^{t-d} f(q^\ell) - f(q) \equiv g_2(q) \bmod p^k$$

From condition (i) we obtain

$$f(q) \equiv \frac{g_1(q)}{\ell^{t-2d} - 1} \bmod p^{k-v} \qquad (v := v_p(\ell^{t-2d} - 1))$$

But f mod p^{k-v} is the image $\iota_{j,k-v}(b'')$ where $b'' \in \mathcal{B}_{t/j,k-v}$ is the reduction of b mod p^{k-v}; hence $r_{t/j,k-v}(f) \equiv 0 \bmod p^{k-v}$. But since $g_1(q) \in (M_t)^0_{\mathbb{Z}/p^k\mathbb{Z}}$, we have

$$\begin{aligned} \frac{g_1(q)}{\ell^{t-2d} - 1} &\equiv r_{t/j,k-v}\left(\frac{g_1(q)}{\ell^{t-2d} - 1}\right) \\ &\equiv r_{t/j,k-v}(f) \equiv 0 \bmod p^{k-v} \end{aligned}$$

which implies

$$g_1(q) \equiv 0 \bmod p^k.$$

So condition (i) is

$$(\ell^{t-2d} - 1) f(q) \equiv 0 \bmod p^k$$

which implies $(p-1)p^{k-1}$ divides $t-2d$ (condition (1) of the theorem). Meanwhile, condition (ii) is condition (5) of the theorem statement. Conditions (2)-(4) of the theorem follow evidently.

Conversely, for $f \in (M_{t+j(p-1)})^0_{\mathbb{Z}/p^k\mathbb{Z}}$ satisfying conditions (1)-(5), the lemma implies f is in the image of $\iota_{j,k}$, so it reduces to

$$b \in \frac{(M_{t+j(p-1)})^0_{\mathbb{Z}/p^k\mathbb{Z}}}{(M_t)^0_{\mathbb{Z}/p^k\mathbb{Z}}}.$$

By (2), b has order p^k, and b lies in $\mathcal{B}_{t/j,k}$ since Conditions (1) and (5) imply (i) and (ii). This completes the proof. □

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Department of Mathematics, MIT, Cambridge, MA 02139
Email address: chxu@mit.edu