A homomorphism between elliptic Hall algebra and K-theoretic algebra of surfaces

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Abstract

In this paper, we construct an algebra homomorphism from the positive part of the elliptic Hall algebra to the K-theoretic Hall algebra of surfaces.

1 Introduction

Let S be a smooth projective surface over $\mathbb{C}$. Quot and Flag schemes can be defined on S, which parametrize certain coherent sheaves on S of finite lengths. Let $K_G(X)$ denote the G-equivariant K-theory group of coherent sheaves on X and $Quot_n^\circ$ the moduli space of framed length-n coherent sheaves on S. According to [7], the following graded abelian group

$$K(Quot) = \bigoplus_{n\geq 1} K_{\mathrm{GL}_n}(Quot_n^\circ),$$

can be equipped with a graded algebra structure. This is called the K-theoretic Hall algebra of Quot schemes on S.

Moreover, we also consider the (positive part of) elliptic Hall algebra $\mathcal{A}_{>0}$, which can be explicitly defined by generators and relations. The main result of this paper is the following theorem:

Theorem 1.1. *There exists a homomorphism of algebras $\mathcal{A}_{>0} \to K(Quot)$.*

The motivation of this theorem originates from several previous results. In [4], Nakajima studied the cohomology of Hilbert scheme of points and proved that it carries a Heisenberg algebra action. Generalizing Nakajima's result in [5], Negut studied the moduli space of stable sheaves on certain surfaces and proved that its K-theory group carries an elliptic Hall algebra action. In another perspective, in [6], Schiffmann and Vasserot studied the equivariant K-theory of Hilbert scheme of $\mathbb{A}^2$ and showed that its convolution algebra is isomorphic to the elliptic Hall algebra.

The structure of this paper is as follows. In section 2, we introduce Quot, Flag schemes and study some geometric properties of moduli spaces related to length-2, length-3 coherent sheaves on surfaces. In section 3, we introduce the equivariant K-theory and K-theoretic Hall algebra of Quot schemes of surfaces. We also construct K-theory classes $e_{(d_1,\ldots,d_n)} \in K_{\mathrm{GL}_n}(Quot_n^\circ)$ corresponding to certain generators $E_{(d_1,\ldots,d_n)}$ of $\mathcal{A}_{>0}$ and compute the commutator relations $[e_{(k)}, e_{(d_1,\ldots,d_n)}]$ for the case $n = 1, 2$. In section 4, we prove Theorem 1.1 based on commutator relations computed in section 3.

Here are some notations in this paper. Let $\mathcal{M}$ be a certain Quot or Flag scheme. $\mathcal{M}^\circ$ denotes the open subscheme of $\mathcal{M}$ where every coherent sheaf $\mathcal{F}$ appeared in quotients or flags is *framed*, i.e., equipped with a framing $\mathcal{O}^{\oplus I} \to \mathcal{F}$ (I is an index set). If H is a closed subgroup of a reductive group G and H acts on $\mathcal{M}$, we often use $\widetilde{\mathcal{M}}$ to denote the orbit space $\mathcal{M} \times_H G$. In the case when $G = \mathrm{GL}_n$ and H is a parabolic subgroup of G, $\mathcal{M} \times_H G$ will be a quotient scheme $(G \times \mathcal{M})/H$ (see Chapter 5.2 of [2]). $\mathbb{C}_x$ denotes the length-1 skyscraper sheaf supported at a single closed point x. "$\twoheadrightarrow$" denotes a surjection map.

2 Moduli spaces and their geometry

Let S be a projective smooth surface over $\mathbb{C}$. For any positive integer d, define $Quot_d$ to be the moduli spaces of length-d coherent sheaves on S. Explicitly, $Quot_d$ has the following functor-of-points description. For any scheme T, there is a one-to-one correspondence between maps $T \to Quot_d$ and the following data:

- A quotient $\mathcal{O}_{S\times T}^{\oplus d} \twoheadrightarrow \mathcal{E}_d$, where $\mathcal{E}_d \in Coh(S \times T)$[1] is flat of length d over T. In other words, for every point $t \in T$, $\mathcal{E}_d|_t$ is a length-d coherent sheaf on $S \times \{t\}$.

There is an open subscheme $Quot_d^\circ \subset Quot_d$, which has T-points corresponding to the above data, with an extra constraint

- The induced map $\mathcal{O}_T^{\oplus d} \to \mathrm{pr}_{T*}\,\mathcal{E}_d$ is an isomorphism, where $\mathrm{pr}_T : S \times T \to T$ is the projection.

Example 2.1. As a basic example, we have $Quot_1 = Quot_1^\circ \cong S$, since a length-1 coherent sheave on S corresponds uniquely to its support (which is a closed point of S).

For a sequence of positive integers $d_\bullet = (d_1, \ldots, d_k)$ with $0 < d_1 < \cdots < d_k$, define the moduli space of flags $Flag_{d_\bullet}^\circ$ to be a subscheme of $Quot_{d_k}^\circ$, such that the set of T-points (i.e., maps from T to $Flag_{d_\bullet}^\circ$) is in bijection with the following data:

[1]Here, $Coh(X)$ denotes the set of coherent sheaves on a scheme X.

- A flag of quotients

$$\begin{array}{ccccccccc}
\mathcal{O}_{S\times T}^{\oplus d_k} & \twoheadrightarrow & \mathcal{O}_{S\times T}^{\oplus d_{k-1}} & \twoheadrightarrow & \cdots & \twoheadrightarrow & \mathcal{O}_{S\times T}^{\oplus d_1} & \twoheadrightarrow & 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \\
\mathcal{E}_{d_k} & \twoheadrightarrow & \mathcal{E}_{d_{k-1}} & \twoheadrightarrow & \cdots & \twoheadrightarrow & \mathcal{E}_{d_1} & \twoheadrightarrow & 0
\end{array}$$

  where $\mathcal{E}_{d_i}$ is flat of length d_i over T.

- The induced map $\mathcal{O}_T^{\oplus d_i} \to \mathrm{pr}_{T*}\,\mathcal{E}_{d_i}$ is an isomorphism for all i.

When $T = Flag^\circ_{d_\bullet}$, the identity map corresponds to a universal flag of quotients over $S \times Flag^\circ_{d_\bullet}$, which we often denote as $\mathcal{U}_{d_k} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{U}_{d_1}$. The pushforward of $\mathcal{U}_{d_i}$ along the projection $S \times Flag^\circ_{d_\bullet} \to Flag^\circ_{d_\bullet}$ is a vector bundle of rank d_i on $Flag^\circ_{d_\bullet}$. Abusing notations, we will also denote this vector bundle $\mathcal{U}_{d_i}$.

In the later sections, we will use the notation $Flag^\circ_{m_k,m_{k-1},\ldots,m_1}$ to denote the scheme $Flag^\circ_{d_\bullet}$ defined in the previous paragraph, where $d_\bullet = (m_1, m_1 + m_2, \ldots, m_1 + \cdots + m_k)$. For example, $Flag^\circ_{m,n} = Flag^\circ_{(n,n+m)}$.

Let $P_{n,m}$ denote the parabolic subgroup of the general linear group GL_{n+m} consisting of lower-triangular block matrices of the form

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}, \text{ where } A \in \mathrm{Mat}_{n\times n}, D \in \mathrm{Mat}_{m\times m}, C \in \mathrm{Mat}_{m\times n}.$$

We observe that $P_{n,m}$ acts on $Flag^\circ_{m,n}$ by acting (in the functor-of-points description) on $\mathcal{O}_{S\times T}^{\oplus(m+n)}$ as well as its quotient $\mathcal{O}_{S\times T}^{\oplus n}$. Furthermore, as a subgroup of GL_{m+n}, the right action of $P_{n,m}$ on GL_{m+n} is free. As a result, $P_{n,m}$ acts on $\mathrm{GL}_{m+n} \times Flag^\circ_{m,n}$ by $h \cdot (g, x) = (gh^{-1}, hx)$. Since this action is free and proper, its orbit space $\mathrm{GL}_{m+n} \times_{P_{n,m}} Flag^\circ_{m,n}$ is equipped with a scheme structure: namely, the quotient $(Flag^\circ_{m,n} \times \mathrm{GL}_{m+n})/P_{n,m}$. We shall denote this scheme as $\widetilde{Flag}^\circ_{m,n}$.

Proposition 2.2. *$\widetilde{Flag}^\circ_{m,n}$ has the following functor-of-points description. For any scheme T, the set of T-points of $\widetilde{Flag}^\circ_{m,n}$ is in bijection with the following data:*

(i) $\mathcal{O}_{S\times T}^{\oplus(n+m)} \twoheadrightarrow \mathcal{E}_{n+m} \twoheadrightarrow \mathcal{E}_n$, *where $\mathcal{E}_i$ is flat of length i over T for $i \in \{n, n+m\}$.*

(ii) *Let $\mathrm{pr}_T : S\times T \to T$, then the data above induce an isomorphism $\mathcal{O}_T^{\oplus(n+m)} \cong \mathrm{pr}_{T*}\,\mathcal{E}_{n+m}$.*

Proof. A T-point of $(Flag^\circ_{m,n} \times \mathrm{GL}_{m+n})/P_{n,m}$ consists of the data

- A T-point of $Flag^\circ_{m,n}$, i.e., a flag of quotients

$$\begin{array}{ccccc}
\mathcal{O}_{S\times T}^{\oplus(n+m)} & \twoheadrightarrow & \mathcal{O}_{S\times T}^{\oplus n} & \twoheadrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathcal{E}_{n+m} & \twoheadrightarrow & \mathcal{E}_n & \twoheadrightarrow & 0
\end{array}$$

and isomorphisms $\mathrm{pr}_{T_*} \mathcal{E}_i \cong \mathcal{O}_T^{\oplus i}$.

- A T-point of GL_{n+m}, which acts on $\mathcal{O}_{S\times T}^{\oplus(n+m)}$ on the left.

Combining the two pieces of data, we obtain

$$\mathcal{O}_{S\times T}^{\oplus(n+m)} \to \mathcal{O}_{S\times T}^{\oplus(n+m)} \twoheadrightarrow \mathcal{E}_{n+m} \twoheadrightarrow \mathcal{E}_n$$

where the first arrow comes from the T-points of GL_{n+m} and the other two come from the T-point of $Flag^{\circ}_{m,n}$. This is essentially the data (i) and (ii), and we could check that the above construction is invariant by $P_{n,m}$ actions.

For the other direction, suppose we are given the data (i) and (ii). $\mathcal{E}_{n+m} \twoheadrightarrow \mathcal{E}_n$ induces a surjection of vector bundles $\mathcal{O}_T^{\oplus(n+m)} \twoheadrightarrow \mathrm{pr}_{T*} \mathcal{E}_n$ on T. This is exactly the data of a morphism $f : T \to \mathrm{Gr}_{n,n+m}$, where $\mathrm{Gr}_{r,k}$ is the Grassmannian of r-dimensional quotient subspaces in a k-dimensional vector space. In particular, $\mathrm{pr}_{T*} \mathcal{E}_n = f^*\mathcal{S}_{n,n+m}$, where $\mathcal{S}_{n,n+m}$ is the universal subbundle over $\mathrm{Gr}_{n,n+m}$.

Consider the standard action of GL_{n+m} on $\mathrm{Gr}_{n,n+m}$: this action is transitive with stablizer $P_{n,m}$. Thus, $\alpha : \mathrm{GL}_{n+m} \to \mathrm{Gr}_{n,n+m}$ is a principal $P_{n,m}$-bundle. Let the fiber product

$$\widetilde{T} := T \times_{\mathrm{Gr}_{n,n+m}} \mathrm{GL}_{n+m},$$

which is a principal $P_{n,m}$-bundle over T. We shall define a $P_{n,m}$-equivariant morphism $\widetilde{T} \to Flag^{\circ}_{m,n} \times \mathrm{GL}_{n+m}$. Set the map $\widetilde{T} \to \mathrm{GL}_{n+m}$ to be the projection. To construct $\widetilde{T} \to Flag^{\circ}_{m,n}$, it suffices to find the data

$$\begin{array}{ccc} \mathcal{O}_{S\times\widetilde{T}}^{\oplus(n+m)} & \twoheadrightarrow & \mathcal{F}_{n+m} \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ \mathcal{O}_{S\times\widetilde{T}}^{\oplus n} & \twoheadrightarrow & \mathcal{F}_n \end{array} \tag{1}$$

Denote $\pi : \widetilde{T} \to T$ the projection and we pick $\mathcal{F}_n := (\mathrm{id}_S \times\pi)^*\mathcal{E}_n, \mathcal{F}_{n+m} := (\mathrm{id}_S \times\pi)^*\mathcal{E}_{n+m}$. We already have the data $\mathcal{O}_{S\times\widetilde{T}}^{\oplus(n+m)} \twoheadrightarrow \mathcal{F}_{n+m}$ by pullback. It remains to identify the map $\mathcal{O}_{S\times\widetilde{T}}^{\oplus n} \twoheadrightarrow \mathcal{F}_n$ to make (1) commute.

We claim that $\pi^* \mathrm{pr}_{T*} \mathcal{E}_n \cong \mathcal{O}_{\widetilde{T}}^{\oplus n}$. Assuming this holds, then from the adjunction $\mathrm{pr}_T^* \mathrm{pr}_{T*} \mathcal{E}_n \twoheadrightarrow \mathcal{E}_n$, we have

$$\mathcal{O}_{S\times\widetilde{T}}^{\oplus n} \cong (\mathrm{id}_S \times p_T)^* \mathrm{pr}_T^* \mathrm{pr}_{T*} \mathcal{E}_n \twoheadrightarrow (\mathrm{id}_S \times p_T)^*\mathcal{E}_n = \mathcal{F}_n,$$

as desired. To prove this claim, consider the diagram

$$\begin{array}{ccc} \widetilde{T} & \xrightarrow{\pi} & T \\ \downarrow & & \downarrow f \\ \mathrm{GL}_{n+m} & \xrightarrow{\alpha} & \mathrm{Gr}_{n,n+m} \end{array}$$

Since $\mathrm{pr}_{T*}\mathcal{E}_n = f^*\mathcal{S}_{n,n+m}$, it suffices to show that $\alpha^*\mathcal{S}_{n,n+m} = \mathcal{O}_{\mathrm{GL}_{n+m}}^{\oplus n}$. The standard action of GL_{n+m} on $\mathrm{Gr}_{n,n+m}$ is a morphism $\mathrm{GL}_{n+m} \times \mathrm{Gr}_{n,n+m} \to \mathrm{Gr}_{n,n+m}$. The fiber of the pullback of $\mathcal{S}_{n,n+m}$ along this action over a point $(g, V) \in \mathrm{GL}_{n+m} \times \mathrm{Gr}_{n,n+m}$ is gV; restricting to $\mathrm{GL}_{n+m} \times \{V\}$ for a fixed V with basis $\{v_1, \ldots, v_n\}$, we see that $\alpha^*\mathcal{S}_{n,n+m}$ has everywhere linearly independent global sections gv_i and hence must be $\mathcal{O}_{\mathrm{GL}_{n+m}}^{\oplus n}$.

By the universal property of quotient, the data of a $P_{n,m}$-equivariant morphism from a principal $P_{n,m}$-bundle $\widetilde{T}$ to $Flag^\circ_{m,n} \times \mathrm{GL}_{n+m}$ corresponds to a morphism

$$T \to (Flag^\circ_{m,n} \times \mathrm{GL}_{n+m})/P_{n,m},$$

as desired.

Finally, we can check that these two natural transformations are inverse to each other, so the proof is complete. ☐

Similarly, $P_{n,m,\ell}$ acts on $Flag^\circ_{\ell,m,n}$ and we can form moduli spaces such as

$$\begin{aligned}
\widetilde{Flag}^{\circ-}_{\ell,m,n} &:= (Flag^\circ_{\ell,m,n} \times P_{n+m,\ell})/P_{n,m,\ell} \\
\widetilde{Flag}^{\circ+}_{\ell,m,n} &:= (Flag^\circ_{\ell,m,n} \times P_{n,m+\ell})/P_{n,m,\ell} \\
\widetilde{Flag}^{\circ}_{\ell,m,n} &:= (Flag^\circ_{\ell,m,n} \times \mathrm{GL}_{n+m+\ell})/P_{n,m,\ell}
\end{aligned}$$

Then these moduli spaces have functor-of-points descriptions analogous to Proposition 2.2.

In later sections, we will mostly focus on $Flag^\circ_{1,1}$, $Flag^\circ_{1,1,1}$ and their variations. We shall also introduce a closed subscheme $Flag^\circ_{x,x}$ of $Flag^\circ_{1,1}$ whose T-points are in bijection with the data:

- A commutative diagram

$$\begin{array}{ccccc}
\mathcal{O}_{S\times T}^{\oplus 2} & \twoheadrightarrow & \mathcal{O}_{S\times T} & \twoheadrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathcal{E}_2 & \xrightarrow{x} & \mathcal{E}_1 & \xrightarrow{x} & 0
\end{array},$$

  where $x : S \to T$ is a map such that there exists line bundles $\mathcal{L}_1, \mathcal{L}_2$ on T satisfying

$$\mathcal{E}_1 \cong \Gamma^x_*\mathcal{L}_1, \ker(\mathcal{E}_2 \to \mathcal{E}_1) \cong \Gamma^x_*\mathcal{L}_2.$$

- The quotient maps above induce isomorphisms $\mathcal{O}_T^{\oplus i} \cong \mathrm{pr}_{T*}\mathcal{E}_i$ for $i = 1, 2$.

Alternatively, $Flag^\circ_{x,x}$ is "cut out" by the condition that $\mathcal{E}_2$ is supported on a single point. As before, $P_{1,1}$ acts on $Flag^\circ_{x,x}$ and we define $\widetilde{Flag}^\circ_{x,x}$ to be the quotient $(Flag^\circ_{x,x} \times \mathrm{GL}_2)/P_{1,1}$, which is a closed subscheme of $\widetilde{Flag}^\circ_{1,1}$.

Analogously, we can define a few closed subschemes of $Flag^\circ_{1,1,1}$: $Flag^\circ_{x,x,y}$, $Flag^\circ_{x,y,x}$, $Flag^\circ_{y,x,x}$, and $Flag^\circ_{x,x,x}$. For example, $Flag^\circ_{x,x,y}$ parametrizes the data

$$\begin{array}{ccccccc} \mathcal{O}_{S\times T}^{\oplus 3} & \twoheadrightarrow & \mathcal{O}_{S\times T}^{\oplus 2} & \twoheadrightarrow & \mathcal{O}_{S\times T} & \twoheadrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{E}_3 & \xrightarrow{x} & \mathcal{E}_2 & \xrightarrow{x} & \mathcal{E}_1 & \xrightarrow{y} & 0 \end{array}$$

The parabolic subgroup $P_{1,1,1}$ acts on these schemes and we can define

$$\widetilde{Flag}^{\circ -}_{x,x,y} := (Flag^\circ_{x,x,y} \times P_{2,1})/P_{1,1,1}$$
$$\widetilde{Flag}^{\circ +}_{x,x,y} := (Flag^\circ_{x,x,y} \times P_{1,2})/P_{1,1,1}$$
$$\widetilde{Flag}^{\circ}_{x,x,y} := (Flag^\circ_{x,x,y} \times \mathrm{GL}_3)/P_{1,1,1}$$

and also for variants of $Flag^\circ_{x,y,x}$ and $Flag^\circ_{y,x,x}$.

On each of the moduli spaces $Flag^\circ_{1,1}$, $\widetilde{Flag}^\circ_{1,1}$, $Flag^\circ_{x,x}$, and $\widetilde{Flag}^\circ_{x,x}$, there is a universal vector bundle $\mathcal{U}_2 \twoheadrightarrow \mathcal{U}_1$. This abuse of notation is acceptable, since the pullback of $\mathcal{U}_i$ is still $\mathcal{U}_i$ along any of the following morphisms:

$$\widetilde{Flag}^\circ_{x,x} \to Flag^\circ_{x,x}, \widetilde{Flag}^\circ_{1,1} \to Flag^\circ_{1,1}, Flag^\circ_{x,x} \hookrightarrow Flag^\circ_{1,1}, \widetilde{Flag}^\circ_{x,x} \hookrightarrow \widetilde{Flag}^\circ_{1,1}.$$

On each of these schemes, denote $\mathcal{L}_2 = \ker(\mathcal{U}_2 \to \mathcal{U}_1)$ and $\mathcal{L}_1 = \mathcal{U}_1$, then $\mathcal{L}_i$ is a line bundle. On $Flag^\circ_{x,x}$ and $Flag^\circ_{1,1}$, $\mathcal{L}_i$ is $P_{1,1}$-equivariant; on $\widetilde{Flag}^\circ_{1,1}$ and $\widetilde{Flag}^\circ_{x,x}$, we will see that $\mathcal{L}_i$ is in fact GL_2-equivariant.

On moduli spaces of variants of $Flag^\circ_{1,1,1}$ (including $Flag^\circ_{x,x,y}, \widetilde{Flag}^{\circ -}_{x,x,y}$, etc.), we denote similarly the universal vector bundles as $\mathcal{U}_3 \twoheadrightarrow \mathcal{U}_2 \twoheadrightarrow \mathcal{U}_1$ and line bundles $\mathcal{L}_i = \ker(\mathcal{U}_i \to \mathcal{U}_{i-1})$ (where $\mathcal{U}_0 = 0$). On each of these moduli spaces, $\mathcal{L}_i$ will carry a corresponding equivariant structure.

2.1 Moduli spaces related to length-2 coherent sheaves

As we defined previously, the scheme $Quot^\circ_2$ parametrizes length-2 coherent sheaves on S. To understand its geometry, we consider a stratification of the closed points $[\mathcal{O}_S^{\oplus 2} \twoheadrightarrow \mathcal{E}_2]$ of $Quot^\circ_2$. Suppose $\mathcal{E}_2$ is supported on closed points $x, y \in S$ (x, y could be the same point). Then

- When $x \neq y$, $\mathcal{E}_2 \cong \mathbb{C}_x \oplus \mathbb{C}_y$ and the map $\mathcal{O}_S^{\oplus 2} \twoheadrightarrow \mathcal{E}_2$ is parametrized by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{2\times 2}$: it sends a pair of rational sections (f_1, f_2) in $\mathcal{O}_S^{\oplus 2}$ (defined on an open subset U containing x, y) to $(af_1(x) + bf_2(x), cf_1(y) + df_2(y))$. Since it induces isomorphism on H^0, we require $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2$. The equivalence relations are given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ if $[a : b] = [a' : b']$ and $[c : d] = [c' : d']$ as elements in $\mathbb{P}^1$.

- When $x = y$, $\mathcal{E}_2$ corresponds to a length-2 $\mathcal{O}_{S,x}$-module. There are two cases:
  - $\mathcal{E}_2$ is the direct sum $\mathbb{C}_x \oplus \mathbb{C}_x$. Then the map $\mathcal{O}_S^{\oplus 2} \twoheadrightarrow \mathcal{E}_2$ is parametrized by $\mathrm{GL}_2(\mathbb{C})$ with certain equivalence relations.
  - $\mathcal{E}_2$ is not a direct sum of length-1 modules. Since S is smooth, we can assume that locally S is the affine plane $\mathbb{A}^2$, so $\mathcal{O}_{S,x} \cong \mathbb{C}[X,Y]_{(X,Y)}$. Then $\mathcal{E}_2$ corresponds to a colength-2 ideal in $\mathbb{C}[X,Y]_{(X,Y)}$, which has the form $(X^2, Y^2, aX + bY)$, parametrized by $[a : b] \in \mathbb{P}^1$.

To understand the scheme structure near the locus $x = y$ better, we state the following local description of $Quot_2^\circ$, in the special case that S is birational to $\mathbb{A}^2$.

Proposition 2.3. *Suppose that S is birational to $\mathbb{A}^2$, then $Quot_2^\circ$ is birational to the following affine scheme:*

$$Quot_2^{loc} := \{X, Y \in \mathrm{Mat}_{2\times 2} : XY = YX\}.$$

Proof. Suppose that U is an open subscheme of S isomorphic to an open subscheme of $\mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[X,Y]$. We shall prove that locally on U, the Quot scheme $Quot_2^\circ$ is isomorphic to an open subscheme of $Quot_2^{loc}$. In other words, consider the open subscheme Q of $Quot_2^\circ$ consisting of closed points described by the data $\phi : \mathcal{O}_S^{\oplus 2} \twoheadrightarrow \mathcal{E}_2$ (with isomorphism $\phi_* : H^0(S, \mathcal{O}_S^{\oplus 2}) \cong H^0(S, \mathcal{E}_2)$), such that $\mathrm{supp}(\mathcal{E}_2) \subset U$; we claim that Q is isomorphic to an open subscheme of $Quot_2^{loc}$.

Since $\mathcal{E}_2$ is supported on U, the local sections $X \in \mathcal{O}_S(U)$ gives rise to a map $\mathcal{E}_2 \xrightarrow{\cdot X} \mathcal{E}_2$ and hence a linear isomorphism

$$X : H^0(S, \mathcal{E}_2) \to H^0(S, \mathcal{E}_2).$$

Similarly, the multiplication-by-Y map induces a linear endomorphism Y on $H^0(S, \mathcal{E}_2)$. Apparently, the maps X, Y commute with each other. Note that ϕ determines a basis on $H^0(S, \mathcal{E}_2)$ and we can write X, Y as 2×2 matrices using the basis determined by ϕ. Therefore, this gives an open embedding $Q \subseteq Quot_2^{loc}$.

On the other hand, from the information of a pair of commuting 2×2 matrices X, Y, we can build a length-2 $\mathbb{C}[X,Y]$-module $\mathcal{E}_2 = \mathbb{C}v_1 \oplus \mathbb{C}v_2$, such that (1) the map $\phi : \mathcal{O}_S^{\oplus 2} \twoheadrightarrow \mathcal{E}_2$ is defined on U by $\phi(1,0) = v_1$ and $\phi(0,1) = v_2$; (2) X, Y acts on $\mathcal{E}_2$ as matrices X, Y with respect to the basis (v_1, v_2). These data determine the structure of a closed point of Q. □

Remark 2.4. For general surface S (not necessarily birational to $\mathbb{A}^2$) and any point $p \in S$, we can find an open neighborhood U of p such that a certain deformation of U is isomorphic to an open subscheme of $\mathbb{A}^2$. This implies that locally we can deform $Quot_2^\circ$ into $Quot_2^{loc}$.

We can use such local descriptions to prove properties of $Quot_2^\circ$ which are invariant by deformation, such as the Cohen-Macaulay, normality, and reducedness.

Similar to $Quot_2^\circ$, $Flag_{1,1}^\circ$ and $\widetilde{Flag}_{1,1}^\circ$ have similar stratifications and local descriptions. For example, when S is birational to $\mathbb{A}^2$, $Flag_{1,1}^\circ$ is also birational to

$$Flag_{1,1}^{loc} = \{X, Y \in B_{1,1} : XY = YX\},$$

where $B_{1,1}$ is the set of all lower-triangular 2×2 matrices. If $U \subset S$ is identified with an open subset of $\mathbb{A}^2$, then for a closed point $[\mathcal{O}_S^{\oplus 2} \twoheadrightarrow \mathcal{E}_2 \overset{x}{\twoheadrightarrow} \mathcal{E}_1 \overset{y}{\twoheadrightarrow} 0]$ (here, the label x on the arrow $\mathcal{E}_2 \overset{x}{\twoheadrightarrow} \mathcal{E}_1$ means that $\ker(\mathcal{E}_2 \to \mathcal{E}_1)$ is supported on a single point $x \in S$) such that $x, y \in U$, the corresponding data $(X, Y) \in Flag_{1,1}^{loc}$ will satisfy

$$x = (X_{22}, Y_{22}), y = (X_{11}, Y_{11}) \in \mathbb{A}^2.$$

The $P_{1,1}$-action on $Flag_{1,1}^\circ$ can be locally described as

$$g \cdot (X, Y) = (gXg^{-1}, gYg^{-1}), \text{ for } g \in P_{1,1}, (X, Y) \in Flag_{1,1}^{loc},$$

since it is essentially a change of basis. Thus, we can describe $\widetilde{Flag}_{1,1}^\circ$ locally as

$$\widetilde{Flag}_{1,1}^{loc} = \{X, Y \in B_{1,1}, g \in \mathrm{GL}_2 : XY = YX\}/\sim$$

where $(X, Y, g) \sim (gXg^{-1}, gYg^{-1}, \mathbf{1})$. We also note that there is a morphism $\widetilde{Flag}_{1,1}^\circ \to Quot_2^\circ$ which, by Proposition 2.2, can be functorially defined by the natural transformation

$$[\mathcal{O}_{S\times T}^{\oplus 2} \twoheadrightarrow \mathcal{E}_2 \twoheadrightarrow \mathcal{E}_1] \mapsto [\mathcal{O}_{S\times T}^{\oplus 2} \twoheadrightarrow \mathcal{E}_2].$$

This map can be locally described as

$$[(X, Y, g)] \mapsto (gXg^{-1}, gYg^{-1}).$$

For a fixed point $p = [\mathcal{O}_S^{\oplus 2} \twoheadrightarrow \mathcal{E}_2]$ of $Quot_2^\circ$, the fiber of $\widetilde{Flag}_{1,1}^\circ$ above p is:

- If $\mathcal{E}_2$ is supported on two points $x \neq y$, then $\mathcal{E}_2 \cong \mathbb{C}_x \oplus \mathbb{C}_y$ and hence $\mathcal{E}_1 = \mathbb{C}_x$ or $\mathbb{C}_y$. In this case, the fiber consists of two points.
- If $\mathcal{E}_2$ is supported on a single point x, then $\mathcal{E}_1 = \mathbb{C}_x$. We have two subcases:
  - $\mathcal{E}_2 \cong \mathbb{C}_x \oplus \mathbb{C}_x$. Then the map $\mathcal{E}_2 \twoheadrightarrow \mathcal{E}_1$ is parametrized by $\mathbb{P}^1$.
  - $\mathcal{E}_2$ is not a direct sum of length-1 sheaves supported on x. Then $\mathcal{E}_2$ is a subsheaf of $\mathcal{O}_S$. The quotient map $\mathcal{E}_2 \twoheadrightarrow \mathcal{E}_1$ is unique (up to isomorphism), since $\mathcal{E}_2 \to \mathcal{E}_1$ is determined by the image of the constant section $1 \in \mathcal{E}_2$. In this case, the fiber above p consists of a single point.

We define $\mathcal{Y}$ to be the fiber product

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\mathrm{pr}} & \widetilde{Flag}^{\circ}_{1,1} \\ \downarrow{\scriptstyle \mathrm{pr}'} & & \downarrow{\scriptstyle \phi} \\ \widetilde{Flag}^{\circ}_{1,1} & \xrightarrow{\phi'} & Quot^{\circ}_{2} \times S \times S \end{array} \tag{2}$$

where the map ϕ, ϕ' are defined as follows. Let $p = [\mathcal{O}_S^{\oplus 2} \twoheadrightarrow \mathcal{E}_2 \overset{x}{\twoheadrightarrow} \mathcal{E}_1 \overset{y}{\twoheadrightarrow} 0]$ be a closed point of $\widetilde{Flag}^{\circ}_{1,1}$ as in Proposition 2.2. Set $\phi(p) = ([\mathcal{O}_S^{\oplus 2} \twoheadrightarrow \mathcal{E}_2], x, y)$ and $\phi'(p) = ([\mathcal{O}_S^{\oplus 2} \twoheadrightarrow \mathcal{E}_2], y, x)$.

Proposition 2.5. *$\mathcal{Y}$ has the following functor-of-points description. For any scheme T, the set of T-points of $\mathcal{Y}$ is in bijection with the following data:*

- *A commutative diagram*

$$\begin{array}{ccccc} & & \mathcal{E}_1 & & \\ & \overset{x}{\nearrow} & & \overset{y}{\searrow} & \\ \mathcal{O}^{\oplus 2}_{S\times T} \twoheadrightarrow & \mathcal{E}_2 & & & 0 \\ & \underset{y}{\searrow} & & \underset{x}{\nearrow} & \\ & & \mathcal{E}'_1 & & \end{array}$$

  where $\mathcal{E}_i, \mathcal{E}'_i$ are flat of length i over T, and x, y are maps $T \to S$ such that there exist line bundles $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}'_1, \mathcal{L}'_2$ on T with

$$\ker(\mathcal{E}_2 \to \mathcal{E}_1) \cong \Gamma^x_* \mathcal{L}_2, \mathcal{E}_1 \cong \Gamma^y_* \mathcal{L}_1$$

$$\ker(\mathcal{E}_2 \to \mathcal{E}'_1) \cong \Gamma^y_* \mathcal{L}'_2, \mathcal{E}'_1 \cong \Gamma^x_* \mathcal{L}'_1$$

  (Here Γ^x, Γ^y denote the graph of x, y, respectively.)

- *The projection map induces an isomorphism $\mathcal{O}_T^{\oplus 2} \cong \mathrm{pr}_{T*} \mathcal{E}_2$.*

Proof. This follows from a functor-of-points description of $\widetilde{Flag}^{\circ}_{1,1}$ equivalent to that of Proposition 2.2: its T-point consists of the data

$$\mathcal{O}^{\oplus 2}_{S\times T} \twoheadrightarrow \mathcal{E}_2 \overset{x}{\twoheadrightarrow} \mathcal{E}_1 \overset{y}{\twoheadrightarrow} 0,$$

where $x, y : T \to S$ such that there exists line bundles $\mathcal{L}_1, \mathcal{L}_2$ on T satisfying

$$\ker(\mathcal{E}_2 \to \mathcal{E}_1) \cong \Gamma^x_* \mathcal{L}_2, \mathcal{E}_1 \cong \Gamma^y_* \mathcal{L}_1.$$

Furthermore, the projection map induces an isomorphism $\mathcal{O}_T^{\oplus 2} \cong \mathrm{pr}_{T*} \mathcal{E}_2$. $\square$

By the functor-of-points description, we can find universal vector bundles on $\mathcal{Y}$, denote as $\mathcal{U}_2, \mathcal{U}_1, \mathcal{U}_2', \mathcal{U}_1'$. They are the pushforward of corresponding universal sheaves on $\mathcal{Y} \times S$. Again, the abuse of notation is acceptable since (referring to diagram (2))

$$\mathrm{pr}^* \mathcal{U}_i = \mathcal{U}_i, \mathrm{pr}'^* \mathcal{U}_i = \mathcal{U}_i'.$$

We denote the line bundles $\mathcal{L}_i = \ker(\mathcal{U}_i \to \mathcal{U}_{i-1})$ and $\mathcal{L}_i' = \ker(\mathcal{U}_i' \to \mathcal{U}_{i-1}')$.

We can also analyze the fiber of the projection $\mathcal{Y} \to \widetilde{Flag^\circ_{1,1}}$. Similar to the case of the morphism $Flag^\circ_{1,1} \to Quot^\circ_2$, the fiber is either a single point or $\mathbb{P}^1$. This motivates the following result:

Proposition 2.6. *Let $\mathcal{U}_2$ be the universal rank-2 vector bundle on $\widetilde{Flag^\circ_{1,1}}$. There is a closed embedding $\iota : \mathcal{Y} \to \mathbb{P}(\mathcal{U}_2)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{Y} & \xhookrightarrow{\iota} & \mathbb{P}(\mathcal{U}_2) \\ & \searrow^{\mathrm{pr}} & \downarrow \\ & & \widetilde{Flag^\circ_{1,1}} \end{array}$$

Proof. A morphism $\mathcal{Y} \to \mathbb{P}(\mathcal{U}_2)$ consists of the following data: a map $f : \mathcal{Y} \to \widetilde{Flag^\circ_{1,1}}$, a line bundle $\mathcal{L}$ on $\mathcal{Y}$ and a surjective homomorphism $f^*(\mathcal{U}_2) \twoheadrightarrow \mathcal{L}$. Let $f = \mathrm{pr}$ and $\mathcal{L} = \mathcal{L}_1'$ where $\mathcal{L}_1'$ is the universal line bundle on $\mathcal{Y}$. We claim that corresponding map ι is a closed embedding.

By Theorem 1.7.8 of [3], it suffices to show that above each closed point $p = [\mathcal{O}^{\oplus 2}_{\widetilde{Flag^\circ_{1,1}}} \twoheadrightarrow \mathcal{F}_2 \twoheadrightarrow \mathcal{F}_1]$ of $\widetilde{Flag^\circ_{1,1}}$, the map between the fibers $\iota_p : \mathcal{Y}_p \to \mathbb{P}(\mathcal{U}_2)_p$ is a closed embedding, i.e., the restriction of $\mathcal{L}$ to $\mathcal{Y}_p$ is a very ample line bundle on $\mathcal{Y}_p$. By our previous discussions,

$$\mathcal{Y}_p \cong \begin{cases} \mathbb{P}^1, & \text{if } \mathcal{F}_2 \cong \mathbb{C}_x^{\oplus 2}, \\ \text{a point}, & \text{otherwise.} \end{cases}$$

It suffices to consider the situation $\mathcal{Y}_p \cong \mathbb{P}^1$, i.e., $\mathcal{F}_2 \cong \mathbb{C}_x^{\oplus 2}$ for some $x \in S$. We see that $\mathcal{O}^{\oplus 2}_{\mathcal{Y} \times S} \twoheadrightarrow \mathcal{E}_1'$ induces

$$\mathcal{O}^{\oplus 2}_{\mathcal{Y}_p} \twoheadrightarrow \mathcal{L}_p := \mathcal{L}|_{\mathcal{Y}_p}.$$

Restricting to a closed point $[\mathbb{C}_x^{\oplus 2} \twoheadrightarrow \mathcal{F}_1']$ [2] of $\mathcal{Y}_p$, this map is (canonically)

$$\mathbb{C}^{\oplus 2} = H^0(S, \mathbb{C}_x^{\oplus 2}) \twoheadrightarrow H^0(S, \mathcal{F}_1')$$

As a result, $\mathcal{O}_{\mathcal{Y}_p} \twoheadrightarrow \mathcal{L}_p$ is precisely the universal quotient $\mathcal{O}^{\oplus 2} \twoheadrightarrow \mathcal{Q}_{1,2}$ of the Grassmannian $\mathrm{Gr}_{1,2} \cong \mathbb{P}^1$. Hence, $\mathcal{L}_p \cong \mathcal{Q}_{1,2} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ is very ample on $\mathcal{Y}_p$, as desired. □

[2] By the functor-of-points description, a closed point of $\mathcal{Y}_p$ should correspond to the data of a certain commutative diagram. Since the data $\mathbb{C}_x^{\oplus 2} \twoheadrightarrow \mathcal{F}_1$ is fixed, only the part $\mathbb{C}_x^{\oplus 2} \twoheadrightarrow \mathcal{F}_1'$ will parametrize $\mathcal{Y}_p$.

Proposition 2.7. *The morphism* $\mathrm{pr} : \mathcal{Y} \to \widetilde{Flag^\circ_{1,1}}$ *is proper and satisfies*

$$R^i \,\mathrm{pr}_* \,\mathcal{O}_{\mathcal{Y}} = \begin{cases} \mathcal{O}_{\widetilde{Flag^\circ_{1,1}}}, & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}$$

Proof. (The proof is analogous to Proposition 2.30 of [5].) By Proposition 2.6, we can embed $\mathcal{Y}$ into a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{U}_2)$ over $\widetilde{Flag^\circ_{1,1}}$. Denote the projection $\pi : \mathbb{P}(\mathcal{U}_2) \twoheadrightarrow \widetilde{Flag^\circ_{1,1}}$, then

$$R^i \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{U}_2)}) = 0 \text{ for all } i \geq 1,$$

and for any coherent sheaf $\mathcal{F}$ on $\mathbb{P}(\mathcal{U}_2)$,

$$R^i \pi_*(\mathcal{F}) = 0 \text{ for all } i \geq 2.$$

Now, from the exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbb{P}_2(\mathcal{U}_2)} \to \iota_* \mathcal{O}_{\mathcal{Y}} \to 0,$$

(where $\mathcal{K}$ is the kernel sheaf) we obtain the long exact sequence

$$\cdots \to R^i \pi_*(\mathcal{O}_{\mathbb{P}_2(\mathcal{U}_2)}) \to R^i \,\mathrm{pr}_*(\mathcal{O}_{\mathcal{Y}}) \to R^{i+1} \pi_*(\mathcal{K}) \to \cdots$$

This implies that $R^i \,\mathrm{pr}_*(\mathcal{O}_{\mathcal{Y}}) = 0$ for $i \geq 1$. The $i = 0$ case follows from Stein factorization and the following facts:

- $\widetilde{Flag^\circ_{1,1}}$ is normal. (Lemma 5.3)
- $\mathcal{Y}$ is reduced. (Lemma 5.4)
- pr is proper and all its fibers are either a point or $\mathbb{P}^1$. This is addressed by previous discussions.

□

Proposition 2.8. *The natural map*

$$\ker(\mathcal{E}_2 \to \mathcal{E}_1) \hookrightarrow \mathcal{E}_2 \to \mathcal{E}'_1$$

on $\mathcal{Y} \times S$ *induces a map of line bundles on* $\mathcal{Y}$*:*

$$\mathcal{L}_2 \to \mathcal{L}'_1.$$

The zero section of corresponding line bundle $\mathcal{L}'_1 \otimes \mathcal{L}_2^{-1}$ *consists of the data*

$$\{(\mathcal{E}_1, x) = (\mathcal{E}'_1, y)\} \subset \mathcal{Y},$$

which is isomorphic to $\widetilde{Flag^\circ_{x,x}}$.

Proof. (The proof is analogous to Proposition 2.28 of [5]) By the functorial description, a map from T to the zero locus of $\mathcal{L}_1' \otimes \mathcal{L}_2^{-1}$ corresponds to the data

$$\begin{array}{ccccc} & & & \mathcal{E}_1 & & \\ & & \overset{x}{\nearrow} & & \overset{y}{\searrow} & \\ \mathcal{O}_{S\times T}^{\oplus 2} & \twoheadrightarrow & \mathcal{E}_2 & & & 0 \\ & & \underset{y}{\searrow} & & \underset{x}{\nearrow} & \\ & & & \mathcal{E}_1' & & \end{array}$$

such that

$$\pi_{T*} \ker(\mathcal{E}_2 \to \mathcal{E}_1) \to \pi_{T*}\mathcal{E}_1'$$

is the zero map. Denote $\ker(\mathcal{E}_2 \to \mathcal{E}_1) = \mathcal{K}$ and $\ker(\mathcal{E}_2 \to \mathcal{E}_1') = \mathcal{K}'$. Since π_T is flat, this means that

$$\mathcal{K} \to \mathcal{E}_1'$$

is also zero. Thus, $\mathcal{K} \subset \mathcal{K}'$, i.e., $\Gamma_*^x(\mathcal{L}_2) \to \Gamma_*^y(\mathcal{L}_2')$ is injective. This gives $x = y$ and that $\mathcal{L}_2 \to \mathcal{L}_2'$ is injective (since Γ_*^x is exact). From the exact sequence

$$0 \to \mathcal{K}'/\mathcal{K} \to \mathcal{E}_1 = \mathcal{E}_2/\mathcal{K} \to \mathcal{E}_1' = \mathcal{E}_2/\mathcal{K}' \to 0,$$

we also see that $\mathcal{L}_1 \twoheadrightarrow \mathcal{L}_1'$ is surjective. Note that $\mathcal{L}_1 \otimes \mathcal{L}_2 \cong \mathcal{L}_1' \otimes \mathcal{L}_2' \cong \mathcal{O}_T$ since we have an exact sequence

$$0 \to \mathcal{L}_2 \to \mathcal{O}_T^{\oplus 2} \to \mathcal{L}_1 \to 0.$$

As a result, we have

$$\mathcal{O}_T \hookrightarrow \mathcal{L}_2'\mathcal{L}_2^{-1} \cong \mathcal{L}_1'\mathcal{L}_1^{-1} \twoheadrightarrow \mathcal{O}_T$$

This shows that $\mathcal{O}_T$ is a direct summand of $\mathcal{L}_1\mathcal{L}_1'^{-1}$ and hence they must be equal. Therefore, $\mathcal{E}_1 = \mathcal{E}_1'$, as desired. ☐

2.2 Moduli spaces related to length-3 coherent sheaves

Similar to $Flag_{1,1}^\circ$, $Flag_{1,1,1}^\circ$ has the following local description:

$$Flag_{1,1,1}^{loc} = \{X, Y \in B_{1,1,1} : XY = YX\}.$$

This also applies to $Flag_{1,2}^\circ, Flag_{2,1}^\circ$, etc.

Define $\mathcal{Y}_+$ to be the fiber product

$$\begin{array}{ccc} \mathcal{Y}_+ & \xrightarrow{\mathrm{pr}_+} & \widetilde{Flag_{x,y,x}^{\circ+}} \\ \downarrow{\scriptstyle \mathrm{pr}_+'} & & \downarrow{\scriptstyle \phi} \\ \widetilde{Flag_{y,x,x}^{\circ+}} & \xrightarrow{\phi'} & Flag_{2,1}^\circ \times S \times S \end{array}$$

where ϕ, ϕ' are defined as follows. For a closed point $p = [\mathcal{E}_3 \overset{x}{\twoheadrightarrow} \mathcal{E}_2 \overset{y}{\twoheadrightarrow} \mathcal{E}_1 \overset{x}{\twoheadrightarrow} 0]$ of $\widetilde{Flag^{\circ+}_{x,y,x}}$ (here the framings of $\mathcal{E}_3$ and $\mathcal{E}_1$ are not written out), set

$$\phi(p) := ([\mathcal{E}_3 \twoheadrightarrow \mathcal{E}_1 \twoheadrightarrow 0](\text{with the same framings}), x, y).$$

In analogy, for a closed point $q = [\mathcal{E}'_3 \overset{y}{\twoheadrightarrow} \mathcal{E}'_2 \overset{x}{\twoheadrightarrow} \mathcal{E}'_1 \overset{x}{\twoheadrightarrow} 0]$ of $\widetilde{Flag^{\circ+}_{y,x,x}}$ (the framings at $\mathcal{E}'_3$ and $\mathcal{E}'_1$ are not written out), set

$$\phi'(q) = ([\mathcal{E}'_3 \twoheadrightarrow \mathcal{E}'_1 \twoheadrightarrow 0](\text{with the same framings}), x, y).$$

We also define $\mathcal{Y}_-$ to be the fiber product

$$\begin{array}{ccc}
\mathcal{Y}_- & \xrightarrow{\mathrm{pr}_-} & \widetilde{Flag^{\circ-}_{x,x,y}} \\
\downarrow{\scriptstyle \mathrm{pr}'_-} & & \downarrow{\scriptstyle \phi} \\
\widetilde{Flag^{\circ-}_{x,y,x}} & \xrightarrow{\phi'} & Flag^{\circ}_{1,2} \times S \times S
\end{array}$$

Proposition 2.9. *$\mathcal{Y}_+$ has the following functor-of-points description. For an scheme T, the set of T-points of $\mathcal{Y}_+$ is in bijection with the data*

- *Commutative diagrams*

$$\begin{array}{ccc}
\mathcal{O}^{\oplus 3}_{S\times T} & \twoheadrightarrow & \mathcal{O}_{S\times T} \\
\downarrow & & \downarrow \\
\mathcal{E}_3 & \twoheadrightarrow & \mathcal{E}_1
\end{array}
\qquad
\begin{array}{ccccccc}
 & & \mathcal{E}_2 & & & & \\
 & {\scriptstyle x}\nearrow & & \searrow{\scriptstyle y} & & & \\
\mathcal{E}_3 & & & & \mathcal{E}_1 & \xrightarrow{x} & 0 \\
 & {\scriptstyle y}\searrow & & \nearrow{\scriptstyle x} & & & \\
 & & \mathcal{E}'_2 & & & &
\end{array} \tag{3}$$

  with maps $x, y : T \to S$ such that there exist line bundles $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}'_2, \mathcal{L}'_3$ on T satisfying

$$\ker(\mathcal{E}_3 \to \mathcal{E}_2) \cong \Gamma^x_* \mathcal{L}_3, \ker(\mathcal{E}_3 \to \mathcal{E}'_2) \cong \Gamma^y_* \mathcal{L}'_3, \ldots$$

- *The above data induce isomorphisms $\mathcal{O}^{\oplus i}_T \cong \mathrm{pr}_{T*} \mathcal{E}_i$ for $i \in \{1, 3\}$.*

A similar functorial description also applies to $\mathcal{Y}_-$, i.e., it is parametrized by the data

$$\begin{array}{ccc}
\mathcal{O}^{\oplus 3} & \twoheadrightarrow & \mathcal{O} \\
\downarrow & & \downarrow \\
\mathcal{E}_3 & \twoheadrightarrow & \mathcal{E}_2
\end{array}
,\quad
\begin{array}{ccccccc}
 & & & & \mathcal{E}_1 & & \\
 & & & {\scriptstyle x}\nearrow & & \searrow{\scriptstyle y} & \\
\mathcal{E}_3 & \xrightarrow{x} & \mathcal{E}_2 & & & & 0 \\
 & & & {\scriptstyle y}\searrow & & \nearrow{\scriptstyle x} & \\
 & & & & \mathcal{E}'_1 & &
\end{array}$$

Proposition 2.10. *On $\widetilde{Flag^{\circ+}_{x,y,x}}$, denote $\mathcal{U} = \ker(\mathcal{U}_3 \twoheadrightarrow \mathcal{U}_1)$, a rank 2 vector bundle. There is a closed embedding $\iota : \mathcal{Y}_+ \hookrightarrow \mathbb{P}(\mathcal{U})$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{Y}_+ & \xhookrightarrow{\iota} & \mathbb{P}(\mathcal{U}) \\ & \searrow^{\mathrm{pr}_+} & \downarrow \\ & & \widetilde{Flag^{\circ+}_{x,y,x}} \end{array}$$

Similar results apply to $\mathcal{Y}_+ \to \widetilde{Flag^{\circ+}_{y,x,x}}$, $\mathcal{Y}_- \to \widetilde{Flag^{\circ-}_{x,y,x}}$, and $\mathcal{Y} \to \widetilde{Flag^{\circ-}_{x,x,y}}$.

Before proving this proposition, we would like to discuss the fiber of $\mathcal{Y}_+$ above a closed point $p = [\mathcal{E}_3 \overset{x}{\twoheadrightarrow} \mathcal{E}_2 \overset{y}{\twoheadrightarrow} \mathcal{E}_1 \overset{x}{\twoheadrightarrow} 0]$ of $\widetilde{Flag^{\circ+}_{x,y,x}}$ (the framings at $\mathcal{E}_3$ and $\mathcal{E}_1$ are not written out). The fiber is parametrized by a length-2 sheaf $\mathcal{E}'_2$ on S which fits into diagram (3). Up to isomorphism, the maps $\mathcal{E}_3 \twoheadrightarrow \mathcal{E}'_2 \twoheadrightarrow \mathcal{E}_1$ have the same data as a length-1 subsheaf $\ker(\mathcal{E}_3 \to \mathcal{E}'_2) \subseteq \ker(\mathcal{E}_3 \to \mathcal{E}_1)$ supported on y. Thus, by previous discussions we have

$$\mathcal{Y}_{+,p} \cong \begin{cases} \mathbb{P}^1, & \text{if } x = y \text{ and } \ker(\mathcal{E}_3 \to \mathcal{E}_1) \cong \mathbb{C}_x^{\oplus 2}, \\ \text{a point}, & \text{otherwise.} \end{cases}$$

Proof of Proposition 2.10. This is similar to the proof of Proposition 2.6. $\square$

Proposition 2.11. *The morphism $\mathrm{pr}_+ : \mathcal{Y}_+ \to \widetilde{Flag^{\circ+}_{x,y,x}}$ is proper and satisfies*

$$R^i \,\mathrm{pr}_{+*} \mathcal{O}_{\mathcal{Y}_+} = \begin{cases} \mathcal{O}_{\widetilde{Flag^{\circ+}_{x,y,x}}}, & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}$$

Similar results apply to $\mathcal{Y}_+ \to \widetilde{Flag^{\circ+}_{y,x,x}}$, $\mathcal{Y}_- \to \widetilde{Flag^{\circ-}_{x,y,x}}$, and $\mathcal{Y}_- \to \widetilde{Flag^{\circ-}_{x,x,y}}$.

Proof. This is similar to the proof of Proposition 2.7. We first apply Proposition 2.10 to show that $R^i \,\mathrm{pr}_{+*} \mathcal{O}_{\mathcal{Y}_+} = 0$ for $i > 0$. For the $i = 0$ case, we need to verify the following statements:

- $\widetilde{Flag^{\circ+}_{x,y,x}}$ is normal. (Lemma 5.3)
- $\mathcal{Y}_+$ is reduced. (Lemma 5.4)
- The fiber of pr_+ is either a point or $\mathbb{P}^1$. This is already addressed in the previous discussions.

$\square$

As in the previous section, there are universal vector bundles $\mathcal{U}_3, \mathcal{U}_2, \mathcal{U}'_2, \mathcal{U}_1$ and universal line bundles $\mathcal{L}_3, \mathcal{L}_2, \mathcal{L}_1, \mathcal{L}'_3, \mathcal{L}'_2$ on $\mathcal{Y}_+$.

Proposition 2.12. *The natural map*

$$\ker(\mathcal{E}_3 \to \mathcal{E}_2) \to \ker(\mathcal{E}_2' \to \mathcal{E}_1)$$

induces a map of line bundles on $\mathcal{Y}_+$*:*

$$\mathcal{L}_3 \to \mathcal{L}_2'$$

The zero section of corresponding line bundle $\mathcal{L}_2' \otimes \mathcal{L}_3^{-1}$ *consists of*

$$\{(\mathcal{E}_2, x) = (\mathcal{E}_2', y)\} \subset \mathcal{Y}_+,$$

which is isomorphic to $\widetilde{Flag}^{\circ+}_{x,x,x}$.

A similar result applies to $\mathcal{L}_1' \otimes \mathcal{L}_2^{-1}$ *on* $\mathcal{Y}_-$.

Proof. This is analogous to the proof of Proposition 2.8. □

3 K-theoretic Hall algebras and commutator relations

3.1 Equivariant K-theory

In this section, we recall some basic definitions and properties of equivariant K-theory. Let G be a reducitive group and X be a scheme over $\mathbb{C}$. All objects and morphisms in this section will be equipped with G-equivariant structures. Suppose X is a scheme over $\mathbb{C}$.

The Grothendieck group $K_G(X)$ of G-equivariant coherent sheaves on X is an abelian group generated by G-equivariant coherent sheaves on X, modulo the relation $[\mathcal{F}] = [\mathcal{F}_1] + [\mathcal{F}_2]$ if there exists a G-equivariant exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0.$$

Let $f : X \to Y$ be a G-equivariant morphism of quasi-projective schemes. If f is proper, then there is a pushforward map

$$f_* : K_G(X) \to K_G(Y)$$

defined by

$$f_*[\mathcal{F}] = \sum_i (-1)^i [R^i f_* \mathcal{F}].$$

This is well-defined since $R^i f_* \mathcal{F}$ vanishes for large i.

On the other hand, if f has finite Tor dimension (for example, when f is flat or smooth), then there is a pullback map

$$f^* : K_G(Y) \to K_G(X)$$

defined by

$$f^*[\mathcal{G}] = \sum_i (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{G})].$$

3.2 Refined Gysin maps

A morphism $f : X \to Y$ is called a *local complete intersection* (l.c.i.) morphism if f is the composition of a regular embedding and a smooth morphism. In this case, f has finite Tor dimension.

Definition 3.1. Suppose we have a Cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

where f is a l.c.i. morphism. The refined Gysin map $f^! : K(Y') \to K(X')$ is defined by

$$f^!([\mathcal{F}]) = \sum_i (-1)^i [\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F})].$$

An important property of refined Gysin maps is the following.

Lemma 3.2 (Lemma 3.1 of [1])**.** *Suppose we have Cartesian squares*

$$\begin{array}{ccccc} X'' & \xrightarrow{h'} & X' & \longrightarrow & X \\ \downarrow{\scriptstyle f''} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y'' & \xrightarrow{h} & Y' & \longrightarrow & Y \end{array}$$

where h is proper and f is l.c.i. Then

$$f^! h_* = h'_* f^! : K(Y'') \to K(X).$$

The definition and properties of refined Gysin maps carry verbatim to the equivariant K-theory.

3.3 Induction

This section follows from Chapter 5.2 of [2]. Let $H \subset G$ be a closed algebraic subgroup and X be a scheme with H-action. We can form the induced space $G \times_H X$: it is the space of orbits of H acting freely on $G \times X$ by $h \cdot (g, x) = (gh^{-1}, hx)$. $G \times_H X$ can be identified as the quotient scheme $(G \times X)/H$.

The projection $G \times X \to G$ induces a flat morphism $(G \times X)/H \to G/H$ with fiber X. For any G-equivariant coherent sheaf $\mathcal{F}$ on $G \times_H X$, we can restrict $\mathcal{F}$ to the fiber over the base point $e \in G/H$. Such a restriction gives an equivalence of categories

$$\mathrm{res} : Coh_G((G \times X)/H) \to Coh_H(X).$$

This gives a canonical isomorphism

$$\mathrm{res} : K_G((G \times X)/H) \to K_H(X)$$

and the inverse map is called the induction from H to G:

$$\mathrm{ind}_H^G : K_H(X) \to K_G((G \times X)/H).$$

The functoriality of this construction implies:

Corollary 3.3. *Suppose H is a closed algebraic subgroup of H_1, H_2, which are closed algebraic subgroups of G. Let X be an H-scheme.*

(a) We have

$$\mathrm{ind}_{H_1}^G \circ \mathrm{ind}_H^{H_1} = \mathrm{ind}_{H_2}^G \circ \mathrm{ind}_H^{H_2} : K_H(X) \to K_G((G \times X)/H).$$

(b) Suppose $f : X \to Y$ is an H-equivariant, proper morphism. Then we have a commutative diagram

$$\begin{array}{ccc} K_H(X) & \longrightarrow & K_H(Y) \\ \downarrow{\scriptstyle \mathrm{ind}_H^G} & & \downarrow{\scriptstyle \mathrm{ind}_H^G} \\ K_G((G \times X)/H) & \longrightarrow & K_G((G \times Y)/H) \end{array}$$

3.4 Derived fiber squares

Consider a Cartesian diagram

$$\begin{array}{ccc} X' & \overset{g}{\hookrightarrow} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \overset{g'}{\hookrightarrow} & Y \end{array} \tag{4}$$

where $f : X \to Y$ is a l.c.i. morphism and $g' : Y' \to Y$ is a regular closed embedding.

Definition 3.4. In diagram (4), if X' has the expected dimension, i.e., if

$$\dim X' - \dim Y' = \dim X - \dim Y,$$

then we say that (4) is a *derived fiber square.*

A derived fiber square has the following property:

Proposition 3.5. *Suppose (4) is a derived fiber square, then f'^* and $f^!$ agree on locally free sheaves as maps $K(Y') \to K(X')$.*

Proof. Since every l.c.i. morphism can be factored into a composition of a smooth morphism and a regular closed embedding, we can write $f = f_2 \circ f_1$, where $f_1 : X \to Z$ is a regular embedding and $f_2 : Z \to Y$ is smooth. Consider the Cartesian diagram

$$\begin{array}{ccc} X' & \hookrightarrow & X \\ \big\downarrow{\scriptstyle f_1'} & & \big\downarrow{\scriptstyle f_1} \\ Z' & \hookrightarrow & Z \\ \big\downarrow{\scriptstyle f_2'} & & \big\downarrow{\scriptstyle f_2} \\ Y' & \hookrightarrow & Y \end{array}$$

Since relative dimension is preserved under base change of smooth morphism, we have

$$\dim Z - \dim Y = \dim Z' - \dim Y'.$$

This implies that

$$\dim X - \dim Z = \dim X' - \dim Z'.$$

Therefore, we can assume in our original diagram that f is a smooth morphism or a regular closed embedding.

If f is smooth, then f' is also smooth and hence flat. Thus, both f'^* and $f^!$ agree with the usual non-derived pullback.

If f is a regular closed embedding, then X' having the expected dimension implies that both g and f' are also regular embeddings. It suffices to consider the case when X, Y are affine, since the refined Gysin map can be computed affine-locally. By inducting on the codimension of X in Y, it suffices to prove the case when codim $X = 1$, i.e., X is cut out by a single function a of $Y = \operatorname{Spec} A$. In this case,

$$\operatorname{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F}) = 0$$

for all $i \geq 2$ and

$$\operatorname{Tor}_1^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F}) = \ker(\mathcal{F} \xrightarrow{\cdot a} \mathcal{F}).$$

Since f' is also regular, $\ker(\mathcal{F} \xrightarrow{\cdot a} \mathcal{F}) = 0$ when $\mathcal{F}$ is free on X, in which case

$$f^!([\mathcal{F}]) = \operatorname{Tor}_0^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{Y'} = f'^*([\mathcal{F}]),$$

as desired. □

3.5 K-theoretic Hall algebra of Quot schemes

This section mainly follows from [7]. Define

$$K(Quot) := \bigoplus_{n=0}^{\infty} K_{\mathrm{GL}_n}(Quot_n^{\circ})$$

as a graded abelian group. [7] constructed a homomorphism

$$*_{Quot} : K_{\mathrm{GL}_n}(Quot_n^{\circ}) \otimes K_{\mathrm{GL}_m}(Quot_m^{\circ}) \to K_{\mathrm{GL}_{n+m}}(Quot_{n+m}^{\circ})$$

which provides $K(Quot)$ with an algebra structure. $(K(Quot), *_{Quot})$ is called the K-theoretic Hall algebra of Quot schemes. It can be shown that this algebra is associative.

The key ingredient of the contruction of $*_{Quot}$ is a homomorphism

$$\psi^!_{m,n} : K_{\mathrm{GL}_n \times \mathrm{GL}_m}(Quot^\circ_n \times Quot^\circ_m) \to K_{P_{n,m}}(Flag^\circ_{n,m})$$

The map $\psi^!_{m,n}$ is given by a refined Gysin pullback in the following fiber diagram:

$$\begin{array}{ccc} Flag^\circ_{n,m} & \longrightarrow & W_{n,m} \\ \downarrow & & \downarrow \psi_{n,m} \\ Quot^\circ_n \times Quot^\circ_m & \longrightarrow & V_{n,m} \end{array}$$

where $W_{n,m}$ and $V_{n,m}$ are vector bundles over $Quot^\circ_n \times Quot^\circ_m$. With $\psi^!_{m,n}$, we can define $*_{Quot}$ as the composition

$$\begin{aligned} &K_{\mathrm{GL}_n}(Quot^\circ_n) \otimes K_{\mathrm{GL}_m}(Quot^\circ_m) \longrightarrow K_{\mathrm{GL}_n \times \mathrm{GL}_m}(Quot^\circ_n \times Quot^\circ_m) \longrightarrow K_{P_{m,n}}(Quot^\circ_n \times Quot^\circ_m) \\ &\longrightarrow K_{P_{m,n}}(Flag^\circ_{n,m}) \xrightarrow{\psi^!_{n,m}} K_{\mathrm{GL}_{n+m}}(\widetilde{Flag^\circ_{n,m}}) \longrightarrow K_{\mathrm{GL}_{n+m}}(Quot^\circ_{n+m}) \end{aligned} \tag{5}$$

3.6 K-theory classes corresponding to $e_{(d_1,\ldots,d_n)}$

For $n = 1, 2, 3$, we would like to associate a K-theory class corresponding to the element $e_{(d_1,\ldots,d_n)}$ which will appear in the elliptic Hall algebra. We shall define $e_{(d_1,\ldots,d_n)}$ as a class in a certain equivariant K-theory group of $Quot^\circ_n$.

- $n = 1$. Consider the $\mathbb{C}^\times$-equivariant K-theory group $K_{\mathbb{C}^\times}(S)$, where $\mathbb{C}^\times$ acts trivially on S. For each integer d, define $e_{(d)}$ to be the class $[\mathcal{L}^d] \in K_{\mathbb{C}^\times}(S)$, where $\mathcal{L}$ is the structure sheaf $\mathcal{O}_S$ with a weight-1 $\mathbb{C}^\times$-action (i.e., $a \in \mathbb{C}^\times$ acts on a section of $\mathcal{L}$ by multiplication).

- $n = 2$. Let d_1, d_2 be two integers. Consider the subscheme $Flag^\circ_{x,x} \hookrightarrow Flag^\circ_{1,1}$ with $P_{1,1}$-action and $P_{1,1}$-equivariant universal line bundles $\mathcal{L}_1$, $\mathcal{L}_2$ on $Flag^\circ_{x,x}$. We define

  $$e_{(d_1,d_2)} := \mathcal{L}_1^{d_1}\mathcal{L}_2^{d_2} \in K_{P_{1,1}}(Flag^\circ_{x,x})$$

  Via the pushforward $Flag^\circ_{x,x} \hookrightarrow Flag^\circ_{1,1}$, we can think of $e_{(d_1,d_2)}$ as the class

  $$\mathcal{L}_1^{d_1}\mathcal{L}_2^{d_2}[\mathcal{O}_{Flag^\circ_{x,x}}] \in K_{P_{1,1}}(Flag^\circ_{1,1}).$$

  Finally, consider the homomorphism

  $$K_{P_{1,1}}(Flag^\circ_{x,x}) \xrightarrow{\mathrm{ind}^{\mathrm{GL}_2}_{P_{1,1}}} K_{\mathrm{GL}_2}(\widetilde{Flag^\circ_{x,x}}) \to K_{\mathrm{GL}_2}(Quot^\circ_2)$$

where the first arrow is given by the induction from the closed subgroup $P_{1,1}$ of GL_2 and the second arrow induced by the proper morphism $\widetilde{Flag^\circ_{x,x}} \to Quot^\circ_2$. Abusing notations, We also denote the image of $e_{(d_1,d_2)}$ in $K_{\mathrm{GL}_2}(Quot^\circ_2)$ as $e_{(d_1,d_2)}$.

- $n = 3$. Generalizing the idea of $n = 2$ case, we define

$$e_{(d_1,d_2,d_3)} := \mathcal{L}_1^{d_1}\mathcal{L}_2^{d_2}\mathcal{L}_3^{d_3} \in K_{P_{1,1,1}}(Flag^\circ_{x,x,x}).$$

Furthermore, consider the homomorphism

$$K_{P_{1,1,1}}(Flag^\circ_{x,x,x}) \xrightarrow{\mathrm{ind}^{\mathrm{GL}_3}_{P_{1,1,1}}} K_{\mathrm{GL}_3}(\widetilde{Flag^\circ_{x,x,x}}) \to K_{\mathrm{GL}_3}(Quot^\circ_3).$$

We abuse our notations to denote the image of $e_{(d_1,d_2,d_3)}$ along the above composition by $e_{(d_1,d_2,d_3)}$ as well.

For general n, it is possible to construct $e_{(d_1,\dots,d_n)}$ in an analogous way. However, the moduli spaces involved in the construction will be highly singular and it is unclear whether the commutator relations generalizing Proposition 3.6 and Proposition 3.7 will hold.

3.7 Commutator relations

Proposition 3.6. *Suppose d, k are integers with $d \geq k$, then*

$$[e_{(d)}, e_{(k)}] := e_{(d)} * e_{(k)} - e_{(k)} * e_{(d)} = \sum_{a=k}^{d-1} e_{(a,d+k-a)} \in K_{\mathrm{GL}_2}(Quot^\circ_2).$$

Proof. In diagram (5), we consider the image of $e_{(d)} \otimes e_{(k)}$ along the composition:

$$\begin{aligned} e_{(d)} \otimes e_{(k)} &= [\mathcal{L}^d] \otimes [\mathcal{L}^k] \in K_{\mathbb{C}^\times}(S) \otimes K_{\mathbb{C}^\times}(S) \\ &\mapsto [\mathcal{L}^d_{(1)} \otimes \mathcal{L}^k_{(2)}] \in K_{\mathbb{C}^\times \times \mathbb{C}^\times}(S \times S) \\ &\mapsto [\mathcal{L}^d_{(1)} \otimes \mathcal{L}^k_{(2)}] \in K_{P_{1,1}}(S \times S) \end{aligned}$$

where $\mathcal{L}$,$\mathcal{L}_{(1)}$,$\mathcal{L}_{(2)}$ are isomorphic to the structure sheaves. $\mathcal{L}$ has a weight-1 $\mathbb{C}^\times$ action, $\mathcal{L}_{(1)}$ has a weight-$(1,0)$ $\mathbb{C}^\times \times \mathbb{C}^\times$ action, and $\mathcal{L}_{(2)}$ has a weight-$(0,1)$ $\mathbb{C}^\times \times \mathbb{C}^\times$ action. By Proposition 3.5, we have

$$\psi^!_{1,1}[\mathcal{L}^d_{(1)} \otimes \mathcal{L}^k_{(2)}] = [\psi^*_{1,1}(\mathcal{L}^d_{(1)} \otimes \mathcal{L}^k_{(2)})] = \mathcal{L}^d_1 \otimes \mathcal{L}^k_2 \in K_{P_{1,1}}(Flag^\circ_{1,1}).$$

Thus,

$$e_{(d)} \otimes e_{(k)} \mapsto [\mathcal{L}^d_1 \otimes \mathcal{L}^k_2] \in K_{P_{1,1}}(Flag^\circ_{1,1}) \mapsto [\mathcal{L}^d_1 \otimes \mathcal{L}^k_2] \in K_{\mathrm{GL}_2}(\widetilde{Flag^\circ_{1,1}}).$$

Now consider the commutative diagram

$$\begin{array}{ccccc}
\mathcal{Y} & \xrightarrow{\text{pr}} & \widetilde{Flag^\circ_{1,1}} & & \\
\downarrow{\scriptstyle \text{pr}'} & & \downarrow{\scriptstyle \phi} & & \\
\widetilde{Flag^\circ_{1,1}} & \xrightarrow{\phi'} & Quot_2^\circ \times S \times S & & \\
& & & \overset{p}{\searrow} & \\
& & & & Quot_2^\circ
\end{array}$$

By Proposition 2.7 and the projection formula, the map

$$\text{pr}_* \, \text{pr}^* : K_{\text{GL}_2}(\widetilde{Flag^\circ_{1,1}}) \to K_{\text{GL}_2}(\mathcal{Y}) \to K_{\text{GL}_2}(\widetilde{Flag^\circ_{1,1}})$$

is an identity on classes of locally free sheaves. The same applies to $\text{pr}'_* \, \text{pr}'^*$. This shows that we could compute the corresponding element $[e_{(d)}, e_{(k)}]$ on $K_{\text{GL}_2}(\mathcal{Y})$ and then push forward to $K_{\text{GL}_2}(Quot_2^\circ)$. As a result,

$$\begin{aligned}
[e_{(d)}, e_{(k)}] &\mapsto [\mathcal{L}_1^d \mathcal{L}_2^k] - [\mathcal{L}_1^k \mathcal{L}_2^d] \in K_{\text{GL}_2}(\widetilde{Flag^\circ_{1,1}}) \\
&\mapsto [\mathcal{L}_1^d \mathcal{L}_2^k] - [\mathcal{L}_1'^k \mathcal{L}_2'^d] \in K_{\text{GL}_2}(\mathcal{Y}),
\end{aligned}$$

where in the second line, we pullback the class $[\mathcal{L}_1^d \mathcal{L}_2^k]$ via pr and the class $[\mathcal{L}_1^k \mathcal{L}_2^d]$ via pr' - this is acceptable since $\text{pr}_* \, \text{pr}^* = \text{pr}'_* \, \text{pr}'^* = \text{id}$ on locally free sheaves and $p \circ \phi \circ \text{pr} = p \circ \phi' \circ \text{pr}'$ as maps $\mathcal{Y} \to Quot_2^\circ$.

By Proposition 2.8, there is an exact sequence of GL_2-equivariant coherent sheaves on $\mathcal{Y}$:

$$0 \to \mathcal{L}_1'^{-1} \otimes \mathcal{L}_2 \to \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\widetilde{Flag^\circ_{x,x}}} \to 0$$

Since $\mathcal{L}_1 \mathcal{L}_2 \cong \mathcal{L}_1' \mathcal{L}_2'$, on $K_{\text{GL}_2}(\mathcal{Y})$ we have

$$[\mathcal{L}_1] - [\mathcal{L}_2'] = [\mathcal{O}_{\widetilde{Flag^\circ_{x,x}}} \otimes \mathcal{L}_2'] = [\mathcal{O}_{\widetilde{Flag^\circ_{x,x}}}][\mathcal{L}_2']$$

From this, we can calculate on $K_{\text{GL}_2}(\mathcal{Y})$ that

$$\begin{aligned}
[\mathcal{L}_1^d \mathcal{L}_2^k] - [\mathcal{L}_1'^k \mathcal{L}_2'^d] &= [\mathcal{L}_1^k \mathcal{L}_2^k]([\mathcal{L}_1]^{d-k} - [\mathcal{L}_2']^{d-k}) \\
&= [\mathcal{L}_1^k \mathcal{L}_2^k]([\mathcal{L}_1] - [\mathcal{L}_2']) \sum_{i=0}^{d-k-1} [\mathcal{L}_1^i \mathcal{L}_2'^{d-k-i-1}] \\
&= [\mathcal{L}_1^k \mathcal{L}_2^k][\mathcal{L}_2'][\mathcal{O}_{\widetilde{Flag^\circ_{x,x}}}] \sum_{i=0}^{d-k-1} [\mathcal{L}_1^i \mathcal{L}_2'^{d-k-i-1}]
\end{aligned}$$

Again, by Proposition 2.7 and projection formula, we can essentially view this as a class on $K_{\mathrm{GL}_2}(\widetilde{Flag^\circ_{x,x}})$. On $\widetilde{Flag^\circ_{x,x}}$, $\mathcal{L}_1' \cong \mathcal{L}_1$ and $\mathcal{L}_2' \cong \mathcal{L}_2$, so

$$\begin{aligned}[e_{(d)}, e_{(k)}] &\mapsto [\mathcal{L}_1^k \mathcal{L}_2^{k+1}] \sum_{i=0}^{d-k-1} [\mathcal{L}_1^i \mathcal{L}_2^{d-k-i-1}] \in K_{\mathrm{GL}_2}(\widetilde{Flag^\circ_{x,x}}) \\ &= \sum_{i=0}^{d-k-1} [\mathcal{L}_1^{k+i} \mathcal{L}_2^{d-i}] \in K_{\mathrm{GL}_2}(\widetilde{Flag^\circ_{x,x}}) \\ &\mapsto \sum_{a=k}^{d-1} e_{(a,d+k-a)} \in K_{\mathrm{GL}_2}(Quot_2^\circ).\end{aligned}$$

This completes the proof. □

Proposition 3.7. *Suppose d_1, d_2, k are integers, then*

$$[e_{(d_1,d_2)}, e_{(k)}] = \begin{cases} -\sum_{a=d_1}^{k-1} e_{(a,d_1+k-a,d_2)}, & \textit{if } k \geq d_1, \\ \sum_{a=k}^{d_1-1} e_{(a,d_1+k-a,d_2)}, & \textit{if } k < d_1. \end{cases} + \begin{cases} -\sum_{a=d_2}^{k-1} e_{(d_1,a,d_2+k-a)}, & \textit{if } k \geq d_2, \\ \sum_{a=k}^{d_2-1} e_{(d_1,a,d_2+k-a)}, & \textit{if } k < d_2. \end{cases}$$

Proof.

- Step 1. We first show that the following diagram is a Cartesian diagram:

$$\begin{array}{ccc} Flag^\circ_{1,1,1} & \longrightarrow & Flag^\circ_{2,1} \\ \downarrow & & \downarrow \\ Flag^\circ_{1,1} \times S & \longrightarrow & Quot_2^\circ \times S \end{array}$$

Let T be a scheme, then a map from T to the fiber product consists of the following data:

$$\begin{array}{ccc} \mathcal{O}_{S\times T}^{\oplus 3} & \longrightarrow & \mathcal{O}_{S\times T} \\ \downarrow & & \downarrow \\ \mathcal{E}_3 & \longrightarrow & \mathcal{E}_1 \end{array}, \quad \begin{array}{ccc} \mathcal{O}_{S\times T}^{\oplus 2} & \longrightarrow & \mathcal{O}_{S\times T} \\ \downarrow & & \downarrow \\ \mathcal{E}_2' & \longrightarrow & \mathcal{E}_1' \end{array}$$

such that we have exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{S\times T}^{\oplus 2} & \hookrightarrow & \mathcal{O}_{S\times T}^{\oplus 3} & \longrightarrow & \mathcal{O}_{S\times T} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}_2' & \hookrightarrow & \mathcal{E}_3 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & 0 \end{array}$$

Denote $\mathcal{K} = \ker(\mathcal{E}_2' \to \mathcal{E}_1')$, then we can form a commutative diagram

$$\begin{array}{ccccccccc} \mathcal{O}_{S\times T} & \hookrightarrow & \mathcal{O}_{S\times T}^{\oplus 2} & \hookrightarrow & \mathcal{O}_{S\times T}^{\oplus 3} & \longrightarrow & \mathcal{O}_{S\times T} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{K} & \hookrightarrow & \mathcal{E}_2' & \hookrightarrow & \mathcal{E}_3 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & 0 \end{array}$$

Define $\mathcal{E}_2 := \mathcal{E}_3/\mathcal{K}$, then $\mathcal{E}_3 \twoheadrightarrow \mathcal{E}_1$ induces $\mathcal{E}_2 \twoheadrightarrow \mathcal{E}_1$ and we obtain a commutative diagram

$$\begin{array}{ccccccc} \mathcal{O}_{S\times T}^{\oplus 3} & \longrightarrow & \mathcal{O}_{S\times T}^{\oplus 2} & \longrightarrow & \mathcal{O}_{S\times T} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{E}_3 & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & 0 \end{array}$$

which is the data of a map $T \to Flag^{\circ}_{1,1,1}$. This data defines a morphism from the fiber product of $Flag^{\circ}_{2,1}$ and $Flag^{\circ}_{1,1} \times S$ (above the base $Quot^{\circ}_2 \times S$) to $Flag^{\circ}_{1,1,1}$. The construction of the morphism in the other direction is clear. We can also check that these two morphisms are inverses to each other.

- Step 2. Note that we have another Cartesian diagram

$$\begin{array}{ccccc} \widetilde{Flag}^{\circ+}_{x,x,y} & \longrightarrow & \widetilde{Flag}^{\circ+}_{1,1,1} & \longrightarrow & Flag^{\circ}_{1,1,1} \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{Flag}^{\circ}_{x,x} \times S & \longrightarrow & \widetilde{Flag}^{\circ}_{1,1} \times S & \longrightarrow & Flag^{\circ}_{1,1} \times S \end{array}$$

By step 1,

$$\begin{array}{ccccc} \widetilde{Flag}^{\circ+}_{x,x,y} & \xrightarrow{p_+} & Flag^{\circ}_{2,1} & \longrightarrow & W_{2,1} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \psi_{2,1}} \\ \widetilde{Flag}^{\circ}_{x,x} \times S & \xrightarrow{p\times \mathrm{id}} & Quot^{\circ}_2 \times S & \longrightarrow & V_{2,1} \end{array}$$

is a fiber diagram. By projection formula, $e_{(d_1,d_2)} \otimes e_{(k)}$ can be thought of as a class $[\mathcal{L}_1^{d_1} \mathcal{L}_2^{d_2} \boxtimes \mathcal{L}^k]$ on $K_{\mathrm{GL}_2 \times \mathrm{GL}_1}(\widetilde{Flag}^{\circ}_{x,x} \times S)$, where $\mathcal{L}$ is line bundle $\mathcal{O}_S$ with weight-1 $\mathbb{C}^{\times}$-action. Since the morphism $p : \widetilde{Flag}^{\circ}_{x,x} \to Quot^{\circ}_2$ is proper, by Lemma 3.2,

$$\psi^!_{2,1} \circ (p \times \mathrm{id})_* = p_{+*} \circ \psi^!_{2,1} : K_{P_{1,2}}(\widetilde{Flag}^{\circ}_{x,x} \times S) \to K_{P_{1,2}}(Flag^{\circ}_{2,1}).$$

To show that

$$\psi^!_{2,1} : K_{P_{1,2}}(\widetilde{Flag}^{\circ}_{x,x} \times S) \to K_{P_{1,2}}(\widetilde{Flag}^{\circ+}_{x,x,y})$$

agrees with the usual pullback, we note that $V_{2,1}$ and $W_{2,1}$ are vector bundles over $Quot^{\circ}_2 \times S$ and $Quot^{\circ}_2 \times S \to V_{2,1}$ is the inclusion of zero section. Let $\widetilde{V_{2,1}}$ and $\widetilde{W_{2,1}}$ be the pullback of $V_{2,1}$ and $W_{2,1}$ along $\widetilde{Flag}^{\circ}_{x,x} \times$

$S \to Quot_2^\circ \times S$. Then we have a Cartesian diagram

$$\begin{array}{ccc} \widetilde{Flag^{\circ+}_{x,x,y}} & \longrightarrow & \widetilde{W_{2,1}} \\ \downarrow & & \downarrow \widetilde{\psi_{2,1}} \\ \widetilde{Flag^{\circ}_{x,x}} \times S & \longrightarrow & \widetilde{V_{2,1}} \end{array} \tag{6}$$

By [7], $\widetilde{\psi_{2,1}}^! = \psi_{2,1}^!$. Note that

$$\dim \widetilde{Flag^{\circ+}_{x,x,y}} - \dim(\widetilde{Flag^{\circ}_{x,x}} \times S) = 9 - 7 = 2$$

and

$$\dim \widetilde{W_{2,1}} - \dim \widetilde{V_{2,1}} = \dim W_{2,1} - \dim V_{2,1} = \operatorname{rank} \mathcal{O}^{\oplus 2}/\mathcal{E}_2 = 2,$$

so diagram (6) is a derived fiber square. By Proposition 3.5, $\widetilde{\psi_{2,1}}^!$ agrees with the usual pullback. Therefore,

$$\psi_{2,1}^![\mathcal{L}_1^{d_1}\mathcal{L}_2^{d_2} \boxtimes \mathcal{L}^k] = [\mathcal{L}_2^{d_1}\mathcal{L}_3^{d_2}\mathcal{L}_1^k] \in K_{P_{1,2}}(\widetilde{Flag^{\circ+}_{x,x,y}}).$$

- Step 3. In the previous step, we see that $e_{(d_1,d_2)} \otimes e_{(k)} \mapsto [\mathcal{L}_1^k\mathcal{L}_2^{d_1}\mathcal{L}_3^{d_2}] \in K_{P_{1,2}}(\widetilde{Flag^{\circ+}_{x,x,y}})$. We can regard it as a class in $K_{P_{1,1,1}}(Flag^\circ_{x,x,y})$ (also denoted by $[\mathcal{L}_1^k\mathcal{L}_2^{d_1}\mathcal{L}_3^{d_2}]$) via the induction map $\operatorname{ind}_{P_{1,1,1}}^{P_{1,2}}$. Furthermore, by Corollary 3.3, we have a commutative diagram

$$\begin{array}{ccccc} K_{P_{1,1,1}}(Flag^\circ_{x,x,y}) & \xrightarrow{\operatorname{ind}_{P_{1,1,1}}^{P_{1,2}}} & K_{P_{1,2}}(\widetilde{Flag^{\circ+}_{x,x,y}}) & \longrightarrow & K_{P_{1,2}}(Flag^\circ_{2,1}) \\ \downarrow \operatorname{ind}_{P_{1,1,1}}^{P_{2,1}} & & \downarrow \operatorname{ind}_{P_{1,2}}^{\mathrm{GL}_3} & & \downarrow \operatorname{ind}_{P_{1,2}}^{\mathrm{GL}_3} \\ K_{P_{2,1}}(\widetilde{Flag^{\circ-}_{x,x,y}}) & \xrightarrow{\operatorname{ind}_{P_{2,1}}^{\mathrm{GL}_3}} & K_{\mathrm{GL}_3}(\widetilde{Flag^{\circ}_{x,x,y}}) & \longrightarrow & K_{\mathrm{GL}_3}(\widetilde{Flag^{\circ}_{2,1}}) \\ & & & & \downarrow \\ & & & & K_{\mathrm{GL}_3}(Quot_3^\circ) \end{array} \tag{7}$$

The upper-right path agrees with diagram (5). However, we will use the lower-left path to evaluate the commutator $[e_{(d_1,d_2)}, e_{(k)}]$.

As before,

$$\begin{aligned} e_{(d_1,d_2)} \otimes e_{(k)} &\mapsto [\mathcal{L}_1^k\mathcal{L}_2^{d_1}\mathcal{L}_3^{d_2}] \in K_{P_{2,1}}(\widetilde{Flag^{\circ-}_{x,x,y}}) \\ &\mapsto [\mathcal{L}_1^k\mathcal{L}_2^{d_1}\mathcal{L}_3^{d_2}] \in K_{P_{2,1}}(\mathcal{Y}_-) \end{aligned}$$

We consider an auxiliary class $[\mathcal{L}_1^{d_1}\mathcal{L}_2^k\mathcal{L}_3^{d_2}] \in K_{P_{1,1,1}}(Flag^\circ_{x,y,x})$. Denote $e \in K_{\mathrm{GL}_3}(Quot_3^\circ)$ to be the image of $[\mathcal{L}_1^{d_1}\mathcal{L}_2^k\mathcal{L}_3^{d_2}]$ under the composition

$$K_{P_{1,1,1}}(Flag^\circ_{x,y,x}) \xrightarrow{\operatorname{ind}_{P_{1,1,1}}^{\mathrm{GL}_3}} K_{\mathrm{GL}_3}(\widetilde{Flag^{\circ}_{x,y,x}}) \to K_{\mathrm{GL}_3}(Quot_3^\circ).$$

Again, by the projection formula and the fact that $P_{2,1}$ is a $\mathbb{P}^1$-bundle over $P_{1,1,1}$, we can view this auxiliary class as an element in $K_{P_{2,1}}(\widetilde{Flag^{\circ-}_{x,y,x}})$. By Proposition 2.11, we can further view it as an element

$$[\mathcal{L}_1'^{d_1}\mathcal{L}_2'^{k}\mathcal{L}_3^{d_2}] \in K_{P_{2,1}}(\mathcal{Y}_-).$$

By Proposition 2.12, there is an exact sequence on $\mathcal{Y}_-$:

$$0 \to \mathcal{L}_2'\mathcal{L}_1^{-1} \to \mathcal{O}_{\mathcal{Y}_-} \to \mathcal{O}_{\widetilde{Flag^{\circ-}_{x,x,x}}} \to 0.$$

After a similar calculation as in the proof of Proposition 3.6, we obtain

$$[\mathcal{L}_1^k\mathcal{L}_2^{d_1}] - [\mathcal{L}_1'^{d_1}\mathcal{L}_2'^{k}] = \begin{cases} -\sum_{a=d_1}^{k-1}[\mathcal{L}_1^a\mathcal{L}_2^{d_1+k-a}], & \text{if } k \geq d_1, \\ \sum_{a=k}^{d_1-1}[\mathcal{L}_1^a\mathcal{L}_2^{d_1+k-a}], & \text{if } k < d_1. \end{cases}$$

as an element of $K_{P_{2,1}}(\widetilde{Flag^{\circ-}_{x,x,x}})$. Moreover, combining the following commutative diagram

$$\begin{array}{ccc} K_{P_{2,1}}(\widetilde{Flag^{\circ-}_{x,x,x}}) & \longrightarrow & K_{P_{2,1}}(\widetilde{Flag^{\circ-}_{x,x,y}}) \\ \big\downarrow \mathrm{ind}_{P_{2,1}}^{\mathrm{GL}_3} & & \big\downarrow \mathrm{ind}_{P_{2,1}}^{\mathrm{GL}_3} \\ K_{\mathrm{GL}_3}(\widetilde{Flag^{\circ}_{x,x,x}}) & \longrightarrow & K_{\mathrm{GL}_3}(\widetilde{Flag^{\circ}_{x,x,y}}) \end{array}$$

and diagram (7), we see that

$$e_{(d_1,d_2)} * e_{(k)} - e = \begin{cases} -\sum_{a=d_1}^{k-1} e_{(a,d_1+k-a,d_2)}, & \text{if } k \geq d_1, \\ \sum_{a=k}^{d_1-1} e_{(a,d_1+k-a,d_2)}, & \text{if } k < d_1. \end{cases}$$

in $K_{\mathrm{GL}_3}(Quot_3^\circ)$.

- Step 4. We can use similar method to calculate the difference $e - e_{(k)} * e_{(d_1,d_2)}$ in $K_{\mathrm{GL}_3}(Quot_3^\circ)$:

$$e - e_{(k)} * e_{(d_1,d_2)} = \begin{cases} -\sum_{a=d_2}^{k-1} e_{(d_1,a,d_2+k-a)}, & \text{if } k \geq d_2, \\ \sum_{a=k}^{d_2-1} e_{(d_1,a,d_2+k-a)}, & \text{if } k < d_2. \end{cases}$$

Combining this with the calculation of $e_{(d_1,d_2)} * e_{(k)} - e$ in the previous paragraph, we obtain the desired formula of $[e_{(d_1,d_2)}, e_{(k)}]$.

□

4 Homomorphism between elliptic and K-theoretic Hall algebras

4.1 Elliptic Hall algebra

This section follows from section 4 of [5]. Let q_1, q_2 be formal parameters and $q = q_1 q_2$. We define the (positive part of) elliptic Hall algebra $\mathcal{A}_{>0}$ as follows.

Definition 4.1. The algebra $\mathcal{A}_{>0}$ is defined over the ring $\mathbb{Z}[q_1^{\pm}, q_2^{\pm}]^{\text{Sym}}$ and generated by symbols $\{E_k : k \in \mathbb{Z}\}$, modulo the following relations:

- $[[E_{k+1}, E_{k-1}], E_k] = 0$,
- $(z-wq_1)(z-wq_2)\left(z - \frac{w}{q}\right) E(z)E(w) = \left(z - \frac{w}{q_1}\right)\left(z - \frac{w}{q_2}\right)(z-wq)E(w)E(z)$,

where $E(z) = \sum_{k\in\mathbb{Z}} E_k z^{-k}$.

An alternative definition of $\mathcal{A}_{>0}$ uses the elliptic Hall algebra, i.e., it is generated by symbols $E_{n,k}$, where $n, k \in \mathbb{Z}$ and $n < 0$, modulo certain relations (Theorem 4.4 of [5]). The two equivalent definitions of $\mathcal{A}_{>0}$ are identified by $E_k = E_{-1,k}$.

We will also introduce the third definition of $\mathcal{A}_{>0}$. This will be the description that we will use in the later sections.

Proposition 4.2. *There exists a unique collection of elements $E_{(d_1,\ldots,d_n)} \in \mathcal{A}_{>0}$ for all $d_1, \ldots, d_n \in \mathbb{Z}$ such that*

- $E_{(d_1,\ldots,d_n)}E_{(d'_1,\ldots,d'_m)} = E_{(d_1,\ldots,d_n,d'_1,\ldots,d'_m)} - qE_{(d_1,\ldots,d_{n-1},d_n-1,d'_1+1,d'_2,\ldots,d'_m)}$,
- $E_{-n,k} = q^{\gcd(n,k)-1}E_{(d_1,\ldots,d_n)}$, *where* $d_i = \lceil \frac{ki}{n} \rceil - \lceil \frac{k(i-1)}{n} \rceil + \delta_i^n - \delta_i^1$.

An important relation among $E_{(d_1,\ldots,d_n)}$ is the following.

Proposition 4.3 (Proposition 4.7 of [5]). *For any $d_1, \ldots, d_n, k \in \mathbb{Z}$, we have*

$$[E_{(d_1,\ldots,d_n)}, E_{(k)}] = (1-q_1)(1-q_2) \sum_{i=1}^{n} \begin{cases} \sum_{a=d_i}^{k-1} E_{(d_1,\ldots,d_{i-1},a,d_i+k-a,d_{i+1},\ldots d_n)}, & \text{if } d_i \le k, \\ -\sum_{a=k}^{d_i-1} E_{(d_1,\ldots,d_{i-1},a,d_i+k-a,d_{i+1},\ldots d_n)}, & \text{if } d_i > k, \end{cases}$$

4.2 Proof of Theorem 1.1

Before the proof of our main theorem, we state a key proposition which connects the definition of elliptic Hall algebra and the commutator relations.

Proposition 4.4 (Proposition 4.8 of [5]). *The relations in Definition 4.1 follow from the special cases $n = 1, 2$ of Proposition 4.3.*

Proof of Theorem 1.1. Define a map $\mathcal{A}_{>0} \to K(Quot)$ as follows: for $n = 1, 2, 3$,

$$E_{(d_1,\ldots,d_n)} \mapsto -(1-q_1)(1-q_2)e_{(d_1,\ldots,d_n)},$$

and we extend the map linearly (since $E_{(k)}$ already generates $\mathcal{A}_{>0}$). To show that this map is an algebra homomorphism, by Proposition 4.4, it suffices to check the commutator relations

$$[e_{(d_1,\ldots,d_n)}, e_{(k)}] = \sum_{i=1}^{n} \begin{cases} -\sum_{a=d_i}^{k-1} e_{(d_1,\ldots,d_{i-1},a,d_i+k-a,d_{i+1},\ldots d_n)}, & \text{if } d_i \le k, \\ \sum_{a=k}^{d_i-1} e_{(d_1,\ldots,d_{i-1},a,d_i+k-a,d_{i+1},\ldots d_n)}, & \text{if } d_i > k, \end{cases}$$

when $n = 1, 2$. These commutator relations are proven in Proposition 3.6 and Proposition 3.7, as desired. ∎

5 Appendix

Here are several results related to the geometric properties of Quot and Flag schemes.

Lemma 5.1. *The schemes* $Flag^{\circ}_{1,1}$*,* $Flag^{\circ}_{x,x,y}$*,* $Flag^{\circ}_{x,y,x}$*, and* $Flag^{\circ}_{y,x,x}$ *are Cohen-Macaulay.*

Proof. Since Cohen-Macaulay property is deformation invariant, by Remark 2.4 we can use the local descriptions by matrices. For $Flag^{\circ}_{1,1}$, we have

$$Flag^{loc}_{1,1} = \{X, Y \in B_{1,1} : XY = YX\}$$

where $B_{1,1}$ is the Borel subgroup consisting of all lower-triangular 2×2 matrices. Write $X = (X_{ij})_{1\le j\le i\le 2}$ and $Y = (Y_{ij})_{1\le j\le i\le 2}$, then $Flag^{loc}_{1,1}$ is isomorphic to $\mathbb{A}^6$ cut out by the equation

$$X_{21}(Y_{11} - Y_{22}) = Y_{21}(X_{11} - X_{22}).$$

This shows that $\dim Flag^{\circ}_{1,1} = 5$ and that it is l.c.i. In particular, $Flag^{\circ}_{1,1}$ is Cohen-Macaulay.

For $Flag^{\circ}_{x,x,y}$, we have

$$Flag^{loc}_{x,x,y} = \{X, Y \in B_{1,1,1} : XY = YX, X_{22} = X_{33}, Y_{22} = Y_{33}\}.$$

We can compute that $Flag^{loc}_{x,x,y}$ is isomorphic to $\mathbb{A}^{10}$ cut out by equations

$$\begin{aligned} &X_{21}(Y_{11} - Y_{22}) = Y_{21}(X_{11} - X_{22}) \\ &X_{31}(Y_{11} - Y_{22}) - Y_{31}(X_{11} - X_{22}) = Y_{32}X_{21} - X_{32}Y_{21} \end{aligned}$$

It is not difficult to verify that the two equations give rise to a length-2 regular sequence. Therefore, $\dim Flag^{loc}_{x,x,y} = 8$ and $Flag^{\circ}_{x,x,y}$ is l.c.i. By symmetry, $Flag^{\circ}_{y,x,x}$ is also l.c.i.

For $Flag^{\circ}_{x,y,x}$, we can compute that $Flag^{loc}_{x,y,x}$ is isomorphic to $\mathbb{A}^{10}$ cut out by equations

$$\begin{aligned} &X_{21}(Y_{11} - Y_{22}) = Y_{21}(X_{11} - X_{22}) \\ &Y_{32}X_{21} = X_{32}Y_{21} \\ &X_{32}(Y_{22} - Y_{33}) = Y_{32}(X_{22} - X_{33}) \end{aligned}$$

By the lemma below, $Flag^{\circ}_{x,y,x}$ is Cohen-Macaulay of dimension 8. □

Lemma 5.2. *The ring*

$$R = \mathbb{C}[x_1, x_2, x_3, y_1, y_2, y_3]/(x_1y_2 - x_2y_1, x_2y_3 - x_3y_2, x_3y_1 - x_1y_3)$$

is Cohen-Macaulay of dimension 4.

Proof. We claim that $x_1, y_3, x_2 + y_1, x_3 + y_2$ form a regular sequence in R.

- x_1 is not a zero divisor in R.

  For the sake of contradiction, assume that there exists $A, B, C, F \in \mathbb{C}[x_1, \ldots, y_3]$ such that $F \notin I$ and

  $$Fx_1 = A(x_1y_2 - x_2y_1) + B(x_2y_3 - x_3y_2) + C(x_3y_1 - x_1y_3).$$

  We can further assume that A, B, C do not contain the variable x_1. This implies that

  $$\begin{cases} F = Ay_2 - Cy_3, \\ y_1(Cx_3 - Ax_2) + B(x_2y_3 - x_3y_2) = 0. \end{cases}$$

  Note that for any polynomial D, by replacing (A, B, C) with $(A - y_3D, B - y_1D, C - y_2D)$, the above equations are preserved. Therefore, we can assume that B does not contain the variable y_1. This gives $B = 0$ and $Cx_3 = Ax_2$. Furthermore, we can write $C = x_2E$ and $A = x_3E$ for some polynomial E. Thus,

  $$F = Ay_2 - Cy_3 = E(x_3y_2 - x_2y_3) \in I,$$

  a contradiction!

- y_3 is not a zero divisor in $R/(x_1) \cong \mathbb{C}[x_2, x_3, y_1, y_2, y_3]/(x_2y_1, x_2y_3 - x_3y_2, x_3y_1)$.

  For the sake of contradiction, assume that there exists $A, B, C, F \in \mathbb{C}[x_2, \ldots, y_3]$ such that $F \notin I_1$ and

  $$Fy_3 = Ax_2y_1 + B(x_2y_3 - x_3y_2) + Cx_3y_1$$

  We can assume that A, B, C do not contain the variable y_3, so

  $$\begin{cases} F = Bx_2, \\ Ax_2y_1 = x_3(By_2 - Cy_1) \end{cases}$$

  Note that by replacing (A, C) with $(A - x_3D, C + x_2D)$, the above equations are preserved. Thus, we can assume that A does not contain the variable x_3. This gives $A = 0$ and $By_2 = Cy_1$. We can write $B = y_1E$ and $C = y_2E$ for some polynomial E. Thus,

  $$F = Bx_2 = Ey_1x_2 \in I_1,$$

  a contradiction!

- $x_2 + y_1$ is not a zero divisor in $R/(x_1, y_3) \cong \mathbb{C}[x_2, x_3, y_1, y_2]/(x_2y_1, x_3y_2, x_3y_1)$. Denote $x_2 = t - y_1$. We need to show that t is not a zero divisor in $\mathbb{C}[x_3, y_1, y_2, t]/((t - y_1)y_1, x_3y_2, x_3y_1)$

For the sake of contradiction, assume that there exists A, B, C, F such that $F \notin I_2$ and

$$Ft = A(t - y_1)y_1 + Bx_3y_2 + Cx_3y_1.$$

Assume that A, B, C do not contain the variable t, then

$$\begin{cases} F = Ay_1, \\ Ay_1^2 = x_3(By_2 + Cy_1) \end{cases}$$

The second equation implies that $A = x_3A'$ for some polynomial A. Thus,

$$F = Ay_1 = A'x_3y_1 \in I_2,$$

a contradiction!

- $x_3 + y_2$ is not a zero divisor in $R/(x_1, y_3, x_2 + y_1) \cong \mathbb{C}[x_2, x_3, y_1, y_2]/(x_2 + y_1, x_2y_1, x_3y_2, x_3y_1) \cong \mathbb{C}[x_3, y_1, y_2]/(y_1^2, x_3y_2, x_3y_1)$. Denote $y_2 = s - x_3$, then we need to show that s is not a zero divisor in $\mathbb{C}[x_3, y_1, s]/(y_1^2, x_3(s - x_3), x_3y_1)$.

  For the sake of contradiction, assume that there exists A, B, C, F such that $F \notin I_3$ and

$$Fs = Ay_1^2 + Bx_3(s - x_3) + Cx_3y_1.$$

  Assuming A, B, C do not contain the variable s, we obtain

$$\begin{cases} F = Bx_3, \\ Bx_3^2 = y_1(Ay_1 + Cx_3). \end{cases}$$

  The second equation implies that $B = y_1B'$ for some polynomial B. Therefore,

$$F = Bx_3 = B'x_3y_1 \in I_3,$$

  a contradiction!

Finally, it is easy to vefiry that $x_1y_2 - x_2y_1, x_2y_3 - x_3y_2$ form a regular sequence in $\mathbb{C}[x_1, x_2, x_3, y_1, y_2, y_3]$. This shows that $\dim R \leq 4$. However, since we can find a regular sequence of length 4 in R, $\dim R \geq 4$. As a result, $\dim R = 4$ and R is Cohen-Macaulay. □

Lemma 5.3. *$\widetilde{Flag^\circ_{1,1}}$, $\widetilde{Flag^{\circ\pm}_{x,x,y}}$, $\widetilde{Flag^{\circ\pm}_{x,y,x}}$, $\widetilde{Flag^{\circ\pm}_{y,x,x}}$ are normal.*

Proof. Since $\widetilde{Flag^\circ_{1,1}}$ is $\mathbb{P}^1$-bundle over $Flag^\circ_{1,1}$, it suffices to show that $Flag^\circ_{1,1}$ is normal. Here we use the fact that a ring R is normal if and only if the polynomial ring $R[t_1, \ldots, t_n]$ is normal. Furthermore, $Flag^\circ_{1,1}$ is Cohen-Macaulay, so we only need to prove that the singular locus of $Flag^\circ_{1,1}$ is of codimension ≥ 2.

Since normality is deformation invariant, by Remark 2.4 we can use the local matrix descriptions. From the proof of Lemma 5.1, we see that $Flag^{loc}_{1,1}$ is cut out by the equation

$$X_{21}(Y_{11}-Y_{22})-Y_{21}(X_{11}-X_{22}).$$

By taking partial derivatives, we can compute that the singular locus is locally given by

$$X_{21}=Y_{21}=X_{11}-X_{22}=Y_{11}-Y_{22}=0,$$

which is of codimension 4, as desired.

Similarly, for the other cases, it suffices to show that the singular loci of $Flag^{\circ}_{x,x,y}$, $Flag^{\circ}_{x,y,x}$, and $Flag^{\circ}_{y,x,x}$ have codimension ≥ 2. Here we only prove the $Flag^{\circ}_{x,y,x}$ case. In the proof of Lemma 5.1, we computed that $Flag^{loc}_{x,y,x}$ is isomorphic to

$$\mathbb{A}^4\times\mathbb{A}^6/(x_1y_2-x_2y_1,x_2y_3-x_3y_2,x_3y_1-x_1y_3).$$

The Jacobian matrix of this ideal is

$$J=\begin{pmatrix} y_2 & -y_1 & & -x_2 & x_1 & \\ y_3 & & -y_1 & -x_3 & & x_1 \\ & y_3 & -y_2 & & -x_3 & x_2 \end{pmatrix}$$

At general points, the rank of J is 2. The singular locus is the set of points where $\mathrm{rank}(J)\leq 1$. This only happens when $x_i=y_i=0$ for all i. Thus, the singular locus of $Flag^{\circ}_{x,y,x}$ has codimension 4, as desired. ☐

Lemma 5.4. *$\mathcal{Y},\mathcal{Y}_{\pm}$ are reduced.*

Proof. Let $\mathcal{Y}'$ be the fiber product

$$\begin{array}{ccc} \mathcal{Y}' & \longrightarrow & Flag^{\circ}_{1,1}\times \mathrm{GL}_2 \\ \downarrow & & \downarrow \phi \\ Flag^{\circ}_{1,1}\times\mathrm{GL}_2 & \xrightarrow{\phi'} & Quot^{\circ}_2\times S\times S \end{array} \tag{8}$$

Then $\mathcal{Y}'$ is a $P_{1,1}\times P_{1,1}$-bundle over $\mathcal{Y}$. As $P_{1,1}$ is isomorphic to an open subset of $\mathbb{A}^3$, it suffices to show that $\mathcal{Y}'$ is reduced. Here we use the fact that a ring R is reduced if and only if the polynomial ring $R[t_1,\dots,t_n]$ is reduced.

We would like to explicitly describe the maps ϕ and ϕ' locally via matrices. We have discussed that locally

$$Flag^{loc}_{1,1}=\{X,Y\in B_{1,1}:XY=YX\}.$$

Furthermore, for a closed point

$$p=\left[\begin{array}{ccccc} \mathcal{O}_S^{\oplus 2} & \twoheadrightarrow & \mathcal{O}_S & \twoheadrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathcal{E}_2 & \xrightarrow{x} & \mathcal{E}_1 & \xrightarrow{y} & 0 \end{array}\right],$$

of $Flag^{loc}_{1,1}$, the corresponding $X, Y \in B_{1,1}$ satisfies

$$y = (X_{11}, Y_{11}), x = (X_{22}, Y_{22}) \in \mathbb{A}^2.$$

As a result, the map ϕ can be explicitly described as

$$\phi((X, Y), g) = ((gXg^{-1}, gYg^{-1}), (X_{22}, Y_{22}), (X_{11}, Y_{11})).$$

Similarly, ϕ' is defined as

$$\phi'((X, Y), g) := ((gXg^{-1}, gYg^{-1}), (X_{11}, Y_{11}), (X_{22}, Y_{22})).$$

Therefore, the closed points of $\mathcal{Y}'$ can be locally described as:

$$(X, Y, g; X', Y', g') : X, Y, X', Y' \in B_{1,1}, g, g' \in \mathrm{GL}_2$$

such that

$$\begin{cases} (X_{11}, Y_{11}) = (X'_{22}, Y'_{22}), (X_{22}, Y_{22}) = (X'_{11}, Y'_{11}), \\ gXg^{-1} = g'X'g'^{-1}, gYg^{-1} = g'Y'g'^{-1}, \\ XY = YX, X'Y' = Y'X'. \end{cases}$$

At this point, we could have computed the explicit expression for $\mathcal{Y}'$ and verify that it is indeed reduced. A way to simplify this calculation is to observe that there is a free GL_2-action on $\mathcal{Y}'$. GL_2 acts on fiber diagram (8) by acting on each $Flag^{\circ}_{1,1}$, $Quot^{\circ}_2$, and S trivially, while acting on each GL_2 by left multiplication. Locally in terms of matrices, we can describe this action by

$$h \cdot (X, Y, g; X', Y', g') := (X, Y, gh^{-1}; X', Y', g'h^{-1})$$

for $h \in \mathrm{GL}_2$. We can easily identify the quotient $\mathcal{Y}' / \mathrm{GL}_2$ as a scheme: it is the fiber product

$$\begin{array}{ccc} \mathcal{Y}'' & \longrightarrow & Flag^{\circ}_{1,1} \times \mathrm{GL}_2 \\ \downarrow & & \downarrow \phi \\ Flag^{\circ}_{1,1} & \xrightarrow{\varphi} & Quot^{\circ}_2 \times S \times S \end{array}$$

where φ is the map $\phi'(\cdot, \mathbf{1})$. As before, since GL_2 is an open subscheme of $\mathbb{A}^4$, the reducedness of $\mathcal{Y}'$ is equivalent to the reducedness of $\mathcal{Y}''$.

Now, the closed points of $\mathcal{Y}''$ can be described locally as

$$(X, Y, g; X', Y') : X, Y, X', Y' \in B_{1,1}, g \in \mathrm{GL}_2$$

such that

$$\begin{cases} (X_{11}, Y_{11}) = (X'_{22}, Y'_{22}), (X_{22}, Y_{22}) = (X'_{11}, Y'_{11}) \\ gXg^{-1} = X', gYg^{-1} = Y', \\ XY = YX, X'Y' = Y'X'. \end{cases}$$

The condition $X'g = gX$ is equivalent to

$$\begin{cases} g_{11}(X_{22} - X_{11}) = g_{12}X_{21}, \\ g_{11}X'_{21} = g_{22}X_{21}, \\ g_{22}(X_{22} - X_{11}) = g_{12}X'_{21} \end{cases}$$

and similar for $Y'g = gY$. The condition $XY = YX$ is equivalent to

$$X_{21}(Y_{22} - Y_{11}) = Y_{21}(X_{22} - X_{11}).$$

We have two cases (since $g \in \mathrm{GL}_2$):

- $g_{11} \neq 0$ (i.e., in the open subset $\{g_{11} \neq 0\} \cap \mathrm{GL}_2$). We obtain

$$\begin{cases} X_{22} - X_{11} = \frac{g_{12}}{g_{11}}X_{21}, \\ X'_{21} = \frac{g_{22}}{g_{11}}X_{21}, \end{cases} \quad \text{and} \quad \begin{cases} Y_{22} - Y_{11} = \frac{g_{12}}{g_{11}}Y_{21}, \\ Y'_{21} = \frac{g_{22}}{g_{11}}Y_{21}. \end{cases}$$

  These equations cut out an affine space.

- $g_{12} \neq 0$. We obtain

$$\begin{cases} X'_{21} = \frac{g_{22}}{g_{12}}(X_{22} - X_{11}), \\ X_{21} = \frac{g_{11}}{g_{12}}(X_{22} - X_{11}), \end{cases} \quad \text{and} \quad \begin{cases} Y'_{21} = \frac{g_{22}}{g_{12}}(Y_{22} - Y_{11}), \\ Y_{21} = \frac{g_{11}}{g_{12}}(Y_{22} - Y_{11}). \end{cases}$$

  These equations cut out an affine space.

This shows that $\mathcal{Y}''$ is reduced, as desired.

For the scheme $\mathcal{Y}_+$, by applying the same argument, it suffices to show that the following fiber product is reduced:

$$\begin{array}{ccc} \mathcal{Y}'_+ & \longrightarrow & \widetilde{Flag}^{\circ+}_{x,y,x} \\ \big\downarrow & & \big\downarrow \\ Flag^{\circ}_{y,x,x} & \longrightarrow & Flag^{\circ}_{2,1} \times S \times S \end{array}$$

Using the local description

$$\widetilde{Flag}^{loc+}_{x,y,x} = \{X, Y \in B_{1,1,1}, g \in P_{1,2} : XY = YX, X_{11} = X_{33}, Y_{11} = Y_{33}\}$$
$$Flag^{loc+}_{y,x,x} = \{X', Y' \in B_{1,1,1} : X'Y' = Y'X', X'_{11} = X'_{22}, Y'_{11} = Y'_{22}\}$$

we can write $\mathcal{Y}_+$ locally as points $(g, X, Y; X', Y')$ which satisfy:

$$\begin{cases} X_{11} = X_{33} = X'_{11} = X'_{22}, X_{22} = X'_{33}, \\ Y_{11} = Y_{33} = Y'_{11} = Y'_{22}, Y_{22} = Y'_{33}, \\ X' = gXg^{-1}, Y' = gYg^{-1}, \\ XY - YX = X'Y' - Y'X' = 0. \end{cases}$$

The condition $X'g = gX$ is equivalent to

$$\begin{pmatrix} g_{11}X_{11} & & \\ g_{11}X'_{21} + g_{21}X_{11} & g_{22}X_{11} & g_{23}X_{11} \\ g_{11}X'_{31} + g_{21}X'_{32} + g_{31}X_{22} & g_{22}X'_{32} + g_{32}X_{22} & g_{23}X'_{32} + g_{33}X_{22} \end{pmatrix}$$
$$= \begin{pmatrix} g_{11}X_{11} & & \\ g_{21}X_{11} + g_{22}X_{21} + g_{23}X_{31} & g_{22}X_{22} + g_{23}X_{32} & g_{23}X_{11} \\ g_{31}X_{11} + g_{32}X_{21} + g_{33}X_{31} & g_{32}X_{22} + g_{33}X_{32} & g_{33}X_{11} \end{pmatrix}$$

Since $g_{11} \neq 0$, we can solve X'_{21}, X'_{31}:

$$X'_{21} = \frac{1}{g_{11}}(g_{22}X_{21} + g_{23}X_{31})$$

$$X'_{31} = \frac{1}{g_{11}}(g_{31}X_{11} + g_{32}X_{21} + g_{33}X_{31} - g_{21}X'_{32} - g_{31}X_{22})$$

The other equations are:

$$\begin{cases} g_{22}X'_{32} = g_{33}X_{32}, \\ g_{22}(X_{11} - X_{22}) = g_{23}X_{32}, \\ g_{33}(X_{11} - X_{22}) = g_{23}X'_{32}. \end{cases}$$

The condition $XY - YX = 0$ gives

$$\begin{cases} X_{21}(Y_{11} - Y_{22}) = Y_{21}(X_{11} - X_{22}), \\ X_{32}(Y_{11} - Y_{22}) = Y_{32}(X_{11} - X_{22}), \\ X_{21}Y_{32} = X_{32}Y_{21}. \end{cases}$$

Denote $X_0 = X_{11} - X_{22}$ and $Y_0 = Y_{11} - Y_{22}$. Then $\mathcal{Y}'_+$ is locally isomorphic to

$$\operatorname{Spec} \mathbb{C}[X_{11}, X_{21}, X_{31}, X_{32}, X_0, X'_{32}, Y_{11}, Y_{21}, Y_{31}, Y_{32}, Y_0, Y'_{32}, g_{ij}]$$

cut out by the equations

$$\begin{cases} g_{22}X'_{32} = g_{33}X_{32}, \\ g_{22}X_0 = g_{23}X_{32}, \\ g_{33}X_0 = g_{23}X'_{32}. \end{cases}, \begin{cases} g_{22}Y'_{32} = g_{33}Y_{32}, \\ g_{22}Y_0 = g_{23}Y_{32}, \\ g_{33}Y_0 = g_{23}Y'_{32}. \end{cases}, \begin{cases} X_{21}Y_0 = Y_{21}X_0, \\ X_{32}Y_0 = Y_{32}X_0, \\ X_{21}Y_{32} = X_{32}Y_{21}. \end{cases}$$

On the open subset $D(g_{23})$ (i.e., $\{g_{23} \neq 0\} \cap P_{1,2}$), we have $X_{32} = \frac{g_{22}X_0}{g_{23}}$ and $X'_{32} = \frac{g_{33}X_0}{g_{23}}$ and similarly for Y's. Thus, the remaining equation is

$$X_{21}Y_0 = Y_{21}X_0,$$

which is clearly a radical ideal.

On the open subset $D(g_{22}g_{33})$, we have $X'_{32} = \frac{g_{33}X_{32}}{g_{22}}$ and $X_0 = \frac{g_{23}X_{32}}{g_{22}}$ and similarly for Y's. Thus, the remaining equation is

$$X_{21}Y_{32} = X_{32}Y_{21},$$

which is clearly a radical ideal.

Since $\det g \neq 0$, $D(g_{23}) \cup D(g_{22}g_{33}) = \mathcal{Y}'_+$, so the proof of reducedness is complete. □

Remark 5.5. In fact, following the same proof we can deduce that $\mathcal{Y}$ is actually smooth. This is surprising since $\mathcal{Y}$ is the fiber product of singular schemes $\widetilde{Flag}^{\circ}_{1,1}$.

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