CONVEX HULL OF PARKING FUNCTIONS OF LENGTH n UROP+ FINAL PAPER, SUMMER 2020

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ABSTRACT. We consider a polytope P_n, which is the convex hull in $\mathbb{R}^n$ of all parking functions of length n. Stanley found the number of vertices and the number of facets of P_n. In this paper we provide the number of faces of arbitrary dimension, the volume, and the number of integer points in P_n.

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1. Introduction

Let P_n be the convex hull in $\mathbb{R}^n$ of all parking functions of length n. The problem proposed by Stanley in [1] asks to determine

(a) the number of vertices of P_n,

(b) the number of $(n-1)$-dimensional faces, i.e. facets, of P_n,

(c) the number of integer points in P_n, i.e., the number of elements of $\mathbb{Z}^n \cap P_n$,

(d) the n-dimensional volume of P_n.

In a private communication with the author, Stanley showed the proof that the vertices of P_n are the permutations of

$$(\underbrace{1,\dots,1}_{k \text{ ones}}, k+1, k+2, \dots, n),$$

for $1 \le k \le n$. This is proven in two parts. First, any parking function $\alpha = (a_1, \dots, a_n)$ that has a term $a_i > 1$ such that $(a_1, \dots, a_{i-1}, a_i{+}1, a_{i+1}, \dots, a_n)$ is also a parking function, is a convex combination of two other parking functions. Second, if $\alpha = (1, \dots, 1, k+1, k+2, \dots, n)$ is a convex combination of $\beta, \gamma \in P_n$, then by the properties of parking functions, $\beta = \gamma = \alpha$, meaning α is a vertex of P_n.

From this observation, the number of vertices of P_n is

$$n!\left(\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right).$$

Stanley also showed that the defining inequalities of P_n are,

$$\begin{aligned}
1 \le & x_i \le n, \ \le i \le n \\
x_i + x_j & \le (n-1) + n, \ i < j \\
x_i + x_j + x_k & \le (n-2) + (n-1) + n, \ i < j < k \vdots \\
x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}} & \le 3 + 4 + \cdots + n, \ i_1 < i_2 < \cdots < i_{n-2} \\
x_1 + x_2 + \cdots + x_n & \le 1 + 2 + \cdots + n.
\end{aligned}$$

Thus, the number of facets is the number of these inequalities, which is equal to

$$2^n - 1.$$

Organization of the paper. In Section 2, we find the number of edges of P_n by understanding which pairs of vertices create an edge and using the formula obtained by Stanley for the number of vertices of P_n. In Section 3, we consider the general case of d-dimensional faces of P_n, determine their structure, and derive the formula for their number by using Stirling numbers of the second kind. In Section 4, we prove that the sequence $\{V_n\}$ of volumes of P_n satisfies a nice recurrence relation and find the exponential generating function of this sequence. Lastly, in Section 5, we show that the set of lattice points of P_n can be divided into sets of lattice points of several permutohedrons, which have a formula given by Postnikov in [2].

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2. Edges

Theorem 2.1. *The number of edges of P_n is equal to*

$$\frac{n \cdot n!}{2}\left(\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right).$$

Definition 2.2. We know that parking functions which are vertices of P_n are permutations of $(1, 1, \ldots, 1, k+1, k+2, \ldots, n)$ $(1 \le k \le n)$. For every such vertex we define its *layer* to be equal to $n-k$. We denote the layer of x by $L(x)$.

Proposition 2.3. *If v_1 and v_2 are two vertices of P_n such that v_1v_2 is an edge, then $|L(v_1) - L(v_2)| \le 1$, meaning v_1 and v_2 are either from neighboring layers or from the same layer.*

Proof. Let $c \cdot x$ be the dot product $c_1x_1 + \cdots + c_nx_n$ of vectors $c, x \in \mathbb{R}^n$. If v_1v_2 is an edge, then there is c such that $c \cdot v_1 = c \cdot v_2 > c \cdot v$ for any vertex v of P_n distinct from v_1 and v_2. Since P_n is invariant under coordinate permutation, without loss of generality we may assume $c_1 \le \cdots \le c_n$.

Suppose v_1 and v_2 are $t \ge 2$ layers apart from each other, so let v_1 be a permutation of $(1, 1, \ldots, 1, k, k{+}1, \ldots, n)$ and v_2 be a permutation of $(1, 1, \ldots, 1, k{+}t, k{+}t{+}1, \ldots, n)$, where $1 \le k < k+2 \le k+t \le n$. Since v_1 and v_2 are unique permutations of $(1, 1, \ldots, 1, k, k+1, \ldots, n)$ and $(1, 1, \ldots, 1, k, k{+}1, \ldots, n)$, respectively, that maximize $c \cdot v$, by the rearrangement inequality, $v_1 = (1, 1, \ldots, 1, k, k+1, \ldots, n)$, $v_2 = (1, 1, \ldots, 1, k+t, k+t+1, \ldots, n)$, and $c_{k-1} < c_k < \cdots < c_n$. If $c_{k+t-1} \ge 0$, then for $v_3 = (1, 1, \ldots, 1, k{+}t{-}1, k{+}t \ldots, n) \in P_n$ which is distinct from v_1 and v_2, we have $c \cdot v_3 \ge c \cdot v_2$, a contradiction. Otherwise, if $c_{k+t-1} < 0$, we have $c_k < \cdots < c_{k+t-1} < 0$, so $c \cdot v_1 - c \cdot v_2 = c_k(k-1) + c_{k+1}k + \cdots + c_{k+t-1}(k+t-2) < 0$, meaning $c \cdot v_1 < c \cdot v_2$, a contradiction. Thus, v_1 and v_2 are at most one layer apart from each other. $\square$

Proposition 2.4. *For each vertex v of P_n, there are exactly n edges of P_n with v as one of the vertices.*

Proof. Suppose $v = v_k$ is on layer $n-k$. Since P_n is invariant under coordinate permutation, without loss of generality we may assume $v_k = (1, \ldots, 1, k+1, \ldots, n)$. Let $v_k v_k'$ be an edge of P_n, then there is $c \in \mathbb{R}^n$ such that $c \cdot v_k = c \cdot v_k' > c \cdot v$ for any vertex v of P_n distinct from v_k and v_k'. By the rearrangement inequality, $c_i \le c_{k+1} \le \cdots \le c_n$ for any $1 \le i \le k$.

If v_k' is on the same layer as v_k, then v_k' is a permutation of $(1, \ldots, 1, k+1, \ldots, n)$. If $c_{k+1} \le 0$, then changing the $(k+1)$-st coordinate of v_k from $k+1$ to 1 will give another vertex v of P_n for which $c \cdot v \ge c \cdot v_k$, contradiction. Thus $0 < c_{k+1} \le \cdots \le c_n$. If $2 \le k \le n$ and $c_i \ge 0$ for some $1 \le i \le k$, then changing the i-th coordinate of v_k from 1 to k will give another vertex v of P_n for which $c \cdot v \ge c \cdot v_k$, a contradiction. Thus, $c_i < 0$ for $1 \le i \le k$ if $k \ge 2$.

By the rearrangement inequality, for $k = 1$, $c_1 < \cdots < c_j = c_{j+1} < \cdots < c_n$, where $1 \le j \le n-1$, and for $k \ge 2$, $0 < c_{k+1} < \cdots < c_j = c_{j+1} < \cdots < c_n$ where $k+1 \le j \le n-1$. Thus, v_k and v_k' differ from each other by exactly one swap of two neighboring coordinates $(i, i+1)$ where $k+1 \le i \le n-1$ for $2 \le k \le (n-1)$, and $k \le i \le n-1$ for $k = 1$.

Suppose v_k' is 1 layer apart from v_k. Then v_k is the only permutation of $(1, \ldots, 1, k+1, \ldots, n)$ maximizing $c \cdot v$, so $c_i < c_{k+1} < \cdots < c_n$ for any $1 \le i \le k$. Therefore, if $k \ge 2$ and v_k' is a permutation of $(1, \ldots, 1, k, k+1, \ldots, n)$, then its last $n-k$ coordinates are exactly $(k+1, k+2, \ldots, n)$, so the first k coordinates are one of the k permutations of $(1, \ldots, 1, k)$, where $c_i < 0$ for $1 \le i \le k$ corresponding to values 1 and $c_i = 0$ for the $1 \le i \le k$ corresponding to value k. If $k < n$ and v_k' is a permutation of $(1, \ldots, 1, k+2, \ldots, n)$ then it is exactly $(1, \ldots, 1, k+2, \ldots, n)$, where $c_i < 0$ for $1 \le i \le k$ and $c_{k+1} = 0$.

Thus, for $2 \le k \le n-1$, there are $(n-k-1) + k + 1 = n$ edges with v_k as one of the vertices. For $k = 1$, there are $(n-k) + 0 + 1 = n$ edges with v_k as one of the vertices. And for $k = n$, there are $0 + k + 0 = n$ edges with v_k as one of the vertices. □

Proof of Theorem 2.1. By Proposition 2.4, the graph of P_n is an n-regular graph with

$$V = n! \left(\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \right)$$

vertices. Therefore, P_n has $\frac{nV}{2}$ edges. □

3. Faces of higher dimensions

In this section we are going to generalize this approach to understand the nature of faces of higher dimensions. More specifically, we will prove the following theorem.

Theorem 3.1. *Let f_{n-s} be the number of $(n-s)$-dimensional faces for s from 0 to n. Then,*

$$f_{n-s} = \sum_{m=0, m\neq 1}^{s} \binom{n}{m} \cdot (s-m)! \cdot S(n-m+1, s-m+1),$$

where $S(n,k)$ are the Stirling numbers of the second kind.

For each $c \in \mathbb{R}^n$, let F_c be the set of points $x \in P_n$ such that $c \cdot x$ is maximized (for $x \in P_n$). Each face of P_n is equal to F_c for some $c \in \mathbb{R}^n$. Also, denote the set of vertices of P_n lying in F_c by $V(F_c)$.

For each c, define an ordered partition $(B_-, B_0, \ldots, B_k)$ of $\{1, 2, \ldots, n\}$ where B_- is the set of indices i such that $c_i < 0$, B_0 is the set of indices i such that $c_i = 0$, and B_j is the set of indices i such that c_i is the j-th smallest positive value among the coordinates of c. Let $l_j = |B_j|$ for $j = -, 0, 1, \ldots, k$.

Lemma 3.2. *The face F_c is determined by the ordered partition $(B_, \ldots, B_k)$ described above. Then each face of P_n can be uniquely defined by an ordered partition $(B_, B_0, \cdots, B_k)$ that does not satisfy $l_- = 0, l_0 = 1$ or $l_- = 0, l_0 = 0, l_1 = 1$.*

Proof. Consider a vertex v of P_n that maximizes $c \cdot v$. By the rearrangement inequality and the structure of vertices of P_n, it is clear that $v_i = 1$ for $i \in B_-$, $(v_i | i \in B_0)$ is a permutation of $(1, 1, \ldots 1, j+1, \ldots, l_- + l_0)$ for some $j \in [l_-, l_- + l_0]$, $(v_i | i \in B_i)$ is a permutation of $(l_- + l_0 + \cdots + l_{i-1} + 1, l_- + l_0 + \cdots + l_{i-1} + 2, \ldots, l_- + l_0 + \cdots + l_{i-1} + l_i)$ for each i from 1 to k.

From this conclusion, if $l_- = 0$ and $l_0 = 1$, we can change the zero coordinate of c to -1, and the set $V(F_c)$ will not change. Also, if $l_- = 0$, $l_0 = 0$, and $B_1 = \{i\}$, we can change the value of c_i to -1, and $V(F_c)$ will not change. So we do not consider $(B_-, B_0, \ldots, B_k)$ with $l_- = 0$ and $l_0 = 1$ or $l_- = 0$, $l_0 = 0$, and $l_1 = 1$. Other than that, from the conclusion of the previous paragraph, different ordered partitions define different $V(F_c)$'s. $\square$

Lemma 3.3. *The dimension of F_c is equal to $n - k - l_-$.*

Proof. Let d be the dimension of F_c. Then $d = \dim(\text{aff}(V(F_c)))$. If $d = n$ then clearly $F_c = P_n$ and $c = \mathbf{0}$, so indeed $n - k - l_- = n = d$. Now suppose $d < n$. Then $\mathbf{0} \notin \text{aff}(V(F_c))$, so $\dim(\text{aff}(V(F_c) \cup \{\mathbf{0}\})) = d + 1$.

It is clear that $\dim(\text{aff}(V(F_c) \cup \{\mathbf{0}\})$ is the dimension of the vector space W spanned by the vectors from $\mathbf{0}$ to points in $V(F_c)$. Take a vector w from $\mathbf{0}$ to some point of $V(F_c)$. For each j from 1 to k, consider $B_j = \{i_1, i_2, \ldots, i_{l_j}\}$. Let V_j be the set of $l_j - 1$ vectors v in $\mathbb{R}^n$ which are the permutations of $(1, -1, 0, 0, \ldots, 0)$ having $v_{i_k} = 1, v_{i_{k+1}} = -1$, for some $1 \leq k \leq l_j - 1$. Also, let V_0 be the set of l_0 vectors e_i in $\mathbb{R}^n$ which are the permutations of $(1, 0, 0, 0, \ldots, 0)$ having value 1 at one of the coordinates with index $i \in B_0$.

Consider the set $S = \left(\bigcup_{i=0}^{k} V_i\right) \cup w$ of

$$l_0 + \sum_{i=1}^{k}(l_i - 1) + 1 = \sum_{i=0}^{k} l_i - k + 1 = n - l_- - k + 1$$

vectors. We will prove that S spans W.

For any $x \in V(F_c)$, consider the vector $a = x - w - \sum_{i \in B_0}(x_i - w_i)e_i$. Clearly, $a_i = 0$ for $i \in B_- \cup B_0$, and for each $0 < j \leq k$, if $B_j = \{i_1, i_2, \ldots, i_{l_j}\}$, then $\sum_{m=1}^{l_j} a_{i_m} = 0$. Then $(a_{i_1}, a_{i_2}, \ldots, a_{i_{l_j}})$ is a linear combination of

$$((1, -1, 0, \ldots, 0), (0, 1, -1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1, -1)).$$

Therefore, a is a linear combination of vectors in $\bigcup_{i=1}^{k} V_i$. Thus, $x = a + w + \sum_{i \in B_0}(x_i - w_i)e_i$ is a linear combination of vectors in S, so S spans W.

Also, S is linearly independent. If it is not, then there is a linear combination β of vectors in S such that $\beta = bw + \sum_{v \in S \setminus \{w\}} b_v v = 0$ and not all of the b_v and b are zero. If $l_- > 0$, then for some $i \in B_-$, $0 = \beta_i = bw_i$, so $b = 0$. If $k > 0$, then $|B_1| > 0$, so

$$\begin{aligned}
0 &= \sum_{q \in B_1} \beta_q \\
&= \sum_{q \in B_1} \left(bw_q + \sum_{v \in S \setminus \{w\}} b_v v_q \right) \\
&= b \sum_{q \in B_1} w_q + \sum_{q \in B_1} \sum_{v \in S \setminus \{w\}} b_v v_q \\
&= b \sum_{q \in B_1} w_q + \sum_{v \in S \setminus \{w\}} b_v \sum_{q \in B_1} v_q \\
&= b \sum_{q \in B_1} w_q + \sum_{v \in S \setminus \{w\}} b_v \cdot 0 \\
&= b \sum_{q \in B_1} w_q.
\end{aligned}$$

Therefore, $b = 0$. Since $d < n$, we have $l_0 < n$, so either $l_- > 0$ or $k > 0$. In both cases $b = 0$. But then $(b_1, \ldots, b_{n-l_- -k}) \neq 0$, so $\bigcup_{i=0}^{k} V_i$ is linearly dependent, which is clearly not true.

Thus, S spans W and is linearly independent, which means it is a basis of W. Thus $d + 1 = \dim(W) = |S| = n - l_- - k + 1$, so $d = n - k - l_-$. $\square$

Proof of Theorem 3.1. To find the number f_{n-s} of $(n-s)$-dimensional faces we need to find the number of different ordered partitions $(B_-, B_0, \ldots, B_k)$ of $\{1, \ldots, n\}$ such that $l_i > 0$ for $i \geq 1$ and $n - s = n - k - l_-$, i.e., $s = k + l_-$, not satisfying $l_- = 0, l_0 = 1$ or $l_- = 0, l_0 = 0, l_1 = 1$. For convenience, we will denote l_- by m in further computations. We have $s = k + m$, so m takes values from 0 to s.

For each m from 0 to s, we first choose m elements for B_-. Then, if $l_0 = 0$, we partition the remaining $n - m$ elements into $k = s - m$ nonempty ordered groups. If $l_0 \geq 1$, we partition the remaining $n - m$ elements into $k + 1 = s - m + 1$ nonempty ordered groups. Thus we have the corresponding Stirling numbers of the second kind multiplied by the number of permutations of the groups because those are ordered. Note that since we do not consider c with $m = l_- = 0$ and $l_0 = 1$ or $m = l_- = 0$, $l_0 = 0$, and $l_1 = 1$, we need to subtract the number of such partitions. So we subtract $n \cdot k! \cdot S(n-1,k) = \binom{n}{1} \cdot s! \cdot S(n-1,s)$ and $n \cdot (k-1)! \cdot S(n-1,k-1) = \binom{n}{1} \cdot (s-1)! \cdot S(n-1,s-1)$. Therefore,

$$\begin{aligned} f_{n-s} &= \sum_{m=0, m\neq 1}^{s} \binom{n}{m} \cdot ((s-m)! \cdot S(n-m,s-m) + (s-m+1)! \cdot S(n-m,s-m+1)) \\ &= \sum_{m=0, m\neq 1}^{s} \binom{n}{m} \cdot (s-m)! \cdot S(n-m+1,s-m+1). \end{aligned}$$

$\square$

Using this formula to find the number of edges of P_n, we take $n - s = 1$, so $s = n - 1$. Then since $S(a, a-1) = \frac{a(a-1)}{2}$ for any positive integer a,

$$\begin{aligned} f_1 &= \sum_{m=0, m\neq 1}^{n-1} \binom{n}{m} \cdot (n-m-1)! \cdot S(n-m+1, n-m) \\ &= \sum_{m=0, m\neq 1}^{n-1} \binom{n}{m} \cdot (n-m-1)! \cdot \frac{(n-m+1)(n-m)}{2} \\ &= \sum_{m=0, m\neq 1}^{n-1} \frac{n! \cdot (n-m+1)}{2m!} \\ &= \sum_{m=1}^{n-1} \frac{n! \cdot n}{2m!} - \sum_{m=2}^{n-1} \frac{n!}{2(m-1)!} + \sum_{m=1}^{n-1} \frac{n!}{2m!} \\ &= \frac{n}{2} \left(\sum_{m=1}^{n-1} \frac{n!}{m!} \right) - \sum_{m=1}^{n-2} \frac{n!}{2m!} + \sum_{m=1}^{n-1} \frac{n!}{2m!} \\ &= \frac{n}{2}(V-1) + \frac{n!}{2(n-1)!} \\ &= \frac{nV}{2}, \end{aligned}$$

where V is the number of vertices of P_n and is equal to $n! \left(\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \right)$. This again proves Theorem 2.1.

4. Volume

To find the volume of P_n, we split the polytope into n-dimensional pyramids with facets of P_n not containing $I = (1, 1, \ldots, 1)$ as bases and point I as vertex. There are $2^n - n - 1$

such pyramids. Now we will derive a recursive formula for the volume of P_n, as a sum of volumes of these pyramids.

Theorem 4.1. *Define a sequence $\{V_n\}_{n\geq 0}$ by $V_0 = 1$ and $V_n = Vol\,(P_n)$ for all positive integers n. Then, $\{V_n\}_{n\geq 0}$ satisfies the following recurrence relation,*

$$V_n = \frac{1}{n}\sum_{k=0}^{n-1}\binom{n}{k}\frac{(n-k)^{n-k-1}(n+k-1)}{2}V_k,$$

for all $n \geq 2$.

In the proof of this theorem we will use the following "decomposition lemma".

Proposition 4.2 ([3, Proposition 2]). *Let $K_1, \dots, K_n$ be some convex bodies of $\mathbb{R}^n$ and suppose that $K_{n-m+1}, \dots, K_n$ are contained in some m-dimensional affine subspace U of $\mathbb{R}^n$. Let MV_U denote the mixed volume with respect to the m-dimensional volume measure on U, and let $MV_{U^\perp}$ be defined similarly with respect to the orthogonal complement $U^\perp$ of U. Then the mixed volume of $K_1, \dots, K_n$*

$$MV(K_1, \dots, K_{n-m}, K_{n-m+1}, \dots, K_n) = \frac{1}{\binom{n}{m}} MV_{U^\perp}(K'_1, \dots, K'_{n-m}) MV_U(K_{n-m+1}, \dots, K_n),$$

where $K'_1, \dots, K'_{n-m}$ denote the orthogonal projections of $K_1, \dots, K_{n-m}$ onto $U^\perp$, respectively.

Proof of Theorem 4.1. Each pyramid has a base which is a facet F with points of P_n satisfying the equation

$$x_{i_1} + x_{i_2} + \cdots + x_{i_k} = (n-k+1) + (n-k+2) + \cdots + (n-1) + n$$

for some $k \in \{1, 2, \dots, n-2, n\}$ and distinct $i_1 < \cdots < i_k$.

Let $\{j_1, j_2, \dots, j_{n-k}\} = \{1, 2, \dots, n\} - \{i_1, i_2, \dots, i_k\}$. Let P'_{n-k} be the polytope containing all points x' such that $x'_p = 0$ for all $p \in \{i_1, i_2, \dots, i_k\}$ and for some $x \in F$, $x'_p = x_p$ for all $p \in \{j_1, j_2, \dots, j_{n-k}\}$. Then P'_{n-k} is an $(n-k)$-dimensional polytope with the following defining inequalities:

$$\begin{aligned}
1 \leq\, & x'_{j_p} \leq n-k, \ 1 \leq p \leq n-k \\
x'_{j_p} + x'_{j_q} &\leq (n-k-1) + (n-k), \ 1 \leq p < q \leq n-k \\
x'_{j_p} + x'_{j_q} + x'_{j_r} &\leq (n-k-2) + (n-k-1) + (n-k), \ 1 \leq p < q < r \leq n-k \\
&\vdots \\
x'_{j_{p_1}} + x'_{j_{p_2}} + \cdots + x'_{j_{p_{n-k-2}}} &\leq 3 + 4 + \cdots + (n-k), \ 1 \leq p_1 < p_2 < \cdots < p_{n-k-2} \leq n-k \\
x'_{j_{p_1}} + x'_{j_{p_2}} + \cdots + x'_{j_{p_{n-k}}} &\leq 1 + 2 + 3 + 4 + \cdots + (n-k).
\end{aligned}$$

This means P'_{n-k} is congruent to P_{n-k}, so $\mathrm{Vol}_{n-k}(P'_{n-k}) = \mathrm{Vol}_{n-k}(P_{n-k}) = V_{n-k}$.

Let Q_k be the polytope containing all points x' such that for all $p \in \{j_1, j_2, \ldots, j_{n-k}\}$, we have $x'_p = 0$, and for some $x \in F$, for all $p \in \{i_1, i_2, \ldots, i_k\}$, we have $x'_p = x_p$. Then the coordinate values $(x'_{i_1}, x'_{i_2}, \ldots, x'_{i_k})$ of vetrices of Q_k are the permutations of $(n-k+1, n-k+2, \ldots, n)$, meaning Q_n is a $(k-1)$-dimensional polytope equivalent to the permutohedron of order k with $(k-1)$-dimensional volume $k^{k-2}\sqrt{k}$.

Thus, F is a Minkowski sum of two polytopes P'_{n-k} and Q_k which lie in two orthogonal subspaces of $\mathbb{R}^n$. Therefore, by Proposition 4.2, the $(n-1)$-dimensional volume of F is equal to

$$\sum_{p_1,\ldots,p_n=1}^{2} MV(K_{p_1}, K_{p_2}, \ldots, K_{p_n}) = V_{n-k} \cdot k^{k-2}\sqrt{k},$$

where $K_1 = P'_{n-k}$ and $K_2 = Q_k$. Then the volume of the pyramid with F as a base and I as a vertex is equal to

$$\frac{1}{n} h_k \mathrm{Vol}(F) = \frac{1}{n} h_k V_{n-k} \cdot k^{k-2}\sqrt{k},$$

where h_k is the distance from point I to the face F, which is equal to

$$\frac{|1+1+\cdots+1-((n-k+1)+(n-k+2)+\cdots+(n-1)+n)|}{\sqrt{1+1+\cdots+1}} = \frac{k(2n-k-1)}{2\sqrt{k}}.$$

Thus,

$$\mathrm{Vol}(\mathrm{Pyr}(I,F)) = \frac{1}{n} \cdot \frac{k(2n-k-1)}{2\sqrt{k}} V_{n-k} \cdot k^{k-2}\sqrt{k} = \frac{1}{n} \cdot \frac{k(2n-k-1)}{2} k^{k-2} V_{n-k}.$$

Since $V_0 = 1$ and $V_1 = 0$, we get for $n \geq 2$,

$$\begin{aligned} V_n &= \frac{1}{n}\left(\sum_{k=1}^{n-2} \binom{n}{k} \frac{k(2n-k-1)}{2} k^{k-2} V_{n-k}\right) + \frac{1}{n} \cdot \frac{n(n-1)}{2} n^{n-2} \\ &= \frac{1}{n}\left(\sum_{k=2}^{n-1} \binom{n}{n-k} \frac{(n-k)(n+k-1)}{2} (n-k)^{n-k-2} V_k\right) + \frac{1}{n} \cdot \frac{n^{n-1}(n-1)}{2} \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} \frac{(n-k)^{n-k-1}(n+k-1)}{2} V_k. \end{aligned}$$

□

For $n = 1, 2, \ldots, 8$ this formula gives the volume values $0, \frac{1}{2}, 4, \frac{159}{4}, 492, \frac{58835}{8}, 129237, \frac{41822865}{16}$.

Proposition 4.3. *Let $f(x) = \sum_{n \geq 0} \frac{V_n}{n!} x^n$ be the exponential generating function of $\{V_n\}_{n \geq 0}$. Let $g(x) = \sum_{n \geq 1} \frac{n^{n-1}}{n!} x^n$ be the exponential generating function of $\{n^{n-1}\}_{n \geq 1}$. Then,*

$$f(x) = e^{\int \frac{x(g'(x))^2}{2}}.$$

Proof. It is known that $g(x) = xe^{g(x)}$, so

$$(*) \qquad g'(x) = e^{g(x)} + xe^{g(x)} = \frac{g(x)}{x} + g(x)g'(x).$$

From Theorem 4.1,

$$\begin{aligned} n \cdot \frac{V_n}{n!} &= \sum_{k=0}^{n-1} \frac{(n-k)^{n-k-1}(n+k-1)}{2(n-k)!} \cdot \frac{V_k}{k!} \\ &= \sum_{k=0}^{n-1} \frac{(n-k)^{n-k-1}(n-k+2k-1)}{2(n-k)!} \cdot \frac{V_k}{k!} \\ &= \sum_{k=0}^{n-1} \frac{1}{2} \cdot \frac{(n-k)^{n-k}}{(n-k)!} \cdot \frac{V_k}{k!} + \sum_{k=0}^{n-1} \frac{(n-k)^{n-k-1}k}{(n-k)!} \cdot \frac{V_k}{k!} - \sum_{k=0}^{n-1} \frac{1}{2} \cdot \frac{(n-k)^{n-k-1}}{(n-k)!} \cdot \frac{V_k}{k!}. \end{aligned}$$

Therefore,

$$f'(x) = \frac{1}{2}g'(x)f(x) + g(x)f'(x) - \frac{1}{2x}g(x)f(x).$$

Then,

$$f'(x)(1-g(x)) = \frac{1}{2x}(xg'(x) - g(x))f(x) \underset{\text{by}(*)}{=} \frac{1}{2x}xg(x)g'(x)f(x) = \frac{1}{2}g(x)g'(x)f(x),$$

so

$$f'(x) = \frac{g(x)g'(x)f(x)}{2(1-g(x))} \underset{\text{by}(*)}{=} \frac{g(x)g'(x)f(x)}{2(\frac{g(x)}{xg'(x)})} = \frac{x(g'(x))^2}{2}f(x).$$

Thus, $f(x) = ce^{\int \frac{x(g'(x))^2}{2}}$. It is clear that $c = 1$, so $f(x) = e^{\int \frac{x(g'(x))^2}{2}}$. □

5. Lattice Points

In this section we are going to find the number of integer points in P_n.

Proposition 5.1. *Let $P_{n,S}$ be the set of points x in P_n satisfying $x_1 + \cdots + x_n = S$. For each integer S from $n+1$ to $\frac{n(n-1)}{2}$ there is a unique pair of positive integers (r, k) such that $2 \le r \le k+1$,*

$$\underbrace{1 + \cdots + 1}_{k \text{ ones}} + r + (k+2) + \cdots + n = S,$$

and the set of vertices of $P_{n,S}$ is the set of permutations of $(1, \ldots, 1, r, k+2, \ldots, n)$. For the case $S = n$, the set of vertices of $P_{n,n}$ is just one vertex $(1, \ldots, 1)$.

Proof. It is clear that if $S = n$, then the only point x in $P_{n,S}$ satisfies $x_1 = \cdots = x_n = 1$. For this case we can say $k = n$ and r is unnecessary.

Since $1+\cdots+1<1+\cdots+1+n<\cdots<1+2+\cdots+n$, for each S from $n+1$ to $\frac{n(n-1)}{2}$ there is a unique $k \leq (n-1)$ such that

$$1+\cdots+1+(k+2)+\cdots+n<S\leq 1+\cdots+1+(k+1)+\cdots+n.$$

Then $0<S-(1+\cdots+1+(k+2)+\cdots+n)\leq k$, so take

$$r=1+S-(1+\cdots+1+(k+2)+\cdots+n)$$

for which $1<r\leq k+1$. Then indeed $1+\cdots+1+r+(k+2)+\cdots+n=S$.

Suppose there is another (r',k') such that $1+\cdots+1+r'+(k'+2)+\cdots+n=S$. If $k<k'$, then

$$\begin{aligned}1+\cdots+1+r'+(k'+2)+\cdots+n &\leq\\ 1+\cdots+1+(k'+1)+(k'+2)+\cdots+n &\leq\\ 1+\cdots+1+(k+2)+\cdots+n &<\\ 1+\cdots+1+r+(k+2)+\cdots+n&,\end{aligned}$$

contradiction. Thus, $k\geq k'$. Similarly, $k'\geq k$, so $k=k'$, from where it is clear that $r=r'$.

Now we will prove that set of vertices of $P_{n,S}$ is the set of permutations of $(1,\ldots,1,r,k+2,\ldots,n)$. Let $a=(a_1,\ldots,a_n)$ be a vertex of $P_{n,S}$. Since $P_{n,S}$ is invariant under coordinate permutation, we may assume $a_1\leq\cdots\leq a_n$.

If there is no $1\leq k\leq n$ such that $a_k<k$, then clearly $a_i=i$ for all $1\leq i\leq n$. In this case $k=1$, $r=2$, and a is indeed a permutation of $(1,\ldots,1,r,k+2,\ldots,n)=(1,2,\ldots,n)$. Otherwise, take the greatest $1\leq k\leq n$ such that $a_k<k$. Then $a=(a_1,\ldots,a_k,k+1,\ldots,n)$.

Case 1. $a_k=a_{k-1}$.

Suppose $c=a_m=\cdots=a_k\leq k-1$ and $a_{m-1}\neq c$. Then

$$c=\frac{a_m+\cdots+a_k}{k-m+1}\leq\frac{m+\cdots+k}{k-m+1}=\frac{m+k}{2}.$$

Suppose $c>1$. Then there exists $\epsilon>0$ such that $\epsilon\leq\frac{j-m}{2}(k-j+1)$ for each j from $m+1$ to k. Consider $x=(a_1,\ldots,a_{m-1},a_m-\epsilon,a_{m+1},\ldots,a_{k-1},a_k+\epsilon,\ldots,a_n)$. For any $m+1\leq j\leq k$,

$$\begin{aligned}a_j+\cdots+a_k+\epsilon&=c(k-j+1)+\epsilon\\ &\leq\frac{m+k}{2}(k-j+1)+\frac{j-m}{2}(k-j+1)\\ &=\frac{j+k}{2}(k-j+1)\\ &=j+\cdots+k.\end{aligned}$$

This means x satisfies all the defining inequalities of P_n, so $x \in P_{n,S}$. Therefore, $x' = (a_1, \dots, a_{m-1}, a_m + \epsilon, a_{m+1}, \dots, a_{k-1}, a_k - \epsilon, \dots, a_n)$ is also in $P_{n,S}$ since it is just a permutation of x. But then $a = \frac{1}{2}x + \frac{1}{2}x'$, so a is not a vertex of $P_{n,S}$ if $c > 1$.

Therefore, $c = 1$, and since $1 \le a_1 \le \cdots \le a_k = c = 1$, we have $a_1 = \cdots = a_k = 1$ and $S = 1 + \cdots + 1 + (k+1) + \cdots + n$, so $(r, k) = (k+1, k)$ and a is indeed a permutation of $(1, \dots, 1, r, k+2, \dots, n)$.

Case 2. $a_k > a_{k-1}$.

Then, since $a_{k-1} \ge 1$, we have $a_k \ge 2$. Suppose $c = a_m = \cdots = a_{k-1} < a_k \le k-1$ and $a_{m-1} \ne c$. Then

$$\begin{aligned} c &= \frac{a_m + \cdots + a_{k-1}}{k-m} \\ &= \frac{a_m + \cdots + a_{k-1} + a_k - a_k}{k-m} \\ &\le \frac{m + \cdots + k - a_k}{k-m} \\ &= \frac{\frac{1}{2}(m+k)(k-m+1) - a_k}{k-m}. \end{aligned}$$

Suppose $c > 1$. For any j from $m+1$ to k,

$$\begin{aligned} (j + \cdots + k) - (a_j + \cdots + a_k) &= \frac{1}{2}(j+k)(k-j+1) - c(k-j) - a_k \\ &\ge \frac{1}{2}(j+k)(k-j+1) - (k-j)\frac{\frac{1}{2}(m+k)(k-m+1) - a_k}{k-m} - a_k \\ &= \frac{1}{2}(j+k)(k-j+1) - (k-j)\frac{\frac{1}{2}(m+k)(k-m+1)}{k-m} + a_k\left(\frac{m-j}{k-m}\right) \\ &> \frac{1}{2}(j+k)(k-j+1) - (k-j)\frac{\frac{1}{2}(m+k)(k-m+1)}{k-m} + \frac{k(m-j)}{k-m} \\ &= \frac{(k-j)(j-m)}{2} \ge 0. \end{aligned}$$

Then there exists $\epsilon > 0$ such that $\epsilon < (j + \cdots + k) - (a_j + \cdots + a_k)$ for each j from $m+1$ to k.

Consider $x = (a_1, \dots, a_{m-1}, a_m - \epsilon, a_{m+1}, \dots, a_{k-1}, a_k + \epsilon, \dots, a_n)$. For any $m+1 \le j \le k$, $a_j + \cdots + a_k + \epsilon < j + \cdots + k$. This means x satisfies all the defining inequalities of P_n, so $x \in P_{n,S}$. Also, $x' = (a_1, \dots, a_{m-1}, a_m + \epsilon, a_{m+1}, \dots, a_{k-1}, a_k - \epsilon, \dots, a_n)$ is also in $P_{n,S}$. But then $a = \frac{1}{2}x + \frac{1}{2}x'$, so a is not a vertex of $P_{n,S}$ if $c > 1$.

Therefore, $c = 1$ and since $1 \le a_1 \le \cdots \le a_{k-1} = c = 1$, we have $a_1 = \cdots = a_{k-1} = 1$. Then $S = 1 + \cdots + 1 + a_k + (k+1) + \cdots + k$, where $2 \le r = a_k < k$, so a is indeed a permutation of $(1, \dots, 1, r, k+1, \dots, n)$. $\square$

Thus, we have that $P_{n,S}$ is a permutohedron with permutations of $(1,\dots,1,r,k{+}2,\dots,n)$ as its vertices. In the case $S=n$, $P_{n,S}$ is a permutohedron consisting of one point $(1,\dots,1)$. In other words, $P_{n,S}$ is the convex hull of all permutations of vector $(x_1,\dots,x_n)$, where

$$(x_1,\dots,x_n)=\begin{cases}(1,\dots,1,r,k+2,\dots,n), & \text{if } S>n,\\ (1,\dots,1), & \text{if } S=n.\end{cases}$$

Let $N(P)$ denote the number of integer points in a polytope P. Then,

$$N(P_n)=\sum_{S=n}^{\frac{n(n-1)}{2}} N(P_{n,S}).$$

Let From [2, Section 4], $P_{n,S}$ is a generalized permutohedron $P_{n-1}(\mathbf{Y})$ with $Y_I=y_{|I|}$ for any $I\subset[n]$ and

$$\begin{aligned}
y_1&=x_1\\
y_2&=x_2-x_1\\
y_3&=x_3-2x_2+x_1\\
&\vdots\\
y_n&=\binom{n-1}{0}x_n-\binom{n-1}{1}x_{n-1}+\cdots\pm\binom{n-1}{n-1}x_1.
\end{aligned}$$

So by [2, Theorem 4.2],

$$N(P_{n,S})=\frac{1}{(n-1)!}\sum_{(S_1,\dots,S_{n-1})}\left\{Y_{S_1}\cdots Y_{S_{n-1}}\right\},$$

where the summation is over ordered collections of subsets $S_1,\dots,S_{n-1}\subset[n]$ such that for any distinct $i_1,\dots,i_k$, we have $|S_{i_1}\cup\cdots\cup S_{i_k}|\geq k+1$, and

$$\left\{\prod_I Y_I^{a_I}\right\}:=(Y_{[n]}+1)^{\{a_{[n]}\}}\prod_{I\neq[n]}Y_I^{\{a_I\}},\text{ where } Y^{\{a\}}=Y(Y+1)\dots(Y+a-1).$$

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