A Convergence Rate for Extended-Source Internal DLA in the Plane

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Abstract

Internal DLA (IDLA) is an internal aggregation model in which particles perform random walks from the origin and stop upon reaching an unoccupied site. Levine and Peres showed that, when particles start instead from fixed point distributions, the modified IDLA processes have deterministic scaling limits related to a certain obstacle problem. In this paper, we investigate the convergence rate of this "extended source" IDLA in the plane to its scaling limit. We show that, if δ is the lattice size, the IDLA occupied set is $\delta^{3/5}$-close (in the Hausdorff sense) to its scaling limit for all time, with probability at least $1 - e^{-1/\delta^{2/5}}$.

Contents

1 Introduction

Internal Diffusion Limited Aggregation (IDLA) is a probabilistic growth process on the integer lattice $\mathbb{Z}^d$, first proposed by Meakin and Deutch [MD86] to model electro-chemical polishing. Namely, IDLA follows the growth of random sets $A(n)$; we set $A(1) = \{0\}$, and $A(n+1)$ is obtained by adding to $A(n)$ the point at which a centered simple random walk exits $A(n)$. With the right scaling, this process resembles a stream of particles from the origin barraging (and thus smoothing) the inner surface of an origin-centered sphere.

In line with its applications in smoothing processes, the overall smoothness of IDLA has been an active area of investigation. Meakin and Deutch first studied this numerically, finding that variations of $A(n)$ from the smooth ball were of magnitude $\log n$ in dimension 2 [MD86]. Significant progress has also been made in proving these properties. In particular, Lawler, Bramson, and Griffeath [LBG92] proved that $A(n)$ approaches the ball of radius $\sqrt[d]{n/\omega_d}$—where ω_d is the volume of the d-dimensional unit sphere—almost certainly as n increases. Several groups [Law95, AG10, AG13] also found convergence rates for this process. Most recently, Jerison, Levine, and Sheffield proved that the fluctuations away from the disk are bounded by $\log n$ in dimension 2 and $\sqrt{\log n}$ in higher dimensions [JLS12, JLS13b].

Of considerable interest is the extended-source case of IDLA, wherein particles start from a fixed point distribution rather than all from the origin. This generalizes the applicability of IDLA to a much wider range of surfaces, allowing us to see how different geometries interact with this smoothing process. This question was originally investigated by Diaconis and Fulton [DF91] in the context of a "smash sum" of two domains. Levine and Peres [LP08] generalized this notion considerably, proving deterministic scaling limits for IDLA with a piecewise constant density $\sigma : \mathbb{R}^d \to \mathbb{Z}_{\geq 0}$ of starting points. It is worth noting, but beyond the scope of this paper, that another model with Poisson particle sources was proposed and studied by Gravner and Quastel [GQ00].

In this paper, we investigate the convergence rate of the extended-source IDLA of Levine and Peres to its scaling limit in dimension 2, using the techniques of Jerison et al. [JLS12]. Under the additional assumptions that the initial mass distribution is "concentrated" (see Section 2) and the deterministic limit of its IDLA flow is smooth, we show that—if δ is the lattice size—the fluctuations of extended-source IDLA are of order $\delta^{3/5}$ or below, with probability at least $1 - e^{-1/\delta^{2/5}}$.

We believe that this bound is non-optimal, and we discuss possibilities for improvement in Section 7. However, we will see in a sequel to this paper that our bound is strong enough to prove weak scaling limits of the IDLA fluctuations themselves. The latter question has been investigated in the single-source case by Jerison, Levine, and Sheffield [JLS14]—more recently, Eli Sadovnik [Sad16] has shown scaling limits for extended-source fluctuations integrated against harmonic polynomials. In the sequel, we will seek to generalize this result and apply it to fluctuations "through time", to better understand the covariances between fluctuations at different times.

In Sections 2 and 3, we will introduce our main result and provide a background on existing theory needed for our proof. The remaining sections are dedicated to the proof of Theorem 3.1. Sections 4 and 5 set up the necessary theory; the former shows that an early point implies a similarly late point, and the latter shows that a late point implies another very early point. Section 6 combines these results in an iterative argument, recovering the full theorem.

Acknowledgments

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2 Background on extended-source IDLA

We will focus on a specific sort of extended source—a *concentrated mass distribution*—slightly narrowing the definitions introduced in [LP08] in order to capture scaling limits for the partially-completed process. We will give details on various extensions in the final section.

We define a *(concentrated) mass distribution* as a triplet $(D_0, \{Q_i^s\}, \{\sigma_s\})$ with the following properties:

1. $D_0 \subset \mathbb{R}^2$ is a compact, connected domain (i.e., a closure of an open set) with a smooth boundary.
2. For each i and for $0 \le s \le T_i$, the sets $Q_i^s \subset\subset \text{int}(D_0)$ have the following properties:
   a) Q_i^s is a compact domain with $\text{Vol}(Q_i^s) = s$.
   b) $Q_i^{s_0} \subset Q_i^{s_1}$ for $0 \le s_0 \le s_1 \le T_i$.
   c) ∂Q_i^s is rectifiable, with arclength bounded independently of s.
3. For $0 \le s \le T := \sum T_i$, we have $\sigma_s = \mathbf{1}_{D_0} + \sum_{i=1}^N 1_{Q_i^{s_i}}$, where we choose increasing functions $s_i = s_i(s)$ satisfying $\sum s_i = s$.

Given such a mass distribution, we also define the sets D_s, $s \in [0, T]$ to be the Diaconis–Fulton "smash" sums

$$D_s = D_0 \oplus Q_1^{s_1} \oplus \cdots \oplus Q_N^{s_N},$$

where $s_i = s_i(s)$ are as above, and with the smash sum operation as defined in [LP08]:

Definition 2.1. If $A, B \subset \frac{1}{m}\mathbb{Z}^2$, we define the discrete smash sum $A \oplus B$ as follows. Let $C_0 = A \cup B$, and for each $x_i \in \{x_1, ..., x_n\} = A \cap B$, start a simple random walk at x_i and stop it upon exiting C_{i-1}. Let y_i be its final position, and define $C_i = C_{i-1} \cup \{y_i\}$. Then $A \oplus B := C_n$ is a random set.

As proven in [LP08], if we instead take domains $A, B \subset \mathbb{R}^2$, the smash sums $A_m \oplus B_m$ of

$$A_m := \frac{1}{m}\mathbb{Z}^2 \cap A, \qquad B_m := \frac{1}{m}\mathbb{Z}^2 \cap B$$

approach a deterministic limit, which we label $A \oplus B$. An example is pictured in Figure 1.

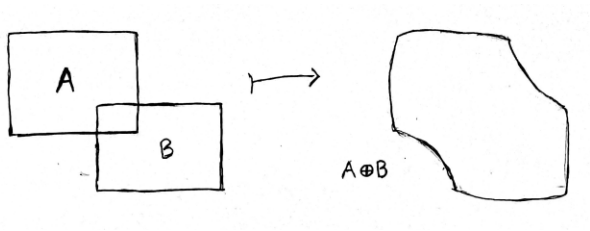

Figure 1: The smash sum $A \oplus B$ is the deterministic limit of an IDLA-type growth process starting from the sets A and B, representing the dispersal of particles in $A \cap B$ to the edges of $A \cup B$. In our setting, we see that IDLA also converges to an iterated smash sum.

Visually, D_s is a smooth outward flow from D_0, with $\mathrm{Vol}(D_s) = s + \mathrm{Vol}(D_0)$; we can think of D_s as the result of allowing the mass at $\{Q_i^{s_i}\}$ to diffuse (in the sense of Brownian motion) to—and accumulate at—the edge of D_0. These sets satisfy the following key property:

Lemma 2.2. *For (D_s, σ_s) arising from a mass distribution, we have*

$$\int_{D_s} h = \int_{\mathbb{R}^2} h\sigma_s$$

for any harmonic $h : D_s \to \mathbb{R}^2$.

Example. Suppose that we take $D_0 = B_1$ to be the unit disk, and we set $Q_i^s = B_{\sqrt{s/\pi}}$ for $s \in [0, \pi/4]$ and $i = 1, ..., N$. Further define $s_i(s) = s/N$, so that all sets Q_i^s are growing at the same rate. Visually, particles are emanating evenly (with density N) from outwardly moving rings of radius $0 \leq r \leq 1/2$, as shown in Figure 2.

Here, $T = N\pi/4$. From symmetry considerations, it is clear that $D_s = B_{\sqrt{1+s/\pi}}$ are outwardly expanding disks, as in the case of a point-source.

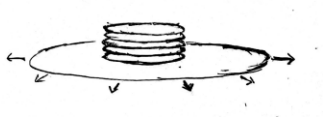

Figure 2: Here, our starting set $D_0 = B_1$ is the unit disk, and our source points $Q_i^s = B_{\sqrt{s/\pi}}$ are identical radially-expanding disks within it (shown in 3D for visual clarity). From symmetry, we see that the occupied sets D_s are also growing disks.

Next, we restrict attention to *smooth* flows:

Definition 2.3. The flow D_s is *smooth* if the flow $s \in [0, T] \mapsto D_s$ is a smooth isotopy from D_0 to D_T.

Note that the disks of 2 form a smooth flow. Now, given a mass distribution, there is a natural way to discretize it. Fix an integer m, and note that $f(s) = \sum_{\frac{1}{m}\mathbb{Z}^2} \sigma_s$ is an increasing, piecewise constant function of s. Let

$$0 = s_{m,0} < s_{m,1} < \cdots < s_{m,N_2} = T$$

be a partition of $[0, T]$ with $s_{m,k+1} - s_{m,k} = O(m^{-2})$ and such that f is constant near $s_{m,k}$ for $k \neq 0, N_2$. Define the sequence $S_m = \{z_{m,1}, ..., z_{m,n_m}\}$ inductively as follows:

1. Let $s_{m,n}$ be the smallest $s_{m,i}$ such that

$$\sum\nolimits_{\frac{1}{m}\mathbb{Z}^2} \sigma_{s_{m,n}} - \sum_{i<n} z_{m,i} > 0.$$

2. Choose $z_{m,n} \in \frac{1}{m}\mathbb{Z}^2$ such that $\sigma_{s_{m,n}}(z_{m,n})$ exceeds the number of times $z_{m,n}$ occurs in $\{z_{m,i}\}_{i<n}$.

Intuitively, we allow the sets $Q_i^{s_i}$ to expand a slight (i.e., $O(m^{-2})$) amount, and then we add all new points to the sequence of $z_{m,i}$. It is possible that multiple points may satisfy this condition for a given time—in the limit $m \to \infty$, the order in which these "nearby" points appear will not matter.

Given these sequences S_m, the (resolution m) *internal DLA (IDLA)* associated to the mass distribution is the following process:

Definition 2.4 (Internal DLA)**.** Suppose we have a concentrated mass distribution with initial set D_0 giving rise to the sequences S_m. The IDLA $A_m(t)$ associated to the mass distribution is as follows. Define the initial set $A_m(0) = \frac{1}{m}\mathbb{Z}^2 \cap D_0$. Then, for each $i \geq 1$, start a random walk at $z_{m,i}$, and let z_i' be the first point in the walk outside the set $A_m(i-1)$—then $A_m(i) := A_m(i-1) \cup \{z_i'\}$.

Importantly, the law of $A_m(i)$ does not depend on the order of $\{z_{m,1}, ..., z_{m,i}\}$, as proven by Diaconis and Fulton [DF91].

We know from Levine and Peres [LP08] that the sets $A_m(m^2 s)$ approach their deterministic limits D_s almost surely. That is, for any $\varepsilon > 0$, we know that

$$d_H(A_m(m^2 s), D_s) \leq \varepsilon$$

almost surely for sufficiently large m. Here and below, we use d_H to denote the Hausdorff distance between sets:

$$d_H(A, B) := \inf_{x \in A} \sup_{y \in B} d(x, y).$$

In the following sections, we will use this result along with an iterative argument to recover a stronger convergence rate on $A_m(m^2 s)$.

3 Main result

We will write $(D_\tau)^\varepsilon$ and $(D_\tau)_\varepsilon$ for the outer- and inner-ε-neighborhoods of D_τ, respectively. Our primary result is the following convergence rate on the IDLA occupied sets $A_m(t)$ to their deterministic scaling limits D_s:

Theorem 3.1. *Suppose D_τ is a smooth flow arising from a concentrated mass distribution. For large enough m and any time $s \in [0, T]$, the fluctuation of the associated IDLA $A_m(t)$ is bounded as*

$$\mathbb{P}\left\{ (D_s)_{C_5 m^{-3/5}} \cap \frac{1}{m}\mathbb{Z}^2 \subset A_m(m^2 s) \subset (D_s)^{C_5 m^{-3/5}} \text{ for all } s \in [0, T] \right\}^c \leq e^{-m^{2/5}}$$

for a constant C_5 depending on the flow. Equivalently,

$$\mathbb{P}\left[d_H(A_m(m^2 s), D_s) > C_5 m^{-3/5} \text{ for any } s \in [0, T] \right] \leq e^{-m^{2/5}},$$

where d_H is the Hausdorff distance.

As mentioned in the introduction, we have reason to believe that the $m^{-3/5}$ convergence rate so described is non-optimal. Indeed, we will see in Lemma 5.2 that this results from a relatively rudimentary L^1 bound on the convergence rate of discrete Green's functions, rather than from the geometry of IDLA. We will discuss suggestions for further research in Section 7.

3.1 Overview of notation

Firstly, we will assume we have fixed a smooth, concentrated mass distribution, and we will use the language of Section 2 to refer to it. That is, T will always refer to the total volume of our source sets, $z_{m,i}$ to the i^{th} source point in the resolution-m discretization of our mass distribution, and D_s to the scaling limit of IDLA started on the density σ_s.

To discuss fluctuations of IDLA away from its scaling limit, we define the following notions of "earliness" and "lateness":

- We say that $z \in \frac{1}{m}\mathbb{Z}^2$ is ε-early if $z \in A(\tau m^2)$ but $z \notin (D_\tau)^\varepsilon$. Let $\mathcal{E}_\varepsilon[t]$ be the event that some point in $A(t)$ is ε-early.
- Similarly, we say that z is ε-late if $z \in (D_\tau)_\varepsilon$, but $z \notin A(\tau m^2)$. Let $\mathcal{L}_\varepsilon[t]$ be the event that some point in $(D_{t/m^2})_\varepsilon$ is ε-late.

As introduced in the preceding subsection, we will write $(A)^\varepsilon$ and $(A)_\varepsilon$ for the outer- and inner-ε-neighborhoods of a set $A \subset \mathbb{R}^2$. That is,

$$(A)^\varepsilon := \{z \in \mathbb{R}^2 \mid d(z, A) < \varepsilon\}, \qquad (A)_\varepsilon := \{z \in A \mid d(z, A^c) > \varepsilon\}.$$

Finally, for convenience and visual clarity, we will use $m^2 s$ [resp., $m^2 T$, etc.] in place of $\lfloor m^2 s \rfloor$ in places where the meaning is clear. In particular, $A(sm^2) := A(\lfloor sm^2 \rfloor)$.

3.2 Required lemmas

A number of existing results are necessary in the proof of Theorem 3.1; we collect many of them here.

Firstly, we use the following two estimates on IDLA. The first bounds the probability of so-called "thin tentacles"—shown in Figure 3—and is simply a transcription of Lemma 2 of [JLS12] in our setting. The second is a part of the estimate of Levine and Peres [LP08] earlier described, demonstrating that extremely late points are unlikely.

Lemma 3.2 (Thin Tentacles)**.** *There are positive absolute constants b, C_0, and c_0 such that for all $z \in \frac{1}{m}\mathbb{Z}^2$ with $d(z, D_0) \geq r$,*

$$\mathbb{P}[z \in A(t), \#(A(t) \cap B(z, r)) \leq bm^2 r^2] \leq C_0 e^{-c_0 mr}.$$

Proof. The proof can be taken verbatim from Jerison et al. [JLS12], with our scaling in mind. □

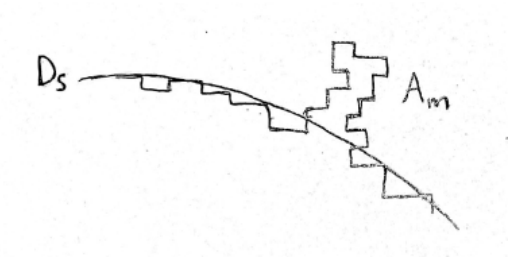

Figure 3: It is conceivable that the IDLA set A_m extends out from its limit D_s in thin tentacles, as pictured; in Lemma 3.2, we show that this is highly unlikely.

Lemma 3.3. *There are absolute constants $C_0, c_0 > 0$ such that for all real $\varepsilon > 0$, $T \geq \tau \geq 0$, and large enough m,*

$$\mathbb{P}\left(\mathcal{L}_\varepsilon[\tau m^2]\right) \leq C_0 e^{-c_0 m^2 / \log m}.$$

Proof. By Levine and Peres [LP08](p. 49), the probability $\mathbb{P}_z$ that some $z \in \frac{1}{m}\mathbb{Z}^2 \cap (D_\tau)_\varepsilon$ is ε-late is bounded as

$$\mathbb{P}_z := \mathbb{P}\left[z \in \frac{1}{m}\mathbb{Z}^2 \cap (D_\tau)_\varepsilon \text{ is } \varepsilon\text{-late}\right] \leq 4e^{-cm^2/\log m}$$

for large enough m, where c depends only on S_m, D_τ, and ε. Now, $\#\{z \in \frac{1}{m}\mathbb{Z}^2 \cap (D_\tau)_\varepsilon\} = O(m^2)$, so we can bound the total probability of $\mathcal{L}_\varepsilon[\tau m^2]$ as

$$\mathbb{P}\left(\mathcal{L}_\varepsilon[\tau m^2]\right) \leq \sum_{z \in \frac{1}{m}\mathbb{Z}^2 \cap (D_\tau)_\varepsilon} \mathbb{P}_z \leq C_0 m^2 e^{-cm^2/\log m}$$

for some C_0. Choosing some $c_0 < c$, we have $m^2 e^{-cm^2/\log m} \leq e^{-c_0 m^2/\log m}$ for large enough m, and the lemma follows. □

The next two lemmas control the flow $s \mapsto D_s$. In short, the first shows that the arclength of ∂D_s is uniformly bounded on both sides, and the second shows that D_s grows at a linear rate at all points. The first follows directly from the smoothness of D_s.

Lemma 3.4. *For $s \in [0,T]$, the arclength of ∂D_s is bounded as*

$$u \leq \text{Len}(\partial D_s) \leq U,$$

where $u, U > 0$ are constants depending only on the flow.

Lemma 3.5. *For a smooth flow D_s and any times $T \geq s_1 \geq s_0 \geq 0$,*

$$v(\sqrt{1+s_1} - \sqrt{1+s_0}) \leq d(D_{s_1}^c, D_{s_0}) := \inf\{d(x,y) \mid x \in D_{s_1}^c, y \in D_{s_0}\}$$

and

$$V(\sqrt{1+s_1} - \sqrt{1+s_0}) \geq d_H(D_{s_1}, D_{s_0}) := \inf_{x \in D_{s_1}} \sup_{y \in D_{s_0}} d(x,y)$$

where $v, V > 0$ are constants depending only on S_m.

Proof. The upper bound follows from the smoothness of D_s and the compactness of the interval $[0,T]$.

For the lower bound, we will exploit the fact that D_s is also the scaling limit of divisible sandpile processes on $\{z_{m,1}, ..., z_{m,n_m}\}$ with starting set D_0. We will not give details on the divisible sandpile process here; see [LP08] for more details on scaling limits of divisible sandpiles.

Choose an $s \in [0,T]$, and let $D_s^{1/m}(t)$ be the fully occupied set of the divisible sandpile on the lattice $\frac{1}{m}\mathbb{Z}^2$ with starting density

$$\mathbf{1}_{D_s} + \sum_{i=m^2 s}^{m^2 s + t} \mathbf{1}_{z_{m,i}}.$$

In the interval $[s, s+\varepsilon]$, a total of εm^2 particles are released—in fact, one particle is started at $z_{m,n}$ at each time n/m^2. From Lemmas 5.1(d) and 5.2(a,b), we can bound the exit probability as

$$\mathbb{P}\left(z_{m,n} \text{ exits } \frac{1}{m}\mathbb{Z} \cap D_s \text{ at } z' \in \partial D_s\right) \geq \frac{c}{m},$$

which tells us that, in the divisible sandpile model, we need m/c particles to ensure that the new set $D_s^{1/m}(m/c)$ is outside the m^{-1}-ball around $\frac{1}{m}\mathbb{Z}^2 \cap D_s$:

$$D_s^{1/m}(m/c) \not\subset \left(\frac{1}{m}\mathbb{Z}^2 \cap D_s\right)^{1/m}.$$

Now, we can apply the same estimate to the expanded set

$$\frac{1}{m}\mathbb{Z}^2 \cap (D_s)^{1/m} \subset D_s^{1/m}(m/c).$$

That is, if z' is in the boundary of both $\frac{1}{m}\mathbb{Z}^2 \cap (D_s)^{1/m}$ and $D_s^{1/m}(m/c)$, we have

$$\mathbb{P}\left(z_{m,n} \text{ exits } D_s^{1/m}(m/c) \text{ at } z'\right) \geq \mathbb{P}\left(z_{m,n} \text{ exits } \frac{1}{m}\mathbb{Z} \cap (D_s)^{1/m} \text{ at } z' \in \partial D_s\right) \geq \frac{c}{m},$$

and thus

$$D_s^{1/m}(2m/c) \not\subset \left(\left(\frac{1}{m}\mathbb{Z}^2 \cap D_s\right)^{1/m}\right)^{1/m} \supset \left(\frac{1}{m}\mathbb{Z}^2 \cap D_s\right)^{\sqrt{2}/m}.$$

Continuing in this manner, we find that

$$D_s^{1/m}(hm/c) \not\subset \left(\frac{1}{m}\mathbb{Z}^2 \cap D_s\right)^{h/\sqrt{2}m},$$

and in particular that

$$D_s^{1/m}(\varepsilon m^2) \not\subset \left(\frac{1}{m}\mathbb{Z}^2 \cap D_s\right)^{\varepsilon c/\sqrt{2}}.$$

Now, $D_{s+\varepsilon}$ is the scaling limit of these sets $D_s^{1/m}(\varepsilon m^2)$, so we find that $d(D_{s+\varepsilon}^c, D_s) \geq \varepsilon c/\sqrt{2}$. Then $\partial_\varepsilon d(D_{s+\varepsilon}^c, D_s) \geq c/\sqrt{2}$ for all $\varepsilon > 0$, which implies the claim. □

Finally, the following two lemmas control the exit times of Brownian motion from an interval $[a, b]$. These are restatements of Lemmas 5 and 6 in [JLS12], so we omit the proofs here. Below, let $B(s)$ be centered, one-dimensional Brownian motion, and denote

$$\tau(-a, b) = \inf\{s > 0 \mid B(s) \notin [-a, b]\}.$$

Lemma 3.6. *Let $0 < a \leq b$. If $a + b \leq 3$, then*

$$\mathbb{E}e^{\tau(-a,b)} \leq 1 + 10ab.$$

Lemma 3.7. *For any $k, s > 0$,*

$$\mathbb{P}\left\{\sup_{s' \in [0,s]} B(s') \geq ks\right\} \leq e^{-k^2 s/2}.$$

3.3 The recurrent potential kernel

Key to much of our analysis will be the so-called *recurrent potential kernel* $g : \mathbb{Z}^2 \to \mathbb{R}$, which acts as a free Green's function for the discrete Poisson equation. We define it in probabilistic terms as

$$g(z) := \sum_{n=0}^{\infty} (P_n(0) - P_n(z)),$$

where $P_n(z)$ is the probability that an n-step simple random walk from the origin in $\mathbb{Z}^2$ ends at z. Importantly,

$$\Delta_h g(x) := \tfrac{1}{4}(g(x+1) + g(x-1) + g(x+i) + g(x-i)) - g(x) = \delta_{x,0}.$$

That is, $\Delta_h g(0) = 1$, but $\Delta_h g(x \neq 0) = 0$. We will also use the first few terms of the asymptotic expansion of g:

$$\left|g(z) - \lambda - \frac{2}{\pi}\log|z|\right| \leq \frac{C_1}{|z|^2}.$$

A complete expansion was discovered by Kozma and Schreiber [KS04], but we will not use it here.

We also consider discrete derivatives of g. Without loss of generality, choose a unit vector $\hat{\mathbf{n}} = \alpha_1\hat{\mathbf{x}} + \alpha_2(\hat{\mathbf{x}} + \hat{\mathbf{y}})$ in the "east-northeast" half-quadrant—i.e., with $1 \geq \alpha_1, \alpha_2 \geq 0$. Then define

$$\partial_{\hat{\mathbf{n}}} g := \alpha_1 g(z-1) + \alpha_2 g(z-(1+i)) - (\alpha_1+\alpha_2)g(z), \tag{1}$$

which is discrete harmonic away from $\{0, 1, 1+i\}$. Now, extend both g and $\partial_{\hat{\mathbf{n}}} g$ by linear interpolation to the grid

$$\mathcal{G} := \{(x,y) \in \mathbb{R}^2 \mid x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}.$$

Choose a constant $c > 0$ such that $\partial_n g(z) > 0$ on the half-plane $\{z \in \mathcal{G} \mid z \cdot \hat{\mathbf{n}} \leq c\}$; since the arc $[0, \pi/4]$ is compact, we can assume without loss of generality that c holds this property for all $\hat{\mathbf{n}}$. Numerical calculations show that we can take $c \gtrsim 1/5$.

For an integer $m \geq 1$, let $B^-_{R_0} = B_{mR_0}(mR_0\hat{\mathbf{n}})$ be the radius mR_0 disk tangent to the origin in the direction $\hat{\mathbf{n}}$. By Lemma 8(a) of [JLS12], we know that

$$\{z \in \mathcal{G} \mid \partial_{\hat{\mathbf{n}}} g < -1/2mR_0'\} \subset (B^-_{R_0})^{C_2}.$$

By the above discussion, this means that

$$\{z \in \mathcal{G} \mid \partial_{\hat{\mathbf{n}}} g < -1/2mR_0\} \subset \{z \in \mathcal{G} \mid z \cdot \hat{\mathbf{n}} > c\} \cap (B^-_{R_0})^{C_2} \subset B^-_{R_0'}, \tag{2}$$

for any $R_0' \geq 4C_2R_0/c$.

4 Early points imply late points

The following sections make up the proof of Theorem 3.1, split into three parts. First, we will show that the existence of an early point at time t implies that of a similarly late point by the same time. For this, we use a harmonic function $H_\zeta(z)$ that has a pole at the proposed early point, $\zeta \in \frac{1}{m}\mathbb{Z}^2$, and we define a martingale $M_\zeta(t)$ by (roughly) summing the values of $H_\zeta(z) - H_\zeta(z_{m,i})$ over $A_m(t)$. Since $H_\zeta(\zeta)$ is large, the martingale takes a much larger value than expected at time t; we finish up by using Lemma 3.7 to show that this is unlikely.

In the following two sections, we set up the theory necessary for this first proof.

4.1 The discrete harmonic function $H_\zeta(z)$

Choose $\zeta \in \frac{1}{m}\mathbb{Z}^2 \cap (D_T \setminus D_0)$, and let $\tau > 0$ be such that $\zeta \in \partial D_\tau$. This is possible because the sets ∂D_s for $s > 0$ form a foliation of $D_T \setminus D_0$.

Without loss of generality, suppose the outward normal vector $\hat{\mathbf{n}}$ to ∂D_τ at ζ is pointing into the "east-northeast" half-quadrant, or equivalently that $\hat{\mathbf{n}} \cdot \hat{\mathbf{x}} \geq \hat{\mathbf{n}} \cdot \hat{\mathbf{y}} \geq 0$. This subsumes other cases by reflecting the plane appropriately.

Now, write $\hat{\mathbf{n}} = \alpha_1 + \alpha_2(1+i)$. Because of the direction of $\hat{\mathbf{n}}$, both α_1 and α_2 are positive and bounded below 1.

Define

$$H_\zeta(z) = \frac{\pi}{2}[\alpha_1 g(mz - m\zeta - 1) + \alpha_2 g(mz - m\zeta - (1+i)) - (\alpha_1+\alpha_2)g(mz - m\zeta)].$$

We can view this as a directional derivative of the potential kernel in the direction opposite $\hat{\mathbf{n}}$. We will extend this by linear interpolation to the grid $\mathcal{G}_m = \{(x,y) \in \frac{1}{m}\mathbb{Z}^2 \mid x \in \frac{1}{m}\mathbb{Z} \text{ or } y \in \frac{1}{m}\mathbb{Z}\}$.

This function is designed to be a discrete-harmonic approximation of the continuum function

$$F_\zeta(z) = \operatorname{Re}\left(\frac{\hat{\mathbf{n}}/m}{\zeta - z}\right),$$

pictured in Figure 4, where we view $\hat{\mathbf{n}}$ as a complex number.

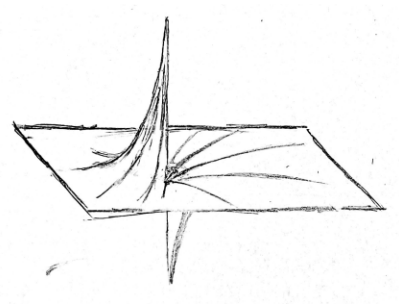

Figure 4: A plot of F_ζ. The discrete harmonic function H_ζ closely approximates this function away from the pole ζ.

Now, choose $R_0' = R_0'(\tau)$ such that the two disks B^+ and B^- of radius R_0' tangent to ∂D_τ at any point lie entirely inside and outside D_τ, respectively. Note that R_0' is bounded away from zero, as $[0, T]$ is compact and $R_0' > 0$ for all time. Let $R_0 = cR_0'/4C_2$, as in Equation 2, and define the following subsets of $\frac{1}{m}\mathbb{Z}^2$:

$$\Omega_\zeta^1 = \mathcal{G}_m \cap D_\tau,$$

$$\Omega_\zeta^2 = \{z \in \mathcal{G}_m \mid H_\zeta(z) > 1/2mR_0\} \setminus \{\zeta + u \mid u \in (0, m^{-1})\}.$$

In short, Ω_ζ^1 is the discretized version of the Hele–Shaw level set D_τ, and Ω_ζ^2 is an approximation of the "inner" radius R_0 circle tangent to ∂D_τ at ζ. We will combine these as

$$\Omega_\zeta = \Omega_\zeta^1 \cup \Omega_\zeta^2.$$

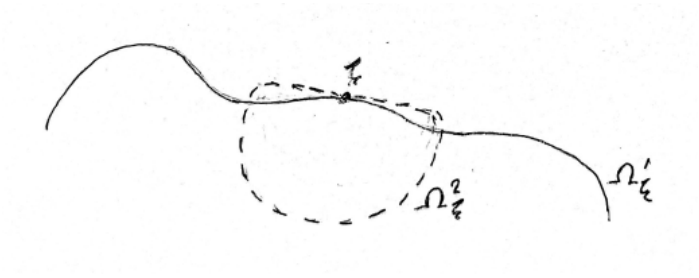

Figure 5: We form our domain Ω_ζ by combining the subsets Ω_ζ^1 and Ω_ζ^2; the latter guarantees that H_ζ is not too large on the boundary, but it may also affect the regularity of the boundary.

An example is pictured in Figure 5. We summarize many of the basic properties of H_ζ and Ω_ζ in the following lemma:

Lemma 4.1. *For any m, H_ζ and Ω_ζ satisfy the following properties:*

(a) *H_ζ is grid-harmonic in the interior of Ω_ζ, and $H_\zeta \geq -\frac{1}{2mR_0}$.*

(b) *$\zeta \in \partial\Omega_\zeta$, and for all $z \in \partial\Omega_\zeta \setminus \{\zeta\}$, we have $|H_\zeta(z)| \leq \frac{1}{2mR_0}$.*

(c) *There is an absolute constant $C_1 < \infty$ such that*

$$|H_\zeta(z) - F_\zeta(z)| \leq C_1 m^{-2} |z - \zeta|^{-2}.$$

In particular, if $R_1 = \inf_{z_{m,i}} d(z_{m,i}, \zeta)$, then

$$|H_\zeta(z_{m,i}) - F_\zeta(z_{m,i})| \leq C_1 m^{-2} R_1^{-2}.$$

(d) $1 \leq H_\zeta(\zeta) \leq 2$.

Proof.

(a) By definition, H_ζ is grid-harmonic everywhere except for ζ, $\zeta + 1/m$, and $\zeta + (1+i)/m$. Firstly, ζ itself lies on the boundary of Ω_ζ by definition. As the normal vector $\hat{\mathbf{n}}$ to ∂D_τ at ζ points into the east-northeast half-quadrant, for large enough m, neither of the remaining points can lie in D_τ (and thus in Ω^1_ζ). Furthermore, H_ζ is negative at both points, as in [JLS12], so they cannot lie in Ω^2_ζ. Thus, they cannot lie in Ω_ζ, so H_ζ is grid-harmonic in that set.

The lower bound follows from Equation 2.

(b) As in part (a), the lower bound $H_\zeta(z) \geq -1/2mR_0$ is clear from Equation 2. The upper bound follows from the inclusion of Ω^2_ζ, as the boundary of Ω_ζ must lie at or outside the boundary of Ω^2_ζ.

(c, d) The last points are exactly Lemma 7(c, d) in [JLS12], as our notions of H_ζ and F_ζ are simply rotations of theirs. □

Lemma 4.2.

(a) *There is an absolute constant $C_2 < \infty$ such that*

$$\frac{1}{m}\mathbb{Z}^2 \cap D_\tau \subset \Omega_\zeta \subset \frac{1}{m}\mathbb{Z}^2 \cap (D_\tau)^{C_2/m}.$$

(b) *For any $U \subset \Omega_\zeta$, then*

$$-\frac{1}{2mR_0} \leq H_\zeta(z) \leq \frac{1}{md(\zeta, U) - C_2}$$

whenever $md(\zeta, U) > C_2$ and $z \in U$.

(c) *For all $0 < s < \tau$,*

$$\left| \sum_{z \in D_s \cap \frac{1}{m}\mathbb{Z}^2} H_\zeta(z) - \sum_{i=0}^{m^2 s} H_\zeta(z_i) \right| \leq C_2 \log m.$$

Proof.

(a) As shown in [JLS12], the level sets of H_ζ differ from the level curves of F_ζ by at most a fixed distance $C_2/m = 2C_1/m$. In particular, $\Omega^2_\zeta \subset (B^+)^{C_2/m}$, where B^+ is the disk of radius R_0 contained within D_τ and tangent to ∂D_τ at ζ.

By construction, $(B^+)^{C_2/m} \subset (D_\tau)^{C_2/m}$. Thus, by adding Ω^2_ζ, we never modify points in Ω^1_ζ outside the narrow strip $(D_\tau)^{C_2/m} \setminus D_\tau$.

(b) The proof of this fact is the same as that of Lemma 8(b) of [JLS12], but now using the fact that $\sup_U F_\zeta \leq \frac{1}{md(\zeta,U)}$.

(c) Let s_0 be maximal such that $D_{s_0} \subset (D_\tau)_{4C_1/m}$—by Lemma 3.5, we know that

$$d_H(\partial D_\tau, \partial D_{s_0}) \leq \frac{V}{v} d(\partial D_\tau, \partial D_{s_0}) \leq 4VC_1/mv. \tag{3}$$

Write $D_s \cap \frac{1}{m}\mathbb{Z}^2 = A \cup B$, where

$$A = \left(D_s \cap \frac{1}{m}\mathbb{Z}^2 \right) \cap D_{s_0} = D_{s \wedge s_0} \cap \frac{1}{m}\mathbb{Z}^2,$$

$$B = \left(D_s \cap \frac{1}{m}\mathbb{Z}^2 \right) \setminus D_{s_0}.$$

Now, choose $R > 0$ with the following properties:

- The union U_R of the disks of radius R centered at $\zeta \pm (R + 2C_1/m)$ and at $\zeta \pm i(R + 2C_1/m)$ is connected.
- The connected component of ζ in $\mathbb{R}^2 \setminus U_R$ is contained within $B_{4C_1/m}(0)$
- We have $B_R(\zeta) \setminus B_{4C_1/m}(\zeta) \subset U_R$.

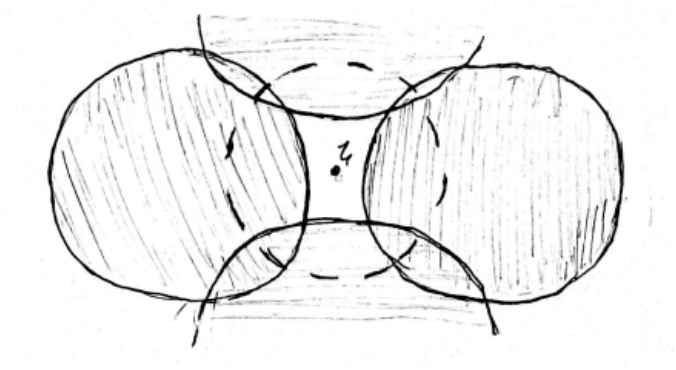

Figure 6: An illustration of U_R. Note that the disks of radius R do not get "too close" to ζ, so the sum $\sum_{z \in U_R} \frac{1}{m^2|z-\zeta|^2}$ is of order $\log m$.

Note that, for any $r > R$, this implies $B_r(0) \setminus B_{4C_1/m}(0) \subset U_r$. Define

$$R' = \max\left(R, \operatorname{diam}(D_0) + 2V\sqrt{1+\tau}\right).$$

Importantly, by Lemma 3.5,

$$D_\tau \subset B_{R'}(\zeta).$$

Since D_{s_0} does not intersect $B_{4C_1/m}$, this means that $A \subset D_{s_0} \subset U_{R'}$. By Lemma 4.1(c),

$$\begin{aligned}
\left|\sum_{z\in A}(H_\zeta(z) - F_\zeta(z)) - \sum_{i=1}^{m^2(s\wedge s_0)}(H_\zeta(z_i) - F_\zeta(z_i))\right| &\le \sum_{z\in A}\frac{C_1}{m^2|z-\zeta|^2} + \frac{s_0}{|\zeta|^2} \\
&\le \sum_{z\in U_{R'}\cap\frac{1}{m}\mathbb{Z}^2}\frac{C_1}{m^2|z-\zeta|^2} + \frac{s_0}{|\zeta|^2} \\
&\le 32C_1\log mR' + \frac{s_0}{|\zeta|^2} \\
&\le \frac{C_2}{6}\log m,
\end{aligned}$$

for an appropriately chosen C_2. Using similar arguments, we bound

$$\left|m^2\int_{D_{s\wedge s_0}} F_\zeta - \sum\nolimits_A F_\zeta\right| \le C\int_{D_{s\wedge s_0}}\frac{dA}{|z-\zeta|^2} \le C\int_{U_{R'}}\frac{dA}{|z-\zeta|^2} \le 8\pi C\log mR' \le \frac{C_2'}{6}\log m,$$

and finally

$$\left|m^2\int F_\zeta\sigma_{s\wedge s_0} - \sum_{i=1}^{m^2(s\wedge s_0)}F_\zeta(z_i)\right| \le \frac{C_2'}{6}\log m.$$

By Lemma 2.2, however,

$$\int_{D_{s\wedge s_0}} F_\zeta(1-\sigma_{s\wedge s_0}) = 0,$$

and we thus find

$$\left|\sum_{z\in A} H_\zeta(z) - \sum_{i=1}^{m^2(s\wedge s_0)} H_\zeta(z_i)\right| \le \frac{C_2'}{2}\log m.$$

Finally, we must show that the contribution of B (if it is nonempty) to the sum is negligible. From Equation 3, we know that $D_s \setminus D_{s_0} \subset D_\tau \setminus (D_\tau)_{4VC_1/mv}$ and thus that $B \subset D_\tau \setminus (D_\tau)_{4VC_1/mv}$. Thus, there are at most $Um \cdot 4VC_1/mv$ points in B; since H_ζ decreases as $1/md(z,\zeta)$ around the edge of D_τ, we find that

$$\left|\sum_{z\in B} H_\zeta(z)\right| \le \frac{4UVC_1}{v}\cdot C\log m \le \frac{C'}{4}\log m.$$

Similarly, there are at most $4VUC_1/v$ source points between times s_0 and s, so we bound the final term as

$$\left|\sum_{i=m^2 s_0}^{m^2 s} H_\zeta(z)\right| \le \frac{4VUC_1}{v}\cdot \sup_{z_{m,i}} H_\zeta(z_{m,i}) = O(m^{-1}) \le \frac{C'}{4}\log m.$$

Putting these contributions together implies the lemma. □

4.2 The martingale $M_\zeta(t)$

The harmonic function H_ζ gives rise to a natural martingale associated to our IDLA process, using the concept of a *grid Brownian motion*:

Definition 4.3. A *grid Brownian motion* starting at the point $x \in \frac{1}{m}\mathbb{Z}^2$ is a random process $t \mapsto W_t \in \mathcal{G}_m$ defined as follows.

Let B_t be an origin-centered Brownian motion, and for each integer $n \ge 1$, let $\tau_n^* > 0$ be the n^{th} time that B_t visits a point in $\frac{1}{m}\mathbb{Z}$. For each n, choose a uniform random direction $\hat{\mathbf{u}}_n \in \{1, i\}$. For $t \in [\tau_n^*, \tau_{n+1}^*]$, define

$$W_t = x + \sum_{j=1}^{n-1}\left(B_t(\tau_{j+1}^*) - B_t(\tau_j^*)\right)\hat{\mathbf{u}}_j + \left(B_t(t) - B_t(\tau_n^*)\right)\hat{\mathbf{u}}_n.$$

In short, W_t is simply the process B_t, but turning in a random direction at each lattice point.

For $k \in \{0, 1, 2, ...\}$, let $\tilde{\beta}_k(s)$ be independent Brownian motions on the grid $\mathcal{G}_m$, starting at the source points $z_{m,k}$. We will define a modified IDLA process $A_\zeta(t)$ by induction. Let $A_\zeta(0) = A_m(0) = \frac{1}{m}\mathbb{Z}^2 \cap D_0$, and let

$$s_k^* = \inf\left\{s \ge 0 \mid \tilde{\beta}_k(s) \in \left(\tfrac{1}{m}\mathbb{Z}^2 \setminus A_\zeta(k)\right) \cup \left(\mathcal{G}_m \setminus \Omega_\zeta\right)\right\}.$$

Then set $\beta_k(s) = \tilde{\beta}_k\big(\min\left(\frac{s}{1-s}, s_k^*\right)\big)$, and set $A_\zeta(k+s) = A_\zeta(k) \cup \{\beta_k(s)\}$ for $0 \le s \le 1$.

Since H_ζ is grid-harmonic, the process

$$M_\zeta(t) := \sum_{\ell=0}^{\lfloor t\rfloor - 1}\left(H_\zeta(\beta_\ell(1)) - H_\zeta(z_{m,\ell})\right) + H_\zeta(\beta_\ell(t - \lfloor t\rfloor)) - H_\zeta(z_{m,\lfloor t\rfloor})$$

is a continuous-time martingale adapted to $\mathscr{F}_t = \sigma\{A_\zeta(s) \mid 0 \le s \le t\}$. By the Dubins–Schwarz theorem [RY91, Theorem V.1.6], we can write $M_\zeta(t) = B_\zeta(S_\zeta(t))$, where $S_\zeta(t) = \langle M_\zeta, M_\zeta\rangle_t$ is the quadratic variation of M_ζ and B_ζ is a standard Brownian motion.

For each k, $S_\zeta(k)$ is a stopping time w.r.t. the filtration $\{\mathscr{F}_{T_\zeta(s)}\}_{s\ge 0}$, where $T_\zeta(s) = \inf\{t \mid S_\zeta(t) > s\}$. Further, $B_\zeta(s)$ is adapted to this filtration. By the strong Markov property, the processes

$$\tilde{B}_\zeta^k(u) := B_\zeta(S_\zeta(k) + u) - B_\zeta(S_\zeta(k))$$

are independent Brownian motions started at zero.

Finally, for $-a < 0 < b$, write $\tau_k(-a, b) = \inf\{u > 0 \mid \tilde{B}_\zeta^k(u) \notin [-a, b]\}$. We will use these exit times in accordance with the following lemma, which is just a restatement of Lemma 9 of [JLS12] in our setting:

Lemma 4.4. *Fix $\zeta \in \frac{1}{m}\mathbb{Z}^2 \setminus D_0$, and let*

$$-a_k = \min_{z \in \partial A_\zeta(k)} (H_\zeta(z) - H_\zeta(z_{m,k+1})), \qquad b_k = \max_{z \in \partial A_\zeta(k)} (H_\zeta(z) - H_\zeta(z_{m,k+1})).$$

Then

$$S_\zeta(k+1) - S_\zeta(k) \le \tau_k(-a_k, b_k).$$

We now proceed with the technique mentioned above. That is, we will use the martingales M_ζ to "detect" the presence of a late or early point at ζ; if either is the case, then M_ζ will be either much larger or much smaller than its mean. In turn, Lemma 3.7 will imply that this scenario is unlikely for small times S_ζ. With the following two lemmas, we will be able to show that S_ζ is small on the event $\mathcal{E}_{a/m}(t)^c$, allowing the above argument to go through.

Lemma 4.5. *Suppose D_s is a smooth flow arising from an initial mass distribution. For*

$$m \ge \max(3a + C_2, 2C_2/\inf_\zeta R_1),$$

all $s \in [0,T]$, and $\zeta \notin (D_s)^{(4a+2C_2)/m}$, we have

$$\mathbb{E}\left[e^{S_\zeta(m^2 s)} \mathbf{1}_{\mathcal{E}_{(a+1)/m}(m^2 s)^c}\right] \le m^K,$$

where K is a constant depending only on the flow.

Proof. On the event $\mathcal{E}_{(a+1)/m}[t]^c$, we have $A(n) \subset (D_{n/m^2})^{(a+1)/m}$ for all $n \le m^2 s$. Since $\zeta \notin (D_s)^{(4a+2C_2)/m}$, Lemma 3.5 tells us that

$$d(\zeta, A(n)) \ge \frac{4a+2C_2}{m} + d(\partial D_s, \partial D_{n/m^2}) - \frac{a+1}{m} \ge \frac{1}{m}(3a + 2C_2 - 1) + v\big(\sqrt{1+s} - \sqrt{1+n/m^2}\big),$$

and thus (using Lemmas 4.1(b) and 4.2(b)) that

$$-\frac{1}{2mR_0} - \frac{1}{mR_1 - C_2} \le H_\zeta(z) - H_\zeta(z_{m,i}) \le \frac{1}{3a + C_2 + mv\big(\sqrt{1+s} - \sqrt{1+n/m^2}\big) - 1} + \frac{1}{2mR_0}.$$

Now, choose m large enough that $mR_1 - C_2 \ge mR_1/2$, and write $R_2 = \min(R_0/2, R_1/4)$. Then we know from Lemma 4.4 that

$$(S_\zeta(n) - S_\zeta(n-1))\mathbf{1}_{\mathcal{E}_{(a+1)/m}[t]^c} \le \tau_n(-c, b_n),$$

with

$$c = \frac{1}{mR_2}, \qquad b_n = \frac{1}{3a + C_2 + mv\big(\sqrt{1+s} - \sqrt{1+n/m^2}\big) - 1} + \frac{1}{2mR_0}.$$

Using Lemma 3.6 along with the fact that $\tau_n(-c, b_n)$ are independent,

$$\log \mathbb{E}\left[e^{S_\zeta(t)} \mathbf{1}_{\mathcal{E}_{(a+1)/m}[t]^c}\right] = \sum_{n=1}^{t} \log \mathbb{E} e^{\tau_n(-c,b_n)} \le \sum_{n=1}^{t} \log(1 + 10cb_n) \le \sum_{n=1}^{t} 10cb_n.$$

Now, write $r_1 = 3a + 2C_2 + mv\sqrt{1+s} - 1$, and calculate

$$\begin{aligned}
\sum_{n=1}^{m^2 s} b_n &\le \int_0^t \frac{dn}{3a + C_2 + mv\big(\sqrt{1+s} - \sqrt{1+n/m^2}\big) - 1} + \frac{t}{2mR_0} \\
&= \int_0^{m^2 s} \frac{dn}{r_1 - v\sqrt{m^2+n}} + \frac{t}{2mR_0} \\
&= 2v^{-1} \int_{vm}^{vm\sqrt{1+s}} \frac{x\,dx}{r_1 - x} + \frac{t}{2mR_0} \\
&= 2v^{-1} \int_{r_1 - vm\sqrt{1+s}}^{r_1 - mv} \frac{(r_1 - y)dy}{y} + \frac{t}{2mR_0} \\
&\le 2v^{-1} r_1 \log\left(\frac{r_1 - mv}{r_1 - vm\sqrt{1+s}}\right) + \frac{t}{2mR_0} \\
&\le 4v^{-1} m\sqrt{1+s}\log(vm\sqrt{1+s}/C_2) + z,
\end{aligned}$$

so long as $mv \ge 3a + 2C_2$. We thus find that

$$\log \mathbb{E}\left[e^{S_\zeta(t)} \mathbf{1}_{\mathcal{E}_{(a+1)/m}[t]^c}\right] \le \frac{40\sqrt{1+s}}{vR_2}\log(mv\sqrt{1+s}/C_2) + \frac{s}{2R_0R_2}.$$

The theorem follows with $K > 40\sqrt{1+T}/vR_2$. □

Lemma 4.6. *Suppose D_s is smooth, and fix $a \ge 2C_2 + 2$, $\ell \le a$, and $s \in [0,T]$. For*

$$m \ge \max(3a + C_2, 5a/\inf_\zeta R_1)$$

and $\zeta \in \frac{1}{m}\mathbb{Z}^2 \cap \big((D_s)_{\ell/m} \setminus D_0\big)$, we have

$$\mathbb{E}\left[e^{S_\zeta(m^2 s)} \mathbf{1}_{\mathcal{E}_{(a+1)/m}(m^2 s)^c}\right] \le m^K e^{K'a},$$

where K is as in Lemma 4.5 and $K' > 0$ is another absolute constant.

Proof. Recall that $R_1 = \inf_{z_{m,i}} d(z_{m,i}, \zeta)$. Since $mR_1 \ge 5a$ and $a \ge 2C_2 + 2$, we can choose $t_0 \ge 1$ such that $D_{t_0/m^2} \subset (D_\tau)_{(4a+2C_2+1)/m}$. By Lemma 3.5, we can take t_0 to satisfy

$$m\sqrt{1+s} - \sqrt{m^2 + t_0} \le 2v^{-1}(4a + 2C_2 + 1),$$

and thus

$$m^2 s - t_0 \le 2m\sqrt{1+T}(m\sqrt{1+s} - \sqrt{m^2+t_0}) \le 4v^{-1} m\sqrt{1+T}(4a + 2C_2 + 1),$$

Further, since $\zeta \notin (D_{t_0/m^2})^{(4a+2C_2)/m}$, Lemma 4.5 gives

$$\mathbb{E}\left[e^{S_\zeta(t_0)} \mathbf{1}_{\mathcal{E}_{a/m}(t_0)^c}\right] \le \mathbb{E}\left[e^{S_\zeta(t_0)} \mathbf{1}_{\mathcal{E}_{(a+1)/m}(t_0)^c}\right] \le m^K.$$

As discussed in the proof of Lemma 4.5, for $m \ge 2C_2/\inf_\zeta R_1$, we have

$$-\frac{1}{mR_2} \le H_\zeta(z) - H_\zeta(z_{m,i}),$$

with $R_2 = \min(R_0/2, R_1/4)$. We also know that $H_\zeta(z) - H_\zeta(z_{m,i}) \le 2 + \frac{1}{mR_1 - C_2} \le 5/2$ from Lemma 4.2(b), so we get

$$S_\zeta(n) - S_\zeta(n-1) \le \tau_n(-1/mR_2, 5/2)$$

and thus

$$\begin{aligned}\log \mathbb{E}\left[e^{S_\zeta(t)}\mathbf{1}_{\mathcal{E}_{(a+1)/m}(t)^c}\right] &= \log \mathbb{E}\left[e^{S_\zeta(t_0)}\mathbf{1}_{\mathcal{E}_{(a+1)/m}(t_0)^c}\right] + \sum_{n=t_0+1}^{m^2 s} \log \mathbb{E}\left[e^{S_\zeta(n)-S_\zeta(n-1)}\mathbf{1}_{\mathcal{E}_{(a+1)/m}(m^2 s)^c}\right] \\ &\le K\log m + \sum_{n=t_0+1}^{m^2 s} \log \mathbb{E}\left[e^{\tau_n(-1/mR_2,5/2)}\right] \\ &\le K\log m + (m^2 s - t_0)\log(1+25/mR_2) \\ &\le K\log m + 100v^{-1}\sqrt{1+T}(4a+2C_2+1)/R_2 \\ &\le K\log m + 100v^{-1}\sqrt{1+T}(4a+2C_2+1)/R_2.\end{aligned}$$

Now, $2C_2 + 1 \le a$ by hypothesis, so the claim follows. □

4.3 First estimate

Choose constants

$$C_3 = \max\left(\frac{24VC_2}{vb}, \frac{72V}{vb}, 3/c_0\right),$$

$$\alpha = \frac{v^2 b}{288UV^2K''}.$$

Lemma 4.7. *For large enough m, $s \in [0,T]$, $3a + C_2 \ge a \ge C_3 m^{2/5}$, and $\ell \le \alpha a$, we have*

$$\mathbb{P}(\mathcal{E}_{a/m}[m^2 s] \cap \mathcal{L}_{\ell/m}[m^2 s]^c) \le e^{-2m^{2/5}}.$$

Step 1. For each integer $1 \le t \le m^2 s$ and each lattice point z, let

$$Q_{z,t} = \{z \in A(t) \setminus A(t-1)\} \cap \{z \notin (D_{t/m^2})^{a/m}\} \cap \mathcal{E}_{a/m}[t-1]^c$$

be the event that z first joins the cluster at time t and is the first a/m-early point. Now,

$$\bigcup_{t \le T} \bigcup_{z \in (D_0)^{T/m}} Q_{z,t} = \mathcal{E}_{a/m}[T].$$

Fix $z \in (D_0)^{t/m} \cap \frac{1}{m}\mathbb{Z}^2$, and let $\zeta = \zeta(z,t)$ be the nearest point to z in the annulus

$$\frac{1}{m}\mathbb{Z}^2 \cap (D_{t/m^2})^{V(4a+2C_2)/mv+2/m} \setminus (D_{t/m^2})^{V(4a+2C_2)/mv}. \tag{4}$$

Since $\zeta \notin (D_{t/m^2})^{(4a+2C_2)/m}$, we have by Lemma 4.5 that

$$\mathbb{E}\left[e^{S_\zeta(t)}\mathbf{1}_{\mathcal{E}_{(a+1)/m}(t)^c}\right] \le m^K.$$

Let $M = 6m^{2/5}$, so that Markov's inequality gives

$$\mathbb{P}\left(\mathcal{E}_{(a+1)/m}(t)^c \cap \{S_\zeta(t) > M\}\right) \le e^{-6m^{2/5}}\mathbb{E}\left[e^{S_\zeta(t)}\mathbf{1}_{\mathcal{E}_{(a+1)/m}(t)^c}\right] \le e^{-3m^{2/5}}.$$

Now, since $A(t-1) \subset (D_{t/m^2})^{a/m}$ and z is adjacent to $A(t-1)$, we must have $z \in (D_{t/m^2})^{(a+1)/m}$. Thus, $Q_{z,t} \subset \mathcal{E}_{(a+1)/m}(t)^c$, so

$$\mathbb{P}\left(Q_{z,t} \cap \{S_\zeta(t) > M\}\right) \le e^{-3m^{2/5}}.$$

Step 2. On the event $Q_{z,t}$, we know that

$$A[t] \subset (D_{t/m^2})^{(a+1)/m}, \tag{5}$$

as no points are $(m+1)$-early. However, we also know that

$$d_H(D_{t/m^2}, D_\tau) \geq d(D_{t/m^2}, \zeta) \geq V(4a + 2C_2)/mv,$$

which implies by Lemma 3.5 that

$$d(D_{t/m^2}, D_\tau) \geq \frac{V}{v} d_H(D_{t/m^2}, D_\tau) \geq (4a + 2C_2)/m.$$

In turn, Equation 5 implies that

$$d(A(t), D_\tau) \geq d((D_{t/m^2})^{(a+1)/m}, D_\tau) \geq d(D_{t/m^2}, D_\tau) - (a+1)/m \geq (3a + 2C_2 - 1)/m.$$

For $C_2 \geq 1$, this means that $A(t) \subset (D_\tau)_{C_2/m}$, and thus that $A(t)$ does not meet $\partial\Omega_\zeta$ by Lemma 4.2(a). This means that we can replace $A[t]$ by $A_\zeta(t)$, which we partition as

$$A_1 = A_\zeta(t) \cap D_{t_0/m^2}, \qquad A_2 = A_\zeta(t) \cap B_{a/m}(z), \qquad A_3 = A_\zeta(t) \setminus (A_1 \cup A_2),$$

where t_0 is chosen such that $D_{t_0/m^2} \subset (D_{t/m^2})_{\ell/m}$. By Lemma 3.5, we can satisfy $d_H(D_{t_0/m^2}, D_{t/m^2}) \leq 2V\ell/mv$, or

$$(D_{t/m^2})_{2V\ell/mv} \subset D_{t_0/m^2}.$$

On the event $\mathcal{L}_{\ell/m}[t]^c$, no point in $(D_{t/m^2})_{\ell/m}$ is left out of $A_\zeta(t)$, so $A_1 = \frac{1}{m}\mathbb{Z}^2 \cap D_{t_0/m^2}$. Since $A(t)$ has t points and A_1 has at least $t - 4\ell UVv^{-1}m$ points (using Lemma 3.4), we know that $\#(A_2 \cup A_3) \leq 4\ell UVv^{-1}m$. Noting that $H_\zeta(z') - H_\zeta(z_{m,i}) \geq -1/mR_2$ for any $z' \in \Omega_\zeta$, this implies

$$\sum_{z' \in A_3} \left(H_\zeta(z') - H_\zeta(z_{m,i(z')})\right) \geq -\frac{\#A_3}{mR_2} \geq -\frac{4\ell UVm}{mvR_2} = -\frac{4\ell UV}{vR_2},$$

where $z_{m,i(z')}$ is the source point that initially generated the point $z' \in A(t)$. Next, we try to estimate the equivalent sum over A_1. By the discussion above, we know that only $4\ell UVv^{-1}m$ points can be outside the bounds of A_1, meaning that $\{z_{m,i(z')} \mid z' \in A_1\}$ differs from $\{z_{m,i} \mid 0 \leq i \leq t_0\}$ by at most $4\ell UVv^{-1}m$ points. Along with Lemma 4.2(c), this implies

$$\begin{aligned}
\sum_{z' \in A_1} (H_\zeta(z') - H_\zeta(z_{m,i(z')})) &\geq \sum_{z' \in A_1} H_\zeta(z') - \sum_{i=1}^{t_0} H_\zeta(z_i) - 8\ell UVv^{-1}m \max|H_\zeta(z_{m,i})| \\
&\geq -C_2' \log m - 8UVK''\ell/v \\
&\geq -C_2' a/C_3 - 8UVK''\ell/v.
\end{aligned}$$

Adding up the contributions from A_1 and A_3 gives

$$\sum_{z' \in A_1 \cup A_3} (H_\zeta(z') - H_\zeta(z_{m,i(z')})) \geq -C_2' a/C_3 - 12UVK''\ell/v \geq -vba/12V, \tag{6}$$

from the definitions of C_3 and ℓ above.

Now, since $z \notin (D_{t/m^2})^{a/m} \supset (D_0)^{a/m}$, Lemma 3.2 tells us that

$$\mathbb{P}\left(Q_{z,t} \cap \mathcal{L}_{\ell/m}[t]^c \cap \{\#A_2 \leq ba^2\}\right) \leq C_0 e^{-c_0 a} \leq C_0 e^{-3m^{2/5}}$$

for large enough m, using the facts that $a \geq C_3 m^{2/5}$ and $C_3 \geq 3/c_0$.

On the event $Q_{z,t}$, the point z is a/m early but not $(a+1)/m$-early, so $a/m \le d(z, D_{t/m^2}) \le (a+1)/m$. We know that ζ is the nearest point to z in the annulus 4, which means that (for $a > 2C_2$) we have $md(z,\zeta) \le 5Va/v$. Then $F_\zeta(z) = v/5Va + O(a^{-2})$, and so by Lemma 4.1, for all $z' \in B(z,a)$,

$$H_\zeta(z') - H_\zeta(z_{m,i(z')}) \ge \frac{v}{5Va} - \frac{2}{mR_1} + O(a^{-2}) \ge \frac{v}{6Va},$$

as long as m (and hence a) is large enough. On the event $\{\#A_2 > ba^2\}$, this means

$$\sum_{z' \in A_2} (H_\zeta(z') - H_\zeta(z_{m,i(z')})) \ge \frac{v\#A_2}{6Va} > \frac{vba}{6V},$$

and hence (from Equation 6) on event $\{\#A_2 > ba^2\} \cap \mathcal{L}_{\ell/m}[t]^c \cap Q_{z,t}$,

$$M_\zeta(t) = \sum_{z' \in A_\zeta(t)} (H_\zeta(z') - H_\zeta(z_{m,i(z')})) > \frac{vba}{6V} - \frac{vba}{12V} = \frac{vba}{12V}.$$

Thus,

$$\{M_\zeta(t) \le vba/12V\} \subset \{\#A_2 \le ba^2\} \cup \left(\mathcal{L}_{\ell/m}[t]^c \cap Q_{z,t}\right)^c,$$

and so

$$\mathbb{P}(\{M_\zeta(t) < vba/12V\} \cap \mathcal{L}_{\ell/m}[t]^c \cap Q_{z,t}) \le \mathbb{P}\left(Q_{z,t} \cap \mathcal{L}_{\ell/m}[t]^c \cap \{\#A_2 \le ba^2\}\right) \le C_0 e^{-3m^{2/5}}.$$

Step 3. Since $C_3 \ge \frac{72V}{vb}$, we know that

$$\frac{vba}{12V} \ge \frac{vbC_3}{12V} m^{2/5} \ge 6m^{2/5} = M,$$

with M as in part 1. Using Lemma 3.7, we find

$$\mathbb{P}\left(\{S_\zeta \le M\} \cap \{M_\zeta \ge vba/12V\}\right) \le \mathbb{P}\left(\{S_\zeta \le M\} \cap \{B_\zeta(S_\zeta) \ge M\}\right) \le e^{-M/2} = m^{-3m^{2/5}}.$$

Finally, we bound

$$\begin{aligned}\mathbb{P}(Q_{z,t} \cap \mathcal{L}_{\ell/m}[m^2 s]^c) &\le \mathbb{P}(Q_{z,t} \cap \{S_\zeta > M\}) \\ &\quad + \mathbb{P}(Q_{z,t} \cap \{M_\zeta(t) \le vba/12V\} \cap \mathcal{L}_{\ell/m}[t]^c) \\ &\quad + \mathbb{P}(\{S_\zeta \le M\} \cap \{M_\zeta(t) \ge vba/12V\}) \\ &\le (C_2+2)e^{-3m^{2/5}},\end{aligned}$$

which implies

$$\begin{aligned}\mathbb{P}(\mathcal{E}_{a/m}[m^2 s] \cap \mathcal{L}_{\ell/m}[m^2 s]^c) &= \sum_{t=1}^{\lfloor m^2 s \rfloor} \sum_{z \in (D_0)^{ms}} \mathbb{P}(Q_{z,t} \cap \mathcal{L}_{\ell/m}[m^2 s]^c) \\ &\le m^2 s \cdot 10m^2(\mathrm{diam}(D_0) + ms)^2 \cdot (C_2+2)e^{-3m^{2/5}} \le m^{-2m^{2/5}}.\end{aligned}$$

□

5 Late points imply early points

Very roughly, we would like the proof of the second part of Theorem 3.1 to go as follows. If ζ is the first (ℓ/m)-late point in $A_\zeta(t)$, then at the time $T \sim m^2\tau + m\ell$, the set $A_\zeta(t)$ has several particles at every boundary point $z \neq \zeta$ in $\partial\Omega_\zeta$. Since $H_\zeta(\zeta)$ is much larger than $H_\zeta(z \neq \zeta)$, this would tell us in turn that M_ζ would have a much lower value than expected. Combined with Lemmas 4.5 and 4.6, we would be able to recover a strong upper bound of the probability of $\mathcal{L}_{\ell/m}[T] \cap \mathcal{E}_{a/m}[T]^c$.

Unfortunately, we are unable to say that the difference $H_\zeta(z) - H_\zeta(z_{m,i})$ that occurs in the expression for M_ζ is even negative, let alone a large negative number. The problem that occurs in the general source (i.e., non-disk) setting is that we cannot obtain a positive lower bound on $H_\zeta(z_{m,i})$, as the source point $z_{m,i}$ may be "behind" the pole ζ, as shown in Figure 7.

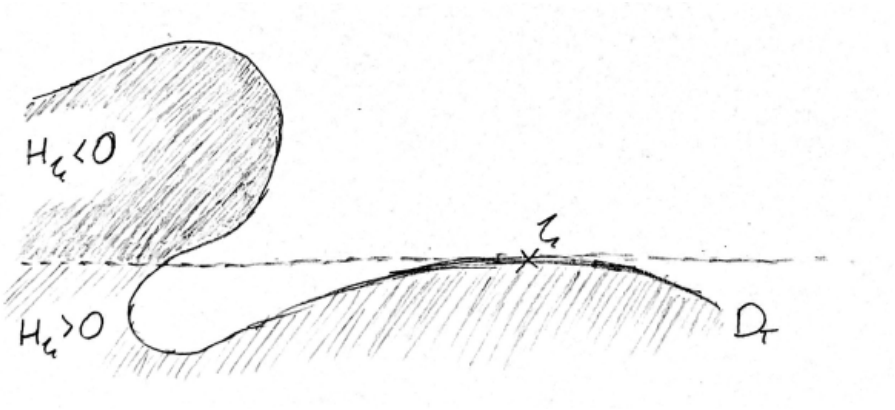

Figure 7: The original harmonic H_ζ is negative on a half-plane cut out by the pole ζ. While this does not come into play in the case of a point-source, it is critical in extended-source IDLA, forcing us to define a new harmonic function $\tilde{H}_\zeta$ to continue with the proof.

To remedy this issue, we introduce a second harmonic function $\tilde{H}_\zeta$, defined to be the discrete Poisson kernel on a slightly modified domain $\tilde{\Omega}_\zeta \approx D_\tau$. We will see that the difference $\tilde{H}_\zeta(z) - \tilde{H}_\zeta(z_{m,i})$ is negative and bounded away from zero, so our program will go through roughly as mentioned above.

On the other hand, we will not be able to get a strong replacement for Lemma 4.1(c), which tells us that H_ζ closely approximates a continuum harmonic function. This leads to an overall $m^{2/5}$ error—rather than the logarithmic errors we saw in Lemma 4.2(c)—when summing $\tilde{H}_\zeta$ over the set D_s, and it eventually creates the $m^{-3/5}$ error of Theorem 3.1.

5.1 The Poisson kernel on Ω_ζ

We introduce a new, positive harmonic function on the new set

$$\tilde{\Omega}_\zeta = \left(D_\tau \cap \frac{1}{m}\mathbb{Z}^2\right) \setminus \{\zeta \pm iu \mid u \in (0, 1/m^2)\}.$$

If $W_z(t)$ is a (grid) Brownian motion in $\tilde{\Omega}_\zeta$ starting at z, and τ^* is the first exit time of $W_\zeta(t)$ from $\tilde{\Omega}_\zeta$, we define

$$\tilde{H}_\zeta(z) := \mathbb{P}[W_z(\tau^*) = \zeta].$$

We can recognize this as the Poisson kernel associated to the set $\tilde{\Omega}_\zeta$. In particular, it satisfies the following key properties:

Lemma 5.1. *For any m, $\tilde{H}_\zeta$ satisfies the following:*

(a) *$\tilde{H}_\zeta$ is grid harmonic in $\tilde{\Omega}_\zeta$, and $H_\zeta \geq 0$.*

(b) $\tilde{H}_\zeta(\zeta) = 1$. *For all* $z \in \partial\tilde{\Omega}_\zeta \setminus \{\zeta\}$, *we have* $\tilde{H}_\zeta(z) = 0$.

(c) *For any* $U \subset \tilde{\Omega}_\zeta$ *with* $md(\zeta, U) > C_2$,

$$\tilde{H}_\zeta(z) \leq \frac{1}{2mR_0} + \frac{1}{md(\zeta, U) - C_2}.$$

(d) *Let* $\zeta' = \zeta - 1/m \in \tilde{\Omega}_\zeta$. *Then*

$$\tilde{H}_\zeta(z) = c_\zeta G_{\tilde{\Omega}_\zeta}(\zeta', z)$$

on $\tilde{\Omega}_\zeta \setminus B_{1/m}(\zeta)$, *where* $1/16 \leq c_\zeta \leq 1$ *and*

$$G_{\tilde{\Omega}_\zeta}(y, z) := \mathbb{E}g(W_z(\tau^*) - y) - g(z - y)$$

is the Green's function associated to $\tilde{\Omega}_\zeta$.

Proof.

(a,b) The first two points follow from the definition of $\tilde{H}_\zeta$.
(c) From 4.1(a,d), we know that

$$H_\zeta + \frac{1}{2mR_0} \geq 0$$

on all of $\tilde{\Omega}_\zeta \subset \Omega_\zeta$, and that $H_\zeta(\zeta) + \frac{1}{2mR_0} \geq 1$. In particular,

$$H_\zeta + \frac{1}{2mR_0} \geq \tilde{H}_\zeta$$

on the boundary of $\tilde{\Omega}_\zeta$, so we know from the maximum principle and Lemma 4.2(b) that

$$\tilde{H}_\zeta|_U \leq \left[H_\zeta + \frac{1}{2mR_0}\right]_U \leq \frac{1}{2mR_0} + \frac{1}{md(\zeta, U) + C_2}.$$

(d) This follows from the last-exit decomposition for simple random walks [LL10, Prop. 4.6.4]. □

Lemma 5.2. *Suppose* D_s *is smooth. Then,*

(a) *For any* $z \in D_\tau \cap \Omega_\zeta$,

$$\left|G_{\tilde{\Omega}_\zeta}(\zeta', z) - G_{D_\tau}(\zeta', z)\right| \leq \frac{C_2}{m^2 d(z, \partial\tilde{\Omega}_\zeta)^2} + \frac{C_2}{m^2 d(z, \zeta')^3},$$

where G_{D_τ} *is the continuous Green's function of* D_τ.

(b) *For any* $z \in D_\tau$,

$$\left|G_{D_\tau}(\zeta', z) - \frac{c'_\zeta}{m} J_\zeta(z)\right| \leq \frac{C_2}{m^2 d(z, \zeta')^2} + \frac{C_2}{m^2 d(z, \zeta)^2},$$

where $c'_\zeta \in [2^{-1/2}, 1]$ *depends only on* ζ *and* J_{D_τ} *is the Poisson kernel on* D_τ.

(c) *The following mean-value property holds:*

$$\left|\sum_{z \in \tilde{\Omega}_\zeta \cap \frac{1}{m}\mathbb{Z}^2} \tilde{H}_\zeta(z) - \sum_{i=1}^{\lfloor m^2\tau \rfloor} \tilde{H}_\zeta(z_{m,i})\right| \leq C'_2 m^{2/5}.$$

Proof. (a) For this, we use the estimate

$$g(x,y) = \log m|x-y| + \lambda + O\left(\frac{1}{m^2|x-y|^2}\right)$$

mentioned in Section 3.3. This implies

$$G_{\Omega_\zeta}(z,\zeta') = \mathbb{E}g(W_{\zeta'}(\tau^*),z) - g(\zeta',z) = \mathbb{E}\log|W_{\zeta'}(\tau^*)-z| - \log|\zeta'-z| + O\left(\frac{1}{m^2 d(z,\partial\Omega_\zeta)^2}\right),$$

as the $\log m$ and λ terms cancel out. Fixing z, we see that $\mathbb{E}\log|W_{z'}(\tau^*)-z|$ is a discrete harmonic function of z', with boundary values $\log|z'-z|$ for $z' \in \partial\tilde{\Omega}_\zeta$. With the possible exception of the points $\zeta \pm im^{-2}$, all boundary points of $\tilde{\Omega}_\zeta$ also lie on the boundary of ∂D_τ; then we can compare $f_0(z') = \mathbb{E}\log|W_{z'}(\tau^*)-z|$ with the continuous harmonic function $f_1(z') = G_{D_\tau}(z',z) + \log|z'-z|$. Indeed, the latter has fourth derivative bounded above by $C/d(z,z')^3$, so we know

$$\left|\Delta_h\left(\mathbb{E}\log|W_{z'}(\tau^*)-z| - (G_{D_\tau}(z',z)+\log|z'-z|)\right)\right| = \left|\Delta_h\left(G_{D_\tau}(z',z)+\log|z'-z|\right)\right| \le \frac{C}{m^2 d(z,z')^3},$$

where Δ_h is the five-point stencil Laplacian. Furthermore, $\mathbb{E}\log|W_{z'}(\tau^*)-z|$ and $G_{D_\tau}(z',z)+\log|z'-z|$ differ by at most $O(m^{-2})$ on the boundary (at $\zeta \pm im^{-2}$), so the maximum principle gives

$$\left|\mathbb{E}\log|W_{z'}(\tau^*)-z| - (G_{D_\tau}(z',z)+\log|z'-z|)\right| \le \Delta_h^{-1}\left(\frac{C}{m^2 d(z,\zeta')^3}\right) + O(m^{-2}).$$

The claim follows, as $\Delta_h^{-1}\left(\frac{C}{m^2 d(z,\zeta')^3}\right)$ approaches ζ' no faster than $O(z^{-3})$.
(b) This follows from the general formula $J_\zeta(z) = \partial_{\hat{\mathbf{n}}} G_{D_\tau}(z',z)|_{z'=\zeta}$, along with the fact that ζ' is at most an angle $\pi/4$ away from the normal direction inwards from ζ.
(c) Set $\alpha_0 = m^{-2/5}$ and $\varepsilon_0 = m^{-1/5}$, and let $B_\gamma^+ \subset D_\tau$ and $B_\gamma^- \subset D_\tau^c$ be the disks of radius R_0 tangent to ∂D_τ at γ. For each $\alpha > 0$, we partition $\overline{\Omega}_\zeta := \tilde{\Omega}_\zeta \cap \frac{1}{m}\mathbb{Z}^2$ by sets A^α, B^α, C^α, and D^α as follows:

$$A^\alpha = \left\{z \in \overline{\Omega}_\zeta \;\middle|\; d(z,\partial\Omega_\zeta) > \alpha\right\}, \qquad B^\alpha = \overline{\Omega}_\zeta \setminus (A^\alpha \cup B_{\varepsilon_0}(\zeta)),$$

$$C^\alpha = \overline{\Omega}_\zeta \setminus (A^\alpha \cup B^\alpha \cup B_\gamma^+), \qquad D^\alpha = \overline{\Omega}_\zeta \setminus (A^\alpha \cup B^\alpha \cup C^\alpha).$$

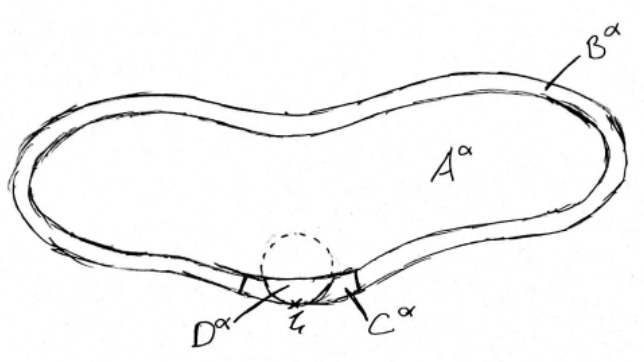

Figure 8: An illustration of our partition $A^\alpha \cup B^\alpha \cup C^\alpha \cup D^\alpha$ of $\overline{\Omega}_\zeta$.

We will bound the error $\left|\sum G_{\tilde{\Omega}_\zeta}(\zeta',\cdot) - m^2\int G_{D_\tau}(\zeta',\cdot)\right|$ over each of these sets (with $\alpha = \alpha_0$) in turn. For any $\alpha > C_2/m$, we know that $A^\alpha \subset D_\tau$ from Lemma 4.1(a), so part (a) implies

$$|G_{\Omega_\zeta}(\zeta',z) - G_{D_\tau}(\zeta',z)| \le C_2/(m\alpha)^2$$

on A^α. This allows us to bound

$$\begin{aligned}\left|\sum_{A^{\alpha_0}} G_{\tilde\Omega_\zeta}(\zeta',z) - m^2\int_{A^{\alpha_0}} G_{D_\tau}(\zeta',z)\right| &\le 2m^2\int_{\alpha_0}^\infty d\alpha\ \frac{C_2}{m^2\alpha^2}\operatorname{Len}(\partial A^\alpha) + 2\int_{(D_\tau)_{\alpha_0}}\frac{C_2}{d(z,\zeta')}\\ &\le 2\int_{\alpha_0}^\infty d\alpha\ \frac{C_2U}{\alpha^2}\sqrt{1+\tau} + \tfrac12 C_2'\alpha_0^{-1}\\ &\le C_2'\alpha_0^{-1} = C_2'm^{2/5},\end{aligned}$$

with C_2' large enough, and using the fact that $\operatorname{Len}(\partial A^\alpha)\le\operatorname{Len}(D_\tau)$. This immediately gives

$$\left|\sum_{A^{\alpha_0}}\tilde H'_\zeta(z) - c_\zeta c'_\zeta m\int_{A^{\alpha_0}} J_\zeta(z)\right| \le \tfrac14 C_2'm^{2/5}$$

from Lemma 5.1(d) and part (b) above.

Next, we control the sum over B^{α_0}. The points outside D_τ contribute only $C_2\log m$ to the total, so we will consider only $B' = B^{\alpha_0}\cap D_\tau$. Since ∂D_τ is smooth, the probability of a point z near the boundary to exit $\tilde\Omega_\zeta$ at ζ is bounded by $C\frac{md(z,\partial\tilde\Omega_\zeta)}{m^2d(z,\zeta)^2}$, which we estimate as

$$\sum_{B^{\alpha_0}}\tilde H_\zeta(z) \le 2Cm^2\cdot\int_{B^{\alpha_0}}\frac{md(z,\partial\tilde\Omega_\zeta)}{m^2d(z,\zeta)^2} \le \tfrac12 C_2'm\int_0^{\alpha_0}\alpha/\varepsilon_0 = \tfrac14 C_2'm\alpha_0^2/\varepsilon_0 = \tfrac14 C_2'm^{2/5}.$$

For the remaining sets, we introduce slice coordinates (x,y) for ∂D_τ near γ, such that $\gamma=(0,0)$. For C^{α_0}, we again consider only $C' = C^{\alpha_0}\cap D_\tau$. These points are bounded outside the disk B_γ^-, so the probability of their associated random walks exiting $\tilde\Omega_\zeta$ at ζ is bounded by

$$\tilde H_\zeta(z) \le \mathbb{P}[W_z \text{ enters } B_\gamma^- \text{ at } \gamma] \le \frac1m\cdot\frac{1}{x^2+y^2}(y+O(x^2+y^2)).$$

Then we find

$$\begin{aligned}\sum_{C^{\alpha_0}}\tilde H_\zeta &\le m^2\cdot\frac4m\int_0^{\varepsilon_0}dx\int_0^{R_0^{-1}x^2}dy\ \left(\frac{y}{x^2+y^2}+C\right)\\ &\le 4m\int_0^{\varepsilon_0}dx\ \left(\log(1+R_0^{-1}x^2)+CR_0^{-1}x^2\right)\\ &\le \tfrac34 mC_2'\int_0^{\varepsilon_0}dx\ x^2 = \tfrac14 C_2'm\varepsilon_0^3 = \tfrac14 C_2'm^{2/5},\end{aligned}$$

and similarly for $c_\zeta c'_\zeta m\int_{C^{\alpha_0}} J_\zeta(z)$.

Finally, for $z\in D^{\alpha_0}$, we can bound

$$\mathbb{P}[W_z \text{ exits } B_\gamma^+ \text{ at } \gamma] \le \tilde H_\zeta(z) \le \mathbb{P}[W_z \text{ enters } B_\gamma^- \text{ at } \gamma],$$

which gives

$$\tilde H_\zeta(z) = \frac1m\cdot\frac{1}{x^2+y^2}(y+O(x^2+y^2))$$

and similarly

$$\frac{c_\zeta c'_\zeta}{m}J_\zeta(z) = \frac1m\cdot\frac{1}{x^2+y^2}(y+O(x^2+y^2)).$$

The error is dominated by the second order part:

$$\left|\sum_{D^{\alpha_0}}\tilde H'_\zeta(z) - c_\zeta c'_\zeta m\int_{D^{\alpha_0}} J_\zeta(z)\right| \le 4m\int_0^{\varepsilon_0}dx\int_0^{\alpha_0}dy\ C\frac{x^2+y^2}{x^2+y^2} = 4mC\varepsilon_0\alpha_0 = \tfrac14 C_2'm^{2/5}.$$

The proof finishes as does Lemma 4.2(c), but including the extra factor c_ζ from Lemma 5.1(d). □

Just as with H_ζ, we associate the following martingale to $\tilde{H}_\zeta$:

$$\tilde{M}_\zeta(t) := \sum_{\ell=0}^{\lfloor t\rfloor -1} \left(\tilde{H}_\zeta(\beta_\ell(1)) - \tilde{H}_\zeta(z_{m,\ell})\right) + \tilde{H}_\zeta(\beta_\ell(t-\lfloor t\rfloor)) - \tilde{H}_\zeta(z_{m,\lfloor t\rfloor}),$$

using the same notation as in Section 4.2. Now, the rescaled function $(1+1/2mR_0)\tilde{H}_\zeta - 1/2mR_0$ satisfies the properties outlined in Lemmas 4.1(a,b,d) and Lemma 4.2(b), so we can prove the following parallels to Lemmas 4.5 and 4.6 exactly as before:

Lemma 5.3. *Suppose D_s is a smooth flow arising from an initial mass distribution. For*

$$m \geq \max(3a + C_2, 2C_2/\inf_\zeta R_1),$$

all $t \geq 1$, and $\zeta \notin (D_{t/m^2})^{(4a+2C_2)/m}$, we have

$$\mathbb{E}\left[e^{\tilde{S}_\zeta(t)}\mathbf{1}_{\mathcal{E}_{(a+1)/m}(t)^c}\right] \leq m^K,$$

where $\tilde{S}_\zeta(t) = \langle \tilde{M}_\zeta, \tilde{M}_\zeta\rangle_t$.

Lemma 5.4. *Suppose D_s is smooth, and fix $a \geq 2C_2+2$, $\ell \leq a$, and $t > 0$. For*

$$m \geq \max(3a + C_2, 5a/\inf_\zeta R_1)$$

and $\zeta \in \frac{1}{m}\mathbb{Z}^2 \cap \left((D_{t/m^2})_{\ell/m} \setminus D_0\right)$, we have

$$\mathbb{E}\left[e^{\tilde{S}_\zeta(t)}\mathbf{1}_{\mathcal{E}_{(a+1)/m}(t)^c}\right] \leq m^K e^{K'a}.$$

5.2 Second estimate

Lemma 5.5. *There is an absolute constant $C_4 > 0$ such that, for large enough m, if $s \in [0,T]$, $\ell > C_4 m^{2/5}$, and $a < \ell^2/C_4 m^{2/5}$, then*

$$\mathbb{P}(\mathcal{L}_{\ell/m}[m^2 s] \cap \mathcal{E}_{a/m}[m^2 s]^c) \leq e^{-2m^{2/5}}.$$

Proof. Without loss of generality, let $a = \ell^2/C_4 m^{2/5} \geq \ell$. We can further suppose that $m \geq \max(3a + C_2, 5a/\inf_\zeta R_1)$. Indeed, otherwise we have $a = \varepsilon m$ for a constant $\varepsilon = \inf(1/4, \inf R_1/5)$; by Lemma 3.3, we know that we can choose m large enough that

$$\mathbb{P}(\mathcal{L}_{\ell/m}[m^2 s] \cap \mathcal{E}_{a/m}[m^2 s]^c) \leq \mathbb{P}(\mathcal{L}_{\ell/m}[m^2 s]) \leq C_0 e^{-c_0 m^2/\log m} \leq e^{-2m^{2/5}}.$$

Fix $\zeta \in \frac{1}{m}\mathbb{Z}^2 \cap ((D_s)_{\ell/m} \setminus D_0)$ and set $T_1 \leq m^2 s$ minimal such that $\zeta \subset (D_{T_1})_{\ell/m}$. Then we know that $d(\zeta, D_{T_1}) = \ell/m$—by Lemma 3.5, this implies that

$$T_1 - m^2\tau \geq 2m^2\sqrt{1+\tau}(\sqrt{1+T_1/m^2} - \sqrt{1+\tau}) \geq \frac{2m\ell}{V}.$$

Let $L[\zeta] = \{\zeta \notin A_m(m^2 T_1)\}$ be the event that ζ is (ℓ/m)-late. Then

$$\mathcal{L}_{\ell/m}[m^2 s] = \bigcup\nolimits_\zeta L[\zeta].$$

On the event $L[\zeta]$, we know that any particles in $A_\zeta(T_1)$ that hit the boundary must do so away from ζ; that is, $\tilde{H}_\zeta \equiv 0$ for these particles. As in [JLS12], this implies that $\tilde{M}_\zeta(T_1)$ is maximized if the interior of $\tilde{\Omega}_\zeta \cap \frac{1}{m}\mathbb{Z}^2$ is fully occupied by $A_\zeta(T_1)$, so we can bound $\tilde{M}_\zeta(T_1)$ as follows:

$$\tilde{M}_\zeta(T_1) \leq \sum_{z \in \partial\tilde{\Omega}_\zeta \cap A_\zeta(T_1)} \left(0 - H_\zeta(z_{m,i(z)})\right) + \sum_{z \in \overline{\Omega}_\zeta} \left(\tilde{H}_\zeta(z) - \tilde{H}_\zeta(z_{m,i(z)})\right),$$

where $z_{m,i(z)}$ is the source point from which the particle landing at $z \in \overline{\Omega}_\zeta$ started, and weighting each term of the first sum by its number of occurrences in the multiset $A_\zeta(T_1)$. First, we reorganize the source terms of the two sums:

$$\begin{aligned}
\tilde{M}_\zeta(T_1) &\le -\sum_{i=m^2\tau+1}^{T_1} \tilde{H}_\zeta(z_{m,i}) + \sum_{z\in\overline{\Omega}_\zeta} \tilde{H}_\zeta(z) - \sum_{i=1}^{m^2\tau} \tilde{H}_\zeta(z_{m,i}) \\
&\le -(T_1 - m^2\tau)\inf_i \tilde{H}_\zeta(z_{m,i}) + \sum_{z\in\overline{\Omega}_\zeta} \tilde{H}_\zeta(z) - \sum_{i=1}^{m^2\tau} \tilde{H}_\zeta(z_{m,i}) \\
&\le -2m\ell V^{-1}\inf_i \tilde{H}_\zeta(z_{m,i}) + \sum_{z\in\overline{\Omega}_\zeta} \tilde{H}_\zeta(z) - \sum_{i=1}^{m^2\tau} \tilde{H}_\zeta(z_{m,i}) \\
&\le -2c\ell + \sum_{z\in\overline{\Omega}_\zeta} \tilde{H}_\zeta(z) - \sum_{i=1}^{m^2\tau} \tilde{H}_\zeta(z_{m,i})
\end{aligned}$$

for a constant $c > 0$ depending only on the flow, using Lemmas 5.2(a,b) to deduce that $\inf_i \tilde{H}_\zeta(z_{m,i}) = \Theta(m^{-1})$. Next, notice that the two right-hand sums are the same that appear in Lemma 5.2(c), implying that

$$\tilde{M}_\zeta(T_1) \le -2c\ell + C_2' m^{2/5} \le -c\ell,$$

so long as $C_4 \ge C_2'/c$.

Choose m large enough that $e^a \ge e^{C_4 m^{2/5}} \ge m$. By Lemma 5.4,

$$\mathbb{E}\left[e^{\tilde{S}_\zeta(m^2T_1)}\mathbf{1}_{\mathcal{E}_{(a+1)/m}(m^2T_1)^c}\right] \le m^K e^{K'a} \le e^{(K+K')a}.$$

Let $M = (K + K' + 1)a$, so Markov's inequality implies

$$\mathbb{P}\left(\{\tilde{S}_\zeta(m^2T_1) > M\} \cap \mathcal{E}_{a/m}[m^2T_1]^c\right) \le e^{-M}\mathbb{E}\left[e^{\tilde{S}_\zeta(m^2T_1)}\mathbf{1}_{\mathcal{E}_{(a+1)/m}(m^2T_1)^c}\right] \le e^{-a}.$$

Since $\tilde{M}_\zeta(m^2T_1) \le -c\ell$ on the event $L[\zeta]$, this means that

$$\begin{aligned}
\mathbb{P}\left(\mathcal{E}_{a/m}[m^2T_1]^c \cap L[\zeta]\right) &\le \mathbb{P}\left(\{\tilde{S}_\zeta(m^2T_1) > M\} \cap \mathcal{E}_{a/m}[m^2T_1]^c\right) + \mathbb{P}\left(\{\tilde{S}_\zeta(m^2T_1) \le M\} \cap L[\zeta]\right) \\
&\le \mathbb{P}\left(\{\tilde{S}_\zeta(m^2T_1) > M\} \cap \mathcal{E}_{a/m}[m^2T_1]^c\right) + \mathbb{P}\left\{\tilde{S}_\zeta(m^2T_1) \le M, \tilde{M}_\zeta(m^2T_1) \le -c\ell\right\} \\
&\le e^{-a} + e^{-c^2\ell^2/2M} = e^{-a} + e^{-c^2C_4m^{2/5}/2(K+K'+1)} \le 2e^{-4m^{2/5}} \le e^{-3m^{2/5}}
\end{aligned}$$

for $C_4 \ge 4(K + K' + 1)/c^2$, using Lemma 3.7 and the fact that $\tilde{M}_\zeta(t) = \tilde{B}_\zeta(\tilde{S}_\zeta(t))$ for a centered Brownian motion $\tilde{B}_\zeta$. We conclude that

$$\mathbb{P}\left(\mathcal{L}_{\ell/m}[m^2s] \cap \mathcal{E}_{a/m}[m^2s]^c\right) \le \sum_{\zeta\in\frac{1}{m}\mathbb{Z}^2\cap((D_s)_{\ell/m}\setminus D_0)} \mathbb{P}\left(\mathcal{E}_{a/m}[m^2T_1]^c \cap L[\zeta]\right) \le 2\,\mathrm{Vol}(D_s)m^2e^{-3m^{2/5}} \le e^{-2m^{2/5}}.$$

□

6 Proof of Theorem 3.1

Choose m large, $\varepsilon \le \alpha/4$, and $s \in [0, T]$. From Lemma 3.3, we know

$$\mathbb{P}(\mathcal{L}_\varepsilon[m^2s]) \le C_0e^{-c_0m^2/\log m} \le e^{-2m^{2/5}}.$$

Set $\ell_0 = \varepsilon m$, and define values a_k, b_k as follows:

$$a_k = \alpha^{-1}\ell_k, \qquad \ell_k = \sqrt{C_4 m^{2/5} a_{k-1}},$$

where $\alpha, C_4 > 0$ are as in Sections 4.3 and 5.2, respectively. Now, if $\ell_k \geq \alpha^{-1}C_4 m^{2/5}$, we know that

$$a_{k-1} = \frac{\ell_k^2}{C_4 m^{2/5}} \geq \alpha^{-2} C_4 m^{2/5}$$

and thus that

$$\ell_{k-1} = \alpha a_{k-1} \geq \alpha^{-1} C_4 m^{2/5}.$$

Thus, if $\ell_k \geq \alpha^{-1}C_4 m^{2/5}$, we know that $\ell_n \geq C_4 m^{2/5}$ and that $a_n \geq C_3 m^{2/5}$ for all $n \leq k$, assuming without loss of generality that $\alpha^{-2}C_4 \geq C_3$. We also know (from the choice of ε) that $a_0 \leq m/4$; in general, if $a_k \leq m/4$, then

$$a_{k+1} = \alpha^{-1}\sqrt{C_4 m^{2/5} a_k} \leq \frac{1}{2}\alpha^{-1}\sqrt{C_4 m^{7/5}} \leq m/4$$

for large enough m. Then the pair (ℓ_n, a_n) satisfies the hypothesis of Lemma 4.7, and similarly for (ℓ_n, a_{n-1}) and Lemma 5.5. By induction, this implies

$$\mathbb{P}(\mathcal{E}_{a_k/m}[m^2 s]) \leq \mathbb{P}(\mathcal{L}_{\ell_k/m}[m^2 s]^c \cap \mathcal{E}_{a_k/m}[m^2 s]) + \mathbb{P}(\mathcal{L}_{\ell_k/m}[m^2 s]) \leq (2k+2)e^{-2m^{2/5}}$$

and

$$\mathbb{P}(\mathcal{L}_{\ell_k/m}[m^2 s]) \leq \mathbb{P}(\mathcal{L}_{\ell_k/m}[m^2 s] \cap \mathcal{E}_{a_{k-1}/m}[m^2 s]^c) + \mathbb{P}(\mathcal{E}_{a_{k-1}/m}[m^2 s]) \leq (2k+1)e^{-2m^{2/5}},$$

so long as $\ell_k \geq \alpha^{-1}C_4 m^{2/5}$.

Now, set $A = \alpha^{-1}C_4 m^{2/5}$, so that $\ell_k = A^{1-2^{-k}}\ell_0^{2^{-k}}$ and $m_k = A^{1-2^{-k}} m_0^{2^{-k}}$. With this formula, we see that the first time $\ell_k < 2A$ occurs is when

$$k = \lceil \log_2 \log_2 (A^{-1}\ell_0) \rceil \leq c \log\log m,$$

for some $c > 0$ independent of m. Fix $k' = \lceil \log_2 \log_2 (A^{-1}\ell_0) \rceil$; iterating k' times, the above calculation shows that

$$\mathbb{P}(\mathcal{L}_{\alpha^{-1}C_4 m^{-3/5}}[m^2 s]) \leq \mathbb{P}(\mathcal{L}_{\ell_{k'}/m}[m^2 s]) \leq (2k'+1)e^{-2m^{2/5}} \leq \frac{1}{2}e^{-m^{2/5}}$$

and

$$\mathbb{P}(\mathcal{E}_{\alpha^{-2}C_4 m^{-3/5}}[m^2 s]) \leq \mathbb{P}(\mathcal{E}_{a_{k'}/m}[m^2 s]) \leq (2k'+2)e^{-2m^{2/5}} \leq \frac{1}{2}e^{-m^{2/5}}.$$

Set $C_5 = \alpha^{-2}C_4$. Putting these bounds together, we get

$$\mathbb{P}\left\{ (D_\tau)_{C_5 m^{-3/5}} \cap \frac{1}{m}\mathbb{Z}^2 \subset A_m(m^2\tau) \subset (D_\tau)^{C_5 m^{-3/5}} \text{ for all } \tau \in [0,s] \right\}^c$$
$$\leq \mathbb{P}(\mathcal{L}_{C_5 m^{-3/5}}[m^2 s]) + \mathbb{P}(\mathcal{E}_{C_5 m^{-3/5}}[m^2 s]) \leq e^{-m^{2/5}}. \qquad \square$$

7 Concluding Remarks

There are a number of possible improvements to the results proven here. Most importantly, it would be interesting to improve the $m^{-3/5}$ bounds on the fluctuations; we expect that fluctuations are truly of order $m^{-1}\log m$, as in the point-source case. Hypothetically, this result could be proven using our technique—the primary obstacle is that we need a stronger version of Lemma 4.2(c), which quantifies how closely $\tilde{H}_\zeta$ approximates a continuum harmonic function. If we were able to get an order $\log m$ bound on the L^1 difference between $\tilde{H}_\zeta$ and its harmonic scaling limit, the final result would improve accordingly.

Furthermore, it would be interesting to lift some of the hypotheses we set on the flow. However, we imagine that it is less likely our techniques would apply without the requirements of a concentrated mass distribution or a smooth flow. Indeed, both hypotheses are necessary to guarantee that R_0 is bounded away from 0, and thus that H_ζ is small enough on the boundary. However, if an independent bound on $\tilde{H}_\zeta$ could be obtained, showing that it satisfies $\tilde{H}_\zeta(z) \leq \frac{1}{md(z,\zeta)-C_2}$ without comparing it to H_ζ, it could be used in place of H_ζ for both parts of the proof.

There are also closely related settings that have not been studied extensively. An interesting example would be to replace the "solid" initial sets Q_i^s with submanifolds of D_0. Since these would be zero volume, they could eject particles evenly from all points rather than having a moving interface ∂Q_i^s. We intend to study this particular question in the sequel.

Finally, another question we will investigate in the sequel is that of the scaling limits of the fluctuations themselves. Jerison, Sheffield, and Levine [JLS14] studied this question for same-time fluctuations in the point-source case, and they found that, when the fluctuations are scaled up by a factor of $m^{d/2}$ (in dimension d), they have a weak limit in law of a certain Gaussian random distribution. They found a similar result in the case of a discrete cylinder $\mathbb{Z}\times\mathbb{Z}/m\mathbb{Z}$ with source points along a fixed-height circumference [JLS13a]; here, they further studied the correlations between fluctuations at different times in the flow. The same question has been studied by Eli Sadovnik [Sad16] in the extended-source case, focusing on same-time fluctuations and using harmonic polynomials as test functions; we are interested in strengthening his result to allow smooth test functions and to investigate correlations between fluctuations at different times.

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