On GKM Description of the Equivariant Cohomology of Flag Varieties and Affine Springer Fibers

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Abstract

Under the torus action of the variety, fixed points and 1-dimensional orbits determine the equivariant cohomology under GKM description. Constructing the GKM graph consisting of fixed points and 1-dimensional orbits, the problem of computing the equivariant cohomology can be translated to a combinatorial problem with the restraints given by GKM condition. Especially when the variety is affine flag or affine Springer fiber, the set of fixed points under the extended torus action becomes the affine Weyl group and the 1-dimensional orbit corresponds to the reflections in the affine Weyl group.

Using the combinatorial properties of Coxeter groups, we provide complete description of equivariant cohomology ring of affine flag varieties and suggest an attempt, with partial results on $sl(2)$, to compute the equivariant cohomology of affine Springer fibers under the GKM description.

1 Introduction and Preliminaries

For any topological spase, there exists the notion of cohomology which embraces topological properties algebraically. However for a variety X endowed with the action of a topological group G, there exists more useful notion of equivariant cohomology which compensates the defects of the theory of cohomology. Given the well defined action of a topological group G, classical results from equivariant cohomology show that the cohomological properties are embedded in zero and one dimensional orbits of G. This section provides the basics of equivariant cohomology and introduces the GKM description of equivariant cohomology, our main methodology of computing the equivariant cohomology ring throughout the paper. We follow the terminology and notation used in [1, 5] and assume the coefficients are $\mathbb{C}$ unless stated explicitly.

By Borel construction, there exists a universal principal G-bundle

$$E_G \to B_G$$

such that $B_G = E_G/G$ and E_G is a contractible G-space where G acts freely. Therefore $X \times E_G$ is homotopy equivalent to X where G acts freely. The equivariant cohomology ring $H^*_G(X)$ is then defined as

$$H^*_G(X) = H^*(X \times E_G/G).$$

By the universality, equivariant cohomology is invariant under the construction of principal G-bundle $E_G \to B_G$. In fact, for 1-dimensional torus S^1, we have principle S^1-bundle

$$S^\infty \to S^\infty/S^1 = \mathbb{CP}^\infty$$

which induces

$$H^*_{S^1}(pt) = H^*(\mathbb{CP}^\infty) = \mathbb{C}[x]$$

where the degree of x is 2. For a torus T in general, we can naturally identify the equivariant cohomology of a point as the symmetric algebra $\mathbb{S}(\mathfrak{t}^*)$ where $\mathfrak{t}$ is a lie algebra of T:

$$H^*_T(pt) = \mathbb{S}(\mathfrak{t}^*).$$

Notice that from the fibration

$$X \times E_G/G \to B_G$$

with fibers X, $H^*_G(X)$ is an algebra over $H^*_G(pt)$. We say X is *equivariantly formal* if the spectral sequence of this fibration collapses at the E_2 level. Then by the spectral sequence, we have a natural isomorphism as module over $H^*_G(pt)$:

$$H^*_G(X) \cong H^*(X) \otimes_{\mathbb{C}} H^*(B_G) = H^*(X) \otimes_{\mathbb{C}} H^*_G(pt).$$

Therefore, when X is equivariantly formal, the equivariant cohomology is a free module over $H^*_G(pt)$. Especially when $G = T$ is a torus, we can compute the cohomology ring of X from the equivariant cohomology ring by the following formula:

$$H^*(X) = H^*_T(X) \otimes_{\mathbb{C}} \mathbb{S}(\mathfrak{t}^*)/(\mathfrak{t}^*)$$

where $(\mathfrak{t}^*)$ is the maximal ideal in the symmetric algebra. Additionally, the classical result by Borel yields the injectivity of the restriction map

$$H^*_T(X) \hookrightarrow H^*_T(X^T)$$

where X^T is the set of fixed points.

Given equivariant formality of algebraic variety X, localization theorem by Goresky, Kottwitz, and MacPherson [3] translates the problem of computing the equivariant cohomology of X. When X has finitely many fixed points and 1-dimensional orbits, localization theorem implies that the closure of one dimensional orbits is a sphere containing two distinct fixed points and the equivariant cohomology ring is embedded in $H^*_T(X^T)$ under the conditions given by 1-dimensional orbits. When $x_1, \cdots, x_k$ are fixed points and each 1-dimensional orbit E_j has two fixed points x_{j_0}, x_{j_∞} in its closure, then the following proposition states the fact formally.

Proposition 1.1. (Goresky, Kottwitz, and MacPherson [3]) *Suppose the algebraic variety X is equivariantly formal. Then the restriction mapping $H_T^*(X) \hookrightarrow H_T^*(X^T)$ is injective and its image is the subalgebra*

$$H = \left\{ (f_1, \cdots, f_k) \in \bigoplus_1^k \mathbb{S}(\mathfrak{t}^*) \middle| f_{j0}|_{\mathfrak{t}_j} = f_{j\infty}|_{\mathfrak{t}_j} \forall j \right\}$$

where $\mathfrak{t}_j \subset \mathfrak{t}$ is the lie algebra of the stabilizer of any point in E_j.

Even when the fixed points and 1-dimensional orbits are locally finite, we can naturally extend this description of equivariant cohomology.

When $X = G/T$ is a classical flag variety, then the fixed points corresponds to the Weyl group W and the 1-dimensional orbits correspond to the reflections s_α connecting w and $s_\alpha w$. Then Proposition 1.1 implies that

$$H_T^*(X) = \{f \in Mor(W, \mathbb{S}(\mathfrak{t}^*)) | f(w) \equiv f(s_\alpha w) \mod \alpha, \forall w \in W, \alpha \in \Phi\}$$

where Φ is the root system.

Define a GKM graph $\Gamma = (V, E)$ where the set of vertices $V = X^T$ and the set of edges $E = \{E_j\}$, or equivalently, the set of one dimensional orbits. When we define a ring

$$H_T^*(\Gamma)$$

as the algebra H given in Proposition 1.1, we have $H_T^*(\Gamma) = H_T^*(X)$.

Classical flag variety Fl endowed with the action of maximal torus T has fixed points corresponding to the Weyl group W and 1-dimensional orbits corresponding to the reflections $(w, s_\alpha w)$ of the Weyl group. Its affine extension to affine flag variety $\tilde{Fl}$ with the action of extended torus $\tilde{T}$ has fixed points corresponding to the affine Weyl group $\tilde{W}$ and one dimensional orbits corresponding to the multiplication by reflections $(w, s_\alpha w)$ of the affine Weyl group. Therefore we can construct the GKM graph of both classical flag variety and affine flag variety.

Throughout the paper, we reformulated the classical results in equivariant cohomology under GKM description and derived new results on the equivariant cohomology of affine flag varieties and affine springer fibers.

In Section 2, we extend the notion of GKM graph of flag variety to the graph of arbitrary Coxeter group. Through the interpretation, we explicitly determine the free module structure of equivarient cohomology over $\mathbb{S}(\mathfrak{t}^*)$. At the end of the section we prove the result is compatible with the classical results on equivariant cohomology.

In Section 3, we analyze the ring structure of the equivariant cohomology rings of affine flag variety $\tilde{Fl}$ and affine Grassmannian Gr under the action of extended torus $\tilde{T}$. In particular, we prove the existence of the generator $g_1, \cdots, g_n$ such that

$$H_{\tilde{T}}^*(Gr) \cong \mathbb{S}(\mathfrak{t}^*) \otimes_{\mathbb{C}} \mathbb{C}[g_1, \cdots, g_n].$$

This result implies that the equivariant cohomology ring is a polynomial ring. Along with the explicit formulation of the equivariant cohomology ring of affine Grassmannian, we prove

$$H_{\tilde{T}}^*(\tilde{Fl}) \cong H_T^*(Fl) \otimes_{H_T^*(pt)} H_{\tilde{T}}^*(Gr).$$

In Section 4, we briefly discuss the equivariant cohomology of affine Springer fibers. We have not yet found the general explicit formula for the equivariant cohomology, but we provide a formula for affine Springer fiber of $sl(2)$. Additionally we suggest a conjecture which connects the equivariant cohomology of affine flag variety and affine Springer fibers.

2 GKM graph over Coxeter system

Let (W, S) be a Coxeter system following the notations of [6] Then we have an $|S|$ dimensional vector space V over $\mathbb{R}$ having a basis $\{\alpha_s \mid s \in S\}$ with symmetric biliniear form

$$< \alpha_s, \alpha_{s'} >= -\cos\frac{\pi}{m(s,s')}$$

where $m(s, s')$ is a minimal integer such that $(ss')^{m(s,s')} = 1$. Then W is generated as a subgroup of $GL(V)$ by reflections $\{s_\alpha | \alpha = \alpha_s, s \in S\}$ with respect to the bilinear form $<,>$. The root system Φ of W is naturally defined as the union of the translation of the basis $\{\alpha_s \ | s \in S\}$ by W-action. Consequenly, we define Π to be the associated positive system. Then every reflections of W corresponds to s_α, a reflection with respect to bilinear form $<,>$ and root $\alpha \in \Pi$.

Consequently, we can naturally construct the GKM graph Γ of the Coxeter system where the vertices are elements of Coxeter group W and edges are reflections $\{(w, s_\alpha w) \mid w \in W, \alpha \in \Phi\}$. Therefore we define GKM ring of the graph as follows:

$$H^*(\Gamma) = \{f \in Mor(W, \mathbb{S}(V)) | f(w) \equiv f(s_\alpha w) \mod \alpha, \forall w \in W, \alpha \in \Phi\}.$$

Here, $Mor(W, \mathbb{S}(V))$ is a set of maps $f : W \to \mathbb{S}(V)$ such that $\sup_{w\in W} \deg F(w) < \infty$ since f should be of finite degree in actual equivariant cohomology ring. Notice that the vertices and edges of GKM graph Γ is undirected Bruhat graph of the Coxeter system.

Apparently, for arbitrary polynomial $P \in \mathbb{S}(V)$, constant element $f \in \bigoplus_W \mathbb{S}(V)$ meaning $f(w) = P$ for all $w \in W$ is an element in $H^*(\Gamma)$. Hence, $\mathbb{S}(V) \hookrightarrow H^*(\Gamma)$ naturally as constant elements. This is analogous to the fibration

$$X \times E_G/G \to B_G$$

which gives natural algebra structure of $H^*_G(X)$ over $\mathbb{S}(\mathfrak{t}^*)$. In this section, we prove $H^*(\Gamma)$ is a free module over $\mathbb{S}(V)$ and specify the generators. Before we proceed, we recall some important lemmas about Coxeter group denoted as the Strong Exchange Condition and Deletion Condition in [6].

Lemma 2.1. *Let $w \in W, \alpha \in \Pi$. Then the following holds.*

$$l(ws_\alpha) > l(w) \textit{ if and only if } w\alpha > 0.$$

$$l(s_\alpha w) > l(w) \textit{ if and only if } w^{-1}\alpha > 0.$$

It is implied in the above lemma that the number of reflections s_α such that $s_\alpha w < w$ in the Bruhat ordering is equal to $l(w)$ since the number of positive roots sent to the negative roots is equal to the length.

Lemma 2.2. *For an element $w \in W$ with reduced expression $s_1 \cdots s_n$ i.e. $l(w) = n$, then the set of elements in the Bruhat interval $[1, w]$ is*

$$\{s_{i_1} \cdots s_{i_k} | 1 \leq i_1 < \cdots < i_k \leq n\}$$

or equivalently, subexpression of the reduced expression $s_1 \cdots s_n$.

Especially, whenever $s_\alpha w < w$, then $s_\alpha = s_1 \cdots \hat{s}_i \cdots s_n$ for some $1 \leq i \leq n$.

To construct the element of $H^*(\Gamma)$ on the Bruhat graph, it is natural to proceed as in [8]. To elaborate, we order the vertices of Bruhat graph according to the length, so that all the elements of length n are aligned on level n. Then we construct the element $f \in Mor(W, \mathbb{S}(\mathfrak{t}^*))$ gradually by the increasing length. For instance, suppose the polynomials $f(v)$ are assigned for $l(v) < n$. Then for an element w in level n, we assign a polynomial $f(w)$ which satisfy the following n restraints:

$$f(w) \equiv f(s_\alpha w) \mod \alpha$$

for all $l(s_\alpha w) < l(w)$. The following theorem specifies the generator of GKM ring as a module over $\mathbb{S}(V)$.

Theorem 2.3. *For an element $w \in W$, there exists a unique class $f_w \in H^*(\Gamma)$ of degree $l(w)$ satisfying the following condition.*

- $f_w(w') = 0$, *for all* $l(w') \leq l(w)$, $w' \neq w$
- $f_w(w) = \prod_{\alpha \in \Pi, l(s_\alpha w) < l(w)} \alpha$

Additionally, f_w is a free module generator of $H^*(\Gamma)$ over $\mathbb{S}(V)$.

Note that there are $l(w)$ distinct $\alpha \in \Pi$ such that $l(s_\alpha w) < l(w)$, so $f(w)$ is indeed of degree $l(w)$.

Before proving Proposition 2.3, we suggest a lemma and operations essential for the proof.

Lemma 2.4. *For $\alpha \in \Pi, w \in W$ and $p, q \in \mathbb{S}(V)$, $p \equiv q \mod \alpha$ if and only if $wp \equiv wq \mod w\alpha$.*

We omit the proof of the lemma since the proof is trivial.

Using Lemma 2.4 we can define two distinct action of W on $f \in H^*(\Gamma)$. Given $w \in W$, the first action $w \cdot f(v) = f(vw)$ for all $v \in W$. An alternative action can be defined as

$$wf(v) = w(f(w^{-1}v)).$$

The lemma implies that the second action is well defined in $H^*(\Gamma)$. We distinguish two actions by $w \cdot f$ and wf. Therefore we can define the following operations analogous to the differentiation in Schubert polynomials on $H^*(\Gamma)$.

Definition 2.5. For a positive root $\alpha \in \Pi$, we define the following differentiations on $f \in H^*(\Gamma)$.

$$\partial_\alpha f(w) = \frac{f(w) - f(ws_\alpha)}{-w\alpha}, \bar{\partial}_\alpha f(w) = \frac{f(w) - s_\alpha f(s_\alpha w)}{\alpha}.$$

One can easily check some properties of the derivatives:

$$\partial_\alpha f(ws_\alpha) = \partial_\alpha f(w), \bar{\partial}_\alpha f(s_\alpha w) = s_\alpha \bar{\partial}_\alpha f(w).$$

This implies that $\partial_\alpha^2 = 0$ and $\bar{\partial}_\alpha^2 = 0$. Using these two properties, it is an easy consequence of Lemma 2.4 that the above operation is well defined.

We say a subgraph of a GKM graph satisfies GKM condition if the polynomials satisfies the restraints given by edges. The restrictions for assigning the polynomial $f(w)$ at the level $n = l(w)$ is contained in the subgraph $[1, w]$. By the following theorem, we can assign a polynomial $f(w)$ if all the other polynomials in the Bruhat interval $[1, w]$ satisfy GKM condition.

Theorem 2.6. *If $f : [1, w] \setminus \{w\} \to \mathbb{S}(V)$ satisfies the GKM condition, then we can extend the function $f : [1, w] \to \mathbb{S}(V)$ so that f satisfies the GKM condition in the subgraph $[1, w]$.*

Proof. We proceed by induction on $l(w) \geq 0$. If $l(w) = 0$, then $w = 1$ which is trivial. Suppose the theorem holds for $l(w) = n - 1$. When $l(w) = n$, let the reduced expressions for $w = s_1 \cdots s_n$ where $s_1 = s_\alpha$ and define $w_1 = s_1 w < w$.

Using Lemma 2.4, $s_1 f$ restricted to $[1, w_1] \setminus \{w_1\}$ satisfy the GKM condition. Therefore,

Consider the subset $s_1[1, w_1] \subset [1, w]$ where $s_1[1, w_1]$ is a set

$$\{s_{i_1} \cdots s_{i_k} | 1 = i_1 < \cdots < i_k \leq n\}$$

by Strong Exchange and Deletion Condition in [6].

To apply the induction hypothesis, we define a function $g : [1, w_1] \setminus \{w_1\} \to \mathbb{S}(V)$ by $g(v) = s_1 f(s_1 v)$. In other words, this is the restriction of s_1 action on the Bruhat interval $[1, w_1]$. By induction hypothesis, g extends to w_1. Therefore, when we let $F = s_1 g(w_1)$, then F satisfy all the restraints on w but one given by $(w, s_1 w)$ in the following sense:

$$F \equiv f(s_\beta w) \mod \beta, \ \ \forall \beta \in \Pi \setminus \{\alpha\} \text{ such that } l(s_\beta w) < l(w).$$

However $F \equiv f(w_1) \mod \alpha$ is not necessarily true, so we need to modify F to assign a polynomial $f(w)$ satisfying GKM condition.

We let $f(w) = F$ temporarily and consider the restriction of derivation $\bar{\partial}_\alpha f$ on the subgraph $[1, w_1] \setminus \{w_1\}$. Again, by induction hypothesis, $\bar{\partial}_\alpha f$ extends to w_1. Let the extension be $\bar{\partial}_\alpha f(w_1) = G$. Then by the restraints on w_1, we have

$$f(w_1) - s_1 F \equiv \alpha G \mod \gamma, \ \forall \gamma \in \Pi \setminus \{\alpha\} \text{ such that } l(s_\gamma w_1) < l(w_1).$$

By Strong Exchange ans Deletion Condition, we have

$$l(s_\gamma w_1) < l(w_1) \text{ if and only if } l(s_{s_1 \gamma} w) < l(w)$$

when $\beta \neq \alpha$. Hence, by Lemma 2.4

$$s_1 f(w_1) - F \equiv -\alpha s_1 G \mod \beta, \ \forall \beta \in \Pi \setminus \{\alpha\} \text{ such that } l(s_\beta w) < l(w).$$

Since $s_1 f(w_1) \equiv f(w_1) \mod \alpha$, there exists a polynomial H such that

$$f(w_1) - F \equiv \alpha H \mod \prod_{\beta \in \Pi \setminus \{\alpha\}, l(s_\beta w) < l(w)} \beta.$$

Therefore we can modify $f(w)$ from F by adding an element multiple of $\prod_{\beta\in\Pi\setminus\{\alpha\}, l(s_\beta w)<l(w)} \beta$ so that $f(w) \equiv f(w_1) \mod \alpha$. Therefore, $f(w) \equiv f(s_\alpha w) \mod \alpha$ for all $l(s_\alpha w) < l(w)$. Hence we can extend the function f to w. □

Now we proceed to the proof of our main theorem.

Proof of Theorem 2.3. When f are given by conditions in the theorem, we can extend f inductively on the length of the elements using Theorem 2.6. This proves the existence of f_w.

For element w' with $l(w') > l(w)$, $f(w')$ should satisfy $l(w')$ modular restraints which is greater than $l(w)$. Therefore $f(w')$ is uniquely determined in each inductive step. Therefore the uniqueness holds.

Suppose $f \in H^*(\Gamma)$. If $f(w') = 0$ for all $l(w') < n$, then given $l(w) = n$, $f(w)$ is multiple of $f_w(w)$ by the GKM condition on w. We can subtract the multiple of f_w by a polynomial in $\mathbb{S}(V)$ from f to induce $f(w) = 0$. Therefore, for any $g \in H^*(\Gamma)$ of degree n, we can inductively subtract the multiple of f_w so that the polynomial assigned to w is 0 whenever $l(w) \leq n$. Concurrently, there exists a linear combination G of f_w over $\mathbb{S}(V)$ such that $G(w) = g(w)$ for all $l(w) \leq n$. Then G is also of degree n and $g = G$ holds by the GKM condition.

From the fact that f_w are linearly independent over $\mathbb{S}(V)$ by the conditions given by Theorem 2.3, $H^*(\Gamma)$ indeed is a free module over $\mathbb{S}(V)$ having f_w as a generator. □

The GKM graph over Coxeter system generalizes the GKM graph of affine flag varieties and classical flag varieties. Indeed V is identified with dual lie algebra $\mathfrak{t}^*$. Therefore, for an (affine) flag variety Fl with (extended) torus action T, we have

$$H_T^*(Fl) \cong H^*(\Gamma)$$

where Γ is a corresponding GKM graph of a Coxeter system. Therefore this approves the following classical result by Poincare which is readily proven by the CW-structure of the flag induced by Schubert cell decomposition.

$$G/B = \amalg_{w\in W} BwB/B.$$

Corollary 2.7. *The Betti number of an (affine) flag variety corresponds to the coefficient of Poincare series of the (affine) Weyl group.*

Proof. From the identity

$$H^*(Fl) = H_T^*(Fl) \otimes_{\mathbb{C}} \mathbb{S}(\mathfrak{t}^*)/(\mathfrak{t}^*),$$

the generator of $H_T^*(Fl)$ as a free module over $\mathbb{S}(\mathfrak{t}^*)$ turns out to be the generator of $H^*(Fl)$ over the coefficient $\mathbb{C}$. Since the number of degree n generators of $H_T^*(Fl)$ is equal to the number of elements in (affine) Weyl group of length n, the n-th Betti number is equal to the n-th coefficient of the Poincare polynomial of the (affine) Weyl group. □

3 Equvariant Cohomology Ring of Affine Flag Variety and Affine Grassmannian

In the previous section, we have investigated the module structure of the equivariant cohomology ring over $\mathbb{S}(\mathfrak{t}^*)$. The ring structure can be determined using the relations of Schubert cells as in [7]. However, the following classical proposition suggests the other method for deriving the equivariant cohomology ring.

Proposition 3.1. *(Borel Description [5]) For a compact semi-simple Lie group G and Cartan subgroup T, we have*

$$H_T^*(G/T) = \mathbb{S}(\mathfrak{t}^*) \otimes_{\mathbb{S}(\mathfrak{t}^*)^W} \mathbb{S}(\mathfrak{t}^*)$$

where W is a Weyl group and $\mathfrak{t}$ is a lie algebra of T.

Notice that the GKM graph Γ of T action on G/T are those described in Section 2 corresponding to the Weyl group W. By GKM, we have $H_T^*(G/T) \cong H^*(\Gamma)$. The explicit mapping $\mathcal{K} : \mathbb{S}(\mathfrak{t}^*) \otimes_{\mathbb{S}(\mathfrak{t}^*)^W} \mathbb{S}(\mathfrak{t}^*) \to H^*(\Gamma)$ is given in [5] by

$$\mathcal{K}(p \otimes q)(w) = p(wq).$$

In this section we prove Borel description of equivariant cohomology in GKM description and find equivariant cohomology ring of affine flag variety using the techniques in the proof.

Lemma 3.2. *Ring homomorphism $\mathcal{L} : \mathbb{S}(\mathfrak{t}^*) \otimes_{\mathbb{C}} \mathbb{S}(\mathfrak{t}^*) \to H^*(\Gamma)$ defined by*

$$\mathcal{L}(p \otimes q)(w) = p(wq)$$

is surjective.

Proof. Let $f \in H^*(\Gamma)$ where f is a homogeneous class, meaning the degree of $f(w)$ is constant. Since $H^*(\Gamma)$ is a graded ring, we can proceed by induction on n, the degree of f.

When $n = 0$, then f is trivially in the image of $\mathcal{L}$. Suppose every element of $H^*(\Gamma)$ with degree less than n is contained in the image. We use the strategy of summing over the W action on f to induce constant function f.

For $\alpha \in \Pi$, $\deg \partial_\alpha f = n - 1$. By induction hypothesis, $\partial_\alpha f$ is contained in the image of $\mathcal{L}$ and hence, $f - s_\alpha \cdot f$ is contained in the image. (Recall that we had a W action on $H^*(\Gamma)$ in Section 2.) Consequently, for any $w \in W$, we have $w \cdot f - ws_\alpha \cdot f$ is contained in the image. Since reflections generate the Weyl group, we have that $f - w \cdot f$ is contained in the image for all $w \in W$.

Since $\sum_{w \in W} w \cdot f$ is constant over the action of W, $\sum_{w \in W} w \cdot f$ is a constant function and hence an element in $\mathbb{S}(\mathfrak{t}^*)$. f can be expressed as a linear combination of $\sum_{w \in W} w \cdot f$ and $f - w \cdot f$ for all $w \in W$ which yields that f is also in the image of $\mathcal{L}$. □

Before proving the proposition, we define additional derivative on $\mathbb{S}(\mathfrak{t}^*)$.

Definition 3.3. For a root $\alpha \in \Phi$, we define the following differentiation D_α on $p \in \mathbb{S}(\mathfrak{t}^*)$.

$$D_\alpha p = \frac{p - s_\alpha p}{-\alpha}.$$

Proof of Proposition 3.1. Since we have a surjection $\mathcal{L}$, it is sufficient for us to prove that the kernel of $\mathcal{L}$ is an ideal I generated by $h \otimes 1 - 1 \otimes h$ for all $h \in \mathbb{S}(\mathfrak{t}^*)^W$.

Suppose $\sum p_i \otimes q_i$ is contained in the kernel of $\mathcal{L}$. Then we again proceed by the induction on n, the degree of $\sum p_i \otimes q_i$. Since initial condition is trivial, suppose the kernel contained in I for degree less than n. $\mathcal{L}(\sum p_i \otimes q_i) = 0$ is equivalent to $\sum p_i(wq_i) = 0$ for all $w \in W$. Therefore, for positive root α, $\mathcal{L}(\sum p_i \otimes D_\alpha q_i) = 0$ and induction hypothesis implies that $\sum p_i \otimes (q_i - s_\alpha q_i)$ is contained in I. As in the proof of Lemma 3.2, from $\sum p_i \otimes (\sum_{w \in W} wq_i) = 0$ we have that $\sum p_i \otimes q_i$ is also contained in I. Therefore, the kernel of $\mathcal{L}$ is I which proves the Borel description. □

Before we proceed to compute the equivariant cohomology of affine flag variety $\tilde{Fl}$, we introduce some basic properties of affine Weyl group $\tilde{W}$. Let $\tilde{T}$ be an extended torus and $\mathfrak{t}^*$ be its dual lie algebra. The root system $\tilde{\Phi}$ of $\tilde{W}$ is $\mathbb{Z}t \oplus \Phi$ where Φ is the root system of the Weyl group W and $t \in \mathfrak{t}^*$ is the vector in the kernel of Coxeter bilinear form. Hence t is invariant under $\tilde{W}$ action.

When we denote $V \subset \mathfrak{t}^*$ as a subspace spanned by Φ, $\tilde{W}$ can be considered as a subgroup of affine group Aff(V). In particular, the root $\alpha + kt$ such that $\alpha \in \Phi$ and $k \in \mathbb{Z}$ satisfies $s_{\alpha+kt} = s_{\alpha,k}$ where $s_{\alpha,k}$ is a reflection with respect to affine hyperplane $H_{\alpha,k} = \{\lambda \in V \mid <\lambda, \alpha >= k\}$ where $<,>$ is a symmetric positive definite bilinear form on V.

Accordingly, we can consider the GKM graph Γ where the vertices are the elements of the affine Weyl group and edges correspond to the affine reflections $(w, s_{\alpha,k}w)$. Then the GKM conditions correspond to

$$f(w) \equiv f(s_{\alpha,k}w) \mod \alpha + kt.$$

To compute $H^*(\Gamma)$, we would like to apply the strategy of summing over the the $\tilde{W}$ action for $f \in H^*(\Gamma)$: $\sum_{w\in\tilde{W}} w \cdot f$. However, since $\tilde{W}$ is infinite group, the sum is not well defined. Instead we take sum over the Weyl group $W \subset \tilde{W}$: $\sum_{w\in\tilde{W}} w \cdot f$. Then the sum is invariant under W action which implies that $\sum_{w\in\tilde{W}} w \cdot f \in Mor(\Lambda, \mathbb{S}(\mathfrak{t}^*))$ where Λ is the lattice of $\tilde{W}$. (Recall $\tilde{W} = \Lambda \rtimes W$.) Therefore we first consider the GKM graph of affine grassmannian Gr whose fixed points corresponds to the lattice Λ.

3.1 Equivariant Cohomology of Affine Grassmannian Gr

Given the extended torus action $\tilde{T}$ on Gr, the set of fixed points of Gr is the lattice, Λ and the edges are affine reflections on Λ. In particular, for $g \in Mor(\Lambda, \mathbb{S}(\mathfrak{t}^*))$ the GKM condition is given by

$$g(\lambda) \equiv g(s_{\alpha,k}\lambda) \mod \alpha + kt$$

for $\lambda \in \Lambda$. Therefore we can compute the equivariant cohomology of Gr through GKM graph Γ_{Gr}: $H^*_{\tilde{T}}(Gr) = H^*(\Gamma_{Gr})$.

Let $\{d_1, \cdots, d_n\}$ be the degrees of the generators of polynomial ring, $\mathbb{S}(V)^W$, where V is a subspace of $\mathfrak{t}^*$ spanned by root system Φ of W. Ginzberg [2] porved that the cohomology ring of affine grassmannian, $H^*(Gr)$, is the polynomial ring generated by the elements of degree $d_i - 1$. From the identity over $\mathbb{S}(\mathfrak{t}^*)$ modules

$$H^*_{\tilde{T}}(Gr) \cong H^*(Gr) \otimes_{\mathbb{C}} \mathbb{S}(\mathfrak{t}^*),$$

we can expect $H^*_{\tilde{T}}(Gr)$ to be a polynomial ring. The following theorem explicitly characterizes the equivariant cohomology $H^*_{\tilde{T}}(Gr)$ as a polynomial ring and determine the generators of $H^*_{\tilde{T}}(Gr)$ in GKM description.

Theorem 3.4. *For $G_i \in \mathbb{S}(V)^W$, $g_i \in Mor(\Lambda, \mathbb{S}(\mathfrak{t}^*))$ defined by*

$$g_i(\lambda) = \frac{\lambda G_i - G_i}{t}$$

satisfies GKM condition and is contained in $H^*(\Gamma_{Gr})$. Additionally, for any choice of algebraically independent generators $G_1, \cdots G_n$ of $\mathbb{S}(V)^W$, $g_1, \cdots, g_n$ are algebraically independent and we have the following ring isomorphism

$$H^*_{\tilde{T}}(Gr) \cong \mathbb{S}(\mathfrak{t}^*) \otimes_{\mathbb{C}} \mathbb{C}[g_1, \cdots, g_n].$$

Proof. We first prove $g_i \in H^*(\Gamma_{Gr})$. Let $\bar{G}_i \in Mor(\Lambda, \mathbb{S}(\mathfrak{t}^*))$ such that $\bar{G}_i(\lambda) = \lambda G_i$. Then we have

$$\bar{G}_i(s_{\alpha,k}\lambda) = s_{\alpha,k}\lambda s_{\alpha,k} G_i = s_{\alpha,k}\bar{G}_i(\lambda)$$

from the fact that G_i is fixed by W action. Therefore $\bar{G}_i$ satisfies the GKM condition.

For a positive root α and its dual $\hat{\alpha} \in \Lambda$, we have $\hat{\alpha} = s_{\alpha,1}s_\alpha$ in $\tilde{W}$ by the property of affine reflection. Therefore, direct computation yields that the Λ action on $\mathbb{S}(\mathfrak{t}^*)$ is invariant modulo t. This induces the fact that $g_i \in H^*(\Gamma_{gr})$.

Now to prove the rest of the theorem, we suggest the following lemma.

Lemma 3.5. *Suppose $q \in \mathbb{S}(V) \subset \mathbb{S}(\mathfrak{t}^*)$ is invariant under the action of $\tilde{W}$. Then $q \in \mathbb{C}$.*

Proof. Since the alcove of the affine Weyl group in V is a compact simplex, q attains maximum and minimum. By the invariance and the transitivity of the alcove over V, q remains bounded in V. Therefore, q is a constant function. □

Using the lemma above we can prove the following lemma.

Lemma 3.6. *Let $x_1, \cdots, x_n$ be the basis of V, span of Φ. Then $\mathbb{S}(\mathfrak{t}^*) = \mathbb{C}[x_1, \cdots, x_n, t]$ and $x_1, \cdots x_n, t, \bar{G}_1, \cdots, \bar{G}_n$ are algebraically independent in $H^*(\Gamma_{Gr})$.*

Proof. Suppose $\sum_{i_0,i_1,\cdots,i_n \geq 0} q_{i_0,i_1,\cdots,i_n} t^{i_0}\bar{G}_1^{i_1}\cdots\bar{G}_n^{i_n} = 0$ where $q_{i_0,i_1,\cdots,i_n} \in \mathbb{S}(V)$. It is sufficient to show that $q_{i_0,i_1,\cdots,i_n} = 0$ for all $i_0, i_1, \cdots, i_n \geq 0$. We proceed by induction on $\deg q = \max_{i_0,i_1,\cdots,i_n \geq 0} \deg q_{i_0,i_1,\cdots,i_n}$. Initial condition when $\deg q = 0$, is trivial since by definition, $t, \bar{G}_1, \cdots, \bar{G}_n$ are algebraically independent over $\mathbb{C}$.

Notice that $\sum q_{i_0,i_1,\cdots,i_n} t^{i_0}\bar{G}_1^{i_1}\cdots\bar{G}_n^{i_n} = 0$ if and only if $\sum q t^{i_0}\lambda(G_1^{i_1}\cdots G_n^{i_n}) = 0$ for all $\lambda \in \Lambda$.(we omit the subscripts for brevity.) Since G_i is invariant under W and $\tilde{W} = \Lambda \rtimes W$, we have $\sum q t^{i_0} w(G_1^{i_1}\cdots G_n^{i_n}) = 0$ for all $w \in \tilde{W}$. Therefore, $\sum q t^{i_0}\bar{G}_1^{i_1}\cdots\bar{G}_n^{i_n} = 0$ if and only if $\sum (wq) t^{i_0} G_1^{i_1}\cdots G_n^{i_n} = 0$ for all $w \in \tilde{W}$. Then $\sum (D_{\alpha+kt}q) t^{i_0} G_1^{i_1}\cdots G_n^{i_n} = 0$ exploits the fact that $D_{\alpha+kt}q = 0$ for all roots $\alpha + kt$ by induction hypothesis. Therefore, q is invariant under the the action of $\tilde{W}$, yielding $q_{i_0,\cdots,i_n} \in \mathbb{C}$ and $\deg q = 0$ by Lemma 3.5. This is the initial case of induction. □

Lemma 3.6 implies the algebraic independence of $x_1, \cdots, x_n, t, g_1, \cdots, g_n$. Therefore, there exists an injection

$$\mathbb{S}(\mathfrak{t}^*) \otimes_{\mathbb{C}} \mathbb{C}[g_1, \cdots, g_n] \hookrightarrow H^*_{\tilde{T}}(Gr).$$

However, the module structure

$$H^*_{\tilde{T}}(Gr) \cong H^*(Gr) \otimes_{\mathbb{C}} \mathbb{S}(\mathfrak{t}^*),$$

implies that the injection above is indeed isomorphism.(For every degree, the dimension as a vector space over $\mathbb{C}$ is identical.) Therefore, the injection is the isomorphism. □

Having the equivariant cohomology ring of affine Grassmannian Gr, we obtain the equivariant cohomology ring of affine flag variety $\tilde{Fl}$ in the following subsection.

3.2 Equivariant Cohomology of Affine Flag Variety $\tilde{Fl}$

Consider the GKM Graph Γ of $\tilde{Fl}$ as discussed in the beginning of the section. Since $\mathbb{S}(V)^W$ is embedded in $H^*(\Gamma_{Gr})$ by translation of lattice element as in the proof of Theorem 3.4, we can think of the natural map $\mathcal{K}$ in the following theorem.

Theorem 3.7. *The ring homomorphism*

$$\mathcal{K} : \mathbb{S}(V) \otimes_{\mathbb{S}(V)^W} H^*(\Gamma_{Gr}) \to H^*(\Gamma)$$

defined by $\mathcal{K}(p \otimes g)(\lambda w) = ((\lambda w)p)g(\lambda)$ *for* $\lambda \in \Lambda, w \in W$ *is an isomorphism.*

By $\tilde{W} = \Lambda \rtimes W$, any element of $\tilde{W}$ is uniquely written as the product λw.

Proof. As in the proof of Proposition 3.1, we proceed in two steps.

First, we prove the surjectivity of the natural map

$$\mathcal{L} : \mathbb{S}(V) \otimes_{\mathbb{C}} H^*(\Gamma_{Gr}) \to H^*(\Gamma)$$

which factors through $\mathcal{K}$. Again, we proceed by induction on the degree of $f \in H^*(\Gamma)$. The initial condition is trivial. For a root of Weyl group $\alpha \in \Phi$, we consider the derivative $\partial_\alpha f$. Then by induction hypothesis, $\partial_\alpha f$ is contained in the image. Additionally, $\sum_{w \in W} w \cdot f$ is in the image of $H^*(\Gamma_{Gr})$ since $\sum_{w \in W} w \cdot f(\lambda v)$ only depend on the lattice factor λ. Therefore f is in the image by the same reason in the proof of Lemma 3.2.

It is sufficient to prove that the kernel of $\mathcal{L}$ is generated by $h \otimes 1 - 1 \otimes h$ for all $h \in \mathbb{S}(V)^W$. Indeed if $\mathcal{L}(\sum p_i \otimes g_i) = 0$, then $\mathcal{L}(\sum D_\alpha p_i \otimes g_i) = 0$ for all $\alpha \in \Phi$. Therefore we only need to prove that $\sum(\sum_{w \in W} wp_i) \otimes g_i$ is generated by $h \otimes 1 - 1 \otimes h$. Since $\sum_{w \in W} wp_i \in \mathbb{S}(V)^W$, the claim holds. Hence the theorem is proven □

Theorem 3.7 yields the following corollaries.

Corollary 3.8.

$$H^*_{\tilde{T}}(\tilde{Fl}) \cong \mathbb{S}(V) \otimes_{\mathbb{S}(V)^W} H^*_{\tilde{T}}(Gr)$$

or equivalently,

$$H^*_{\tilde{T}}(\tilde{Fl}) \cong H^*_T(Fl) \otimes_{H^*_T(pt)} H^*_{\tilde{T}}(Gr).$$

Proof. The proof of the first identity is immediate from GKM description and Theorem 3.7. The second identity is also immediate from the Borel Description. □

In fact, the fibration

$$\tilde{Fl} \to Gr$$

with fiber Fl directly induces the identity

$$H^*(\tilde{Fl}) \cong H^*(Fl) \otimes_{\mathbb{C}} H^*(Gr)$$

from associated spectral sequence. This classical result is compatible with our corollary.

4 Equivariant Cohomology of Affine Springer Fibers $\tilde{Fl}_1$ of $sl(2)$

Having investigated equivariant cohomology of affine flag variety in the previous section, we now investigate the equivariant cohomology of affine Springer fibers $\tilde{Fl}_n$. The fixed points and one dimensional orbits of affine Springer fibers are analyzed in [4] which allows one to construct GKM graph. In this section, we analyze the GKM graph of $\tilde{Fl}_1$ of $sl(2)$ and compute equivariant cohomology of $\tilde{Fl}_1$.

Since $\tilde{Fl}_1$ of $sl(2)$ is union of projective lines, fixed points and one dimensional orbits of torus T and those of extended torus $\tilde{T}$ are identical. Therefore, we can compute the two distinct equivariant cohomology $H^*_T(\tilde{Fl}_1)$ and $H^*_{\tilde{T}}(\tilde{Fl}_1)$ using GKM description. In both cases, the fixed points are the affine Weyl group of type A_1, $\mathbb{Z} \rtimes \{+,-\}$ and the one dimensional orbits are projective lines connecting $(n,+),(n+1,-)$ or $(n,+),(n,-)$ for all $n \in \mathbb{Z}$. Therefore, under GKM description we can easily derive the following theorem.

Theorem 4.1. *Suppose $\tilde{Fl}_1$ is an affine Springer fiber of $sl(2)$. Then, we have*

$$H^*_T(\tilde{Fl}_1) \cong \mathbb{C}[[x_w]]_{w \in \mathbb{Z} \rtimes \{+,-\}}/(x_w x_{w'})_{w \neq w'}.$$

We do not present the proof here which involves dubious computations.

Computing the equivariant cohomology of general affine Springer fiber is not quite explicit as in the previous sections. However, there seems to exist a property which connects the cohomology of affine flag variety and of affine Springer fiber. We leave this as conjecture originally proposed by Professor Bezrukavnikov.

Conjecture 4.2.

$$H^*(\tilde{Fl}) \to H^*(\tilde{Fl}_n)^{\Lambda}$$

is surjective.

Using Theorem 4.1 and the description of the cohomology of affine flag variety, it is not hard to prove the conjecture in the case of the affine Springer fiber of $sl(2)$. We would like to suggest further researchers to solve the conjecture or to use the GKM description to determine the equivariant cohomology of affine Springer fibers.

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References

[1] M. Brion, *Equivariant cohomology and equivariant intersection theory,* in: A. Broer and A. Daigneault (eds.), Representation theories and algebraic geometry, Kluwer, 1998.

[2] V. Ginzburg, *Perverse sheaves on a loop group and Langlands' duality,* arXiv:alg-geom/9511007

[3] M. Goresky, R. Kottwitz and R. MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem,* Invent. Math. 131 (1998), no. 1, 25-83.

[4] M. Goresky, R. Kottwitz and R. Macpherson, *Homology Of Affine Springer Fibers In The Unramified Case,* Duke Math. J. 121 (2004) 509–561.

[5] V. Guillemin, T. Holm and C. Zara, *A GKM description of the equivariant cohomology ring of a homogeneous space,* J. Algebraic Combin. 23 (2006), 21–41.

[6] J. Humphreys, *Reflection groups and Coxeter groups,* Cambridge Univ. Press, 1990.

[7] B. Konstant and S. Kumar, *The nil Hecke ring and cohomology of G/P for a Kac-Moody group G,* Adv. Math. 62 (1986), 187-237.

[8] J. Tymoczko, *An introduction to equivariant cohomology and homology, following Goresky, Kottwitz, and MacPherson,* Snowbird lectures in algebraic geometry, 169-188, Contemp. Math., 388, Amer. Math. Soc., Providence, RI, 2005.