NILPOTENT ORBITS: GEOMETRY AND COMBINATORICS

YUZHOU GU
MENTOR: KONSTANTIN TOLMACHOV
PROJECT SUGGESTED BY: ROMAN BEZRUKAVNIKOV

Abstract. We review the geometry of nilpotent orbits, and then restrict to classical groups and discuss the related combinatorics.

1. Introduction

Throughout this note we work over $\mathbb{C}$. Let G be a connected complex semisimple Lie group and $\mathfrak{g}$ be its corresponding Lie algebra. Let $\mathcal{N}$ be the set of nilpotent elements in $\mathfrak{g}$. Then there is a natural action of G on $\mathcal{N}$ by conjugation. The G-orbits in $\mathcal{N}$ are called **nilpotent orbits**. Nilpotent orbits are important in geometry and representation theory, and are the object of study in this project.

In this note we review the geometry and combinatorics related to nilpotent orbits.

2. Flag Varieties

The **flag variety** $\mathcal{B}$ is the variety of Borel subgroups in G (alternatively, the variety of Borel subalgebras in $\mathfrak{g}$). We have $\mathcal{B} \simeq G/B$ for any Borel subgroup B in G. Let $\mathcal{U}$ be the variety of unipotent elements in G. We have $\mathcal{U} \simeq \mathcal{N}$ by the Springer map. The **Springer resolution** $\tilde{\mathcal{N}}$ is the closed subvariety of $\mathcal{B} \times \mathcal{N}$ consisting of pairs $(\mathfrak{b}, n)$ where $\mathfrak{b}$ is a Borel subalgebra, n is a nilpotent element and $n \in \mathfrak{b}$. Alternatively, $\tilde{\mathcal{N}}$ is the closed subvariety of $\mathcal{B} \times \mathcal{U}$ consisting of pairs (B, u) where B is a Borel subgroup, u is a unipotent element and $u \in B$. There is a natural projection $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ ($\pi : \tilde{\mathcal{N}} \to \mathcal{U}$), which is a resolution of singularities of $\mathcal{N}$. For any $n \in \mathfrak{b}$ (resp. $u \in \mathcal{U}$), let $\mathcal{B}_n$ (resp. $\mathcal{B}_u$) denote the fiber over n (resp. u), i.e. $\pi^{-1}(n)$ (resp. $\pi^{-1}(u)$). These fibers are called the **Springer fiber**.

For nilpotent elements n and n' in the same nilpotent orbit, the Springer fibers $\mathcal{B}_n$ and $\mathcal{B}_{n'}$ are canonically isomorphic. Therefore we sometimes say the Springer fiber over an nilpotent orbit while the actual meaning is the Springer fiber over an nilpotent element in the orbit.

Let G acts on $\mathcal{B}\times\mathcal{B}$ by diagonal. The orbits are indexed by Weyl group elements. Namely, we have the Bruhat decomopsition

$$\mathcal{B} \times \mathcal{B} = \bigsqcup_{w \in W} G(B, wB)$$

where B is any Borel subgroup. This gives a map $\mathcal{B} \times \mathcal{B} \to W$ which maps a pair of Borel subgroups to the Weyl group element corresponding to the G-orbit. This map is called **relative position**. Let $\mathcal{O}_w$ be the orbit corresponding to $w \in W$.

Let $Z = \tilde{N} \times_N \tilde{N}$ be the **Steinberg variety**. It parametrizes triples $(\mathfrak{b}, \mathfrak{b}', n) \in \mathcal{B} \times \mathcal{B} \times \mathcal{N}$ where $n \in \mathfrak{b} \cap \mathfrak{b}'$. Let $p : Z \to \mathcal{B} \times \mathcal{B}$ be the map that forgets the last

Date: May 11, 2017.

summand. Then the irreducible components of Z is are closures of $Z_w = p^{-1}\mathcal{O}_w$ for $w \in W$. Also, for every $w \in W$ we have $\dim Z_w = 2n$, where $n = \dim \mathcal{B}$.

Let u be a unipotent element. Let A_u be the component group of $Z_G(u)$. For any two irreducible components of $\mathcal{B}_u$, two generic elements in them have fixed relative position. Therefore the relative position gives a map $\mathrm{Irr}(\mathcal{B}_u) \times \mathrm{Irr}(\mathcal{B}_u) \to W$. The component group A_u acts on $\mathrm{Irr}(\mathcal{B}_u)$ (therefore on $\mathrm{Irr}(\mathcal{B}_u) \times \mathrm{Irr}(\mathcal{B}_u)$) and fixes the relative position. Therefore we have a map $A_u \backslash (\mathrm{Irr}(\mathcal{B}_u) \times \mathrm{Irr}(\mathcal{B}_u)) \to W$. Using the Steinberg variety, Steinberg [Ste76] proved that if we let u goes over all unipotent orbits, then we get a bijection, i.e.

$$\bigsqcup_u A_u \backslash (\mathrm{Irr}(\mathcal{B}_u) \times \mathrm{Irr}(\mathcal{B}_u)) \simeq W.$$

3. Nilpotent Orbits

There is a natural partial order on the set of nilpotent orbits. For any two nilpotent orbits $\mathcal{O}$ and $\mathcal{O}'$, define $\mathcal{O} \leq \mathcal{O}'$ if $\mathcal{O} \subseteq \overline{\mathcal{O}'}$. Using this partial order we can identify certain orbits of interest.

(1) There is the **zero orbit** consisting of a single element 0. This is the smallest orbit in the partial ordering.
(2) There is a unique smallest orbit larger than the zero orbit, called **the minimal orbit**.
(3) There is a unique largest orbit, called the **regular orbit**.
(4) There is a unique largest orbit smaller than the regular orbit, called **the subregular orbit**.

The **Lusztig-Spaltenstein duality** (Spaltenstein [Spa82]) is an endomorphism of the set of nilpotent orbits, which have many good properties. Nilpotent orbits in the image of the duality are called **special orbits**. The duality is an involution when restricted to the set of special orbits. The duality reverses the orbit order.

The zero orbit, regular orbit and subregular orbit are special orbits. However, the minimal orbit is special only in simply laced cases. In all cases, there is a unique smallest special orbit larger than the zero orbit, which is called the **minimal special orbit**. The duality exchanges the zero orbit with the regular orbit, and exchanges the subregular orbit with the minimal special orbit.

4. Kazhdan-Lusztig Cells

Let (W, S) be a Coxeter system, where W is the Coxeter group and S is the set of simple reflections. Let $l : W \to \mathbb{Z}_{\geq 0}$ be the length function. The **Hecke algebra** $\mathcal{H}$ is the algebra over $A = \mathbb{Z}[q^{1/2}, q^{-1/2}]$ with basis $T_w : w \in W$ subject to relations

$$T_{ww'} = T_w T_{w'} \text{ if } l(ww') = l(w) + l(w'),$$
$$(T_s + 1)(T_s - q) = 0 \text{ for } s \in S.$$

Every T_w is invertible because $T_s^{-1} = q^{-1}(T_s + (1 - q))$ for $s \in S$.

There is a natural involution $a \mapsto \overline{a}$ on A defined by $\overline{q^{1/2}} = q^{-1/2}$. This extends to an involution on $\mathcal{H}$ by $\overline{\sum a_w T_w} = \sum \overline{a_w} T_{w^{-1}}^{-1}$.

For $w \in W$, define $\epsilon_w = (-1)^{l(w)}$ and $q_w = q^{l(w)}$. Equip W with the Bruhat order, i.e. $w \leq w'$ if and only if there is some reduced expression of w that is a substring of a reduced expression of w'.

Theorem 4.1 (Kazhdan-Lusztig, [KL79] Theorem 1.1)**.** *There is an A-basis $\{C_w : w \in W\}$ of $\mathcal{H}$ such that $\overline{C_w} = C_w$ and*

$$C_w = \sum_{w' \leq w} \epsilon_{w'} \epsilon_w q_w^{1/2} q_{w'}^{-1} \overline{P_{w',w}} T_{w'}$$

where $P_{w,w} = 1$ and $P_{y,w} \in A$ is a polynomial in q of degree $\leq \frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$.

The basis $\{C_w : w \in W\}$ is called the **Kazhdan-Lusztig basis** of $\mathcal{H}$.

Using the Kazhdan-Lusztig basis, we can define preorders $\leq_L$, $\leq_R$ and $\leq_{LR}$ on W, where

(1) $w \leq_L w'$ if C_w appears with nonzero coefficient in $C_s C_{w'}$ for some $s \in S$;
(2) $w \leq_R w'$ if C_w appears with nonzero coefficient in $C_{w'} C_s$ for some $s \in S$;
(3) $\leq_{LR}$ is the preorder generated by $\leq_L$ and $\leq_R$.

The preorders $\leq_L$ defines an equivalence relation $\sim_L$ on W, where $w \sim_L w'$ if $w \leq_L w'$ and $w' \leq_L w$. The equivalence classes are called **left cells**. Similarly, we can define equivalence relations $\sim_L$ and $\sim_{LR}$ using $\leq_R$ and $\leq_{LR}$, and the equivalence classes are called **right cells** and **two-sided cells**, respectively. Note that $\leq_L$ (resp. $\leq_R$, $\leq_{LR}$) gives a partial order on the set of left (resp. right, two-sided) cells.

In the case W is finite, let w_0 be the longest element in W. The map $w \mapsto w_0 w$ reverses the preorders $\leq_L$, $\leq_R$, $\leq_{LR}$. So it induces an order-reversing involution on the set of left (resp. right, two-sided) cells.

The Kazhdan-Lusztig conjecture (proven by Beilinson-Bernstein [BB81] and Brylinski-Kashiwara [BK81]) relates Kazhdan-Lusztig polynomials with highest weight modules of complex semisimple Lie algebras. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and W be its Weyl group. Let ρ be the half sum of positive roots. Let M_w be the Verma module of $\mathfrak{g}$ with highest weight $-w\rho - \rho$ and L_w be its irreducible quotient.

Conjecture 4.2 (Kazhdan-Lusztig [KL79])**.** The following two equations hold for all $w \in W$.

$$\mathrm{ch}(L_w) = \sum_{w' \leq w} \epsilon_{w'} \epsilon_w P_{w',w}(1) \mathrm{ch}(M_{w'})$$

$$\mathrm{ch}(M_w) = \sum_{w' \leq w} P_{w_0 w, w_0 w'}(1) \mathrm{ch}(L_{w'})$$

5. Primitive Ideals

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $U(\mathfrak{g})$ be its universal enveloping algebra. Let $Z(U(\mathfrak{g}))$ be the center of $U(\mathfrak{g})$. We have the Harish-Chandra isomorphism $Z(U(\mathfrak{g})) \simeq S(\mathfrak{h})^W$ where $S(\mathfrak{h})$ is the symmetric algebra of a Cartan subalgebra and W is the Weyl group. A **primitive ideal** is the annihilator of an irreducible (left) $U(\mathfrak{g})$-module. Let $\mathrm{Prim}(\mathfrak{g})$ be the set of primitive ideals of $\mathfrak{g}$.

Let $I = \mathrm{Ann}(M)$ be a primitive ideal. Then $Z(U(\mathfrak{g}))$ acts on M by a character $\chi_\lambda : Z(U(\mathfrak{g})) \to \mathbb{C}$. We have $I \cap Z(U(\mathfrak{g})) = \ker \chi_\lambda$. So I determines $\ker \chi_\lambda$, thus determines χ_λ, which in turn determines λ up to an action of W. Let $\mathrm{Prim}_\lambda(\mathfrak{g})$ ($\lambda \in \mathfrak{h}^*/W$) be the set of primitive ideals whose central character is χ_λ. Then we have a decomposition

$$\mathrm{Prim}(\mathfrak{g}) = \bigsqcup_{\lambda\in\mathfrak{h}^*/W} \mathrm{Prim}_\lambda(\mathfrak{g}).$$

The primary examples of irreducible $U(\mathfrak{g})$-modules are the irreducible highest weight modules. Let ρ be the half sum of positive roots. Denote by $M(\lambda)$ the highest weight Verma module with weight $\lambda-\rho$. It has a unique irreducible quotient, denoted by $L(\lambda)$. Every irreducible highest weight module is $L(\lambda)$ for some root λ. We have a map $\mathfrak{h}^* \to \mathrm{Prim}(\mathfrak{g})$ given by $\lambda \mapsto \mathrm{Ann}(L(\lambda))$. Duflo's theorem ([Duf77]) states that this maps is surjective, i.e. every primitive ideal is the annihilator of an irreducible highest weight module.

Now let us consider the primitive ideals in $\mathrm{Prim}_{-\rho}(\mathfrak{g})$. For $w \in W$, let $M_w = M(-w\rho)$, $L_w = L(-w\rho)$ and $I_w = \mathrm{Ann}(L_w)$. This gives a map $W \to \mathrm{Prim}_{-\rho}(\mathfrak{g})$ which maps w to I_w. On W we can define preorders $\leq_L$, $\leq_R$ and $\leq_{LR}$, where

(1) $w \leq_L w'$ if $I_w \subseteq I_{w'}$;
(2) $w \leq_R w'$ if $w'^{-1} \leq_L w^{-1}$;
(3) $\leq_{LR}$ is the preorder generated by $\leq_L$ and $\leq_R$.

A corollary of the Kazhdan-Lusztig conjecture is that the preorders $\leq_L, \leq_R, \leq_{LR}$ defined using primitive ideals is the same as the corresponding preorders defined using the Hecke algebra. This justifies that we use the same symbols to denote the preorders defined in different ways.

The algebra $U(\mathfrak{g})$ has a standard filtration $0 = U_{-1}(\mathfrak{g}) \subseteq U_0(\mathfrak{g}) \subseteq U_1(\mathfrak{g}) \subseteq \cdots$. The associated graded algebra

$$\mathrm{gr}(U(\mathfrak{g})) = \bigoplus_{i\geq 0}(U_i(\mathfrak{g})/U_{i-1}(\mathfrak{g})))$$

is $S(\mathfrak{g})$, the symmetric algebra of $\mathfrak{g}$, and is a polynomial ring. Let I be a primitive ideal. We have the associated graded ideal

$$\mathrm{gr}(I) = \bigoplus_{i\geq 0}(I \cap U_i(\mathfrak{g}))/(I \cap U_{i-1}(\mathfrak{g})).$$

The **associated variety** $\mathrm{Ass}(I)$ is the affine variety defined by $\mathrm{gr}(I)$. Borho-Brylinski [BB82], Joseph [Jos85] and Kashiwara-Tanisaki [KT84] proved that $\mathrm{Ass}(I_w)$ is the closure of a nilpotent orbit in $\mathfrak{g}$, and is irreducible. This relates primitive ideals with nilpotent orbits.

6. Classical Types

The complex simple lie algebras of classical types are A_n ($\mathfrak{sl}_{n+1}$), B_n ($\mathfrak{so}_{2n+1}$), C_n ($\mathfrak{sp}_{2n}$), and D_n ($\mathfrak{so}_{2n}$). The Weyl group for type A_n is the symmetric group S_{n+1}. The Weyl group for type B_n and C_n is the hyperoctahedral group $H_n = C_2^n \rtimes S_n$ where S_n acts on C_2^n by permutation. We can see C_2 as the multiplicative group $\{-1, +1\}$. Therefore H_n is the group of signed permutations. The Weyl group for type D_n is the subgroup of H_n containing elements whose C_2^n part is -1 for an even number of positions. Therefore the Weyl group is the group of even signed permutations. We use a compact notation for hyperoctahedral group elements following van Leeuwen [vL89]. For $(\delta_1, \ldots, \delta_n, p_1, \ldots, p_n) \in H_n$, we write it as $(w_1, \ldots, w_n)$ where $w_i = \delta_i p_i$. Sometimes we use $\overline{p_i}$ to denote $-p_i$.

In classical types the flag variety can be described in terms of flags, justifying its name. In type A_n, we fix a vector space V of dimension $n+1$, and a flag is a sequence $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \subseteq V_{n+1} = V$ where $\dim V_i = i$. The flag variety is just the variety of flags.

In other classical types, we fix a vector space V with basis $\{e_{-n}, \cdots, e_0, \cdots, e_n\}$ in type B_n and $\{e_{-n}, \cdots, e_{-1}, e_1, \cdots, e_n\}$ in type C_n and D_n. Now define a bilinear form $(\cdot, \cdot)$ on V where

$$
\begin{aligned}
(e_i, e_j) &= 0 \text{ when } i + j \neq 0 \\
(e_i, e_{-i}) &= 1 \text{ for } i \geq 0 \\
(e_{-i}, e_i) &= \epsilon \text{ for } i \geq 0
\end{aligned}
$$

where $\epsilon = 1$ in types B and D, and $\epsilon = -1$ in type C. An **isotropic flag** (sometimes simply called a flag) is a sequence $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n$ where $\dim V_i = i$ and $(V_n, V_n) = 0$. A unipotent element $u \in G$ is said to **stabilize** a flag V if $uV_i \subseteq V_i$ for every i. Let $\mathcal{F}$ denote the variety of isotropic flags. We have a map $\mathcal{F} \to \mathcal{B}$ that maps a flag V to the set of elements in G° that stabilizes $\mathcal{F}$ (which forms a Borel subgroup of G). In types B and C, this map is a bijection. In type D, this map is a two-to-one map. $\mathcal{F}$ has two irreducible components and each is isomorphic to $\mathcal{B}$. From the above description, we see that the Springer fiber over a unipotent element $u \in G$ can also be described as the set of flags that is stabilized by u. Denote by $\mathcal{F}_u$ the variety of flags fixed by u.

By considering G-orbits in $\mathcal{F} \times \mathcal{F}$, we can define relative position of flags. The result for classical types other than D are the same. However, in type D, the orbits are not in bijection with W, but in bijection with H_n. Let $\tilde{W}$ denote the group that classifies the G-orbits in $\mathcal{F} \times \mathcal{F}$, i.e. $\tilde{W} = S_{n+1}$ in type A_n and $\tilde{W} = H_n$ in type B_n, C_n, and D_n. The relative position gives a map $A_u \backslash (\mathrm{Irr}(\mathcal{F}_u) \times \mathrm{Irr}(\mathcal{F}_u)) \to \tilde{W}$. If we let u goes over all nilpotent orbits, we get a bijection

$$\bigsqcup_u A_u \backslash (\mathrm{Irr}(\mathcal{F}_u) \times \mathrm{Irr}(\mathcal{F}_u)) \simeq \tilde{W}.$$

When restricted to the classical groups, nilpotent orbits can be studied using combinatorics. Roughly we have the following dictionary:

(1) nilpotent orbits $\sim$ partitions;
(2) irreducible components of the Springer fiber $\sim$ tableaux;
(3) relative position of flags $\sim$ Robinson-Schensted correspondence and its analogues.

7. Nilpotent Orbits in Classical Types

The main reference for this section is Collingwood and McGovern [CM93].

In classical types, a nilpotent element is just a nilpotent matrix that satisfies certain conditions. An obvious invariant of a nilpotent orbit is the Jordan decomposition. Because of nilpotency, diagonals of the Jordan decomposition are zeros. So the Jordan decomposition is the same as a list of the sizes of the Jordan blocks, i.e. a partition. It turns out that in types A, B and C, the partition corresponding to a nilpotent orbit is a complete invariant; and in type D, it is almost a complete invariant.

(1) In type A_n, the nilpotent orbits are in bijection with partitions of $n + 1$;
(2) In type B_n, the nilpotent orbits are in bijection with partitions of $2n + 1$ where even parts occur with even multiplicity;

(3) In type C_n, the nilpotent orbits are in bijection with partitions of $2n$ where odd parts occur with even multiplicity;
(4) In type D_n, the nilpotent orbits are in bijection with partitions of $2n$ where even parts occur with even multiplicity, with exception for **very even partitions**, whose parts are all even, which corresponds to two (instead of one) nilpotent orbits.

For a fixed type, partitions that correspond to nilpotent orbits are called **admissible partitions**.

Therefore we can use partitions to refer to nilpotent orbits. In type D, for a very even partition λ, we use λ^{I} and λ^{II} to refer to the two nilpotent orbits corresponding to λ. (The choice of I and II is not at random, and does matter for e.g. the Lusztig-Spaltenstein dualityf, but we omit it here.)

For a partition λ, we write it as $[a_1^{b_1}, \cdots, a_l^{b_l}]$ where $a_i > a_{i+1}$ and $b_i \geq 1$. This means a_i occurs exactly b_i times in λ. When $b_i = 1$ it is usually omitted. We can also write the partition as $[\lambda_1, \cdots, \lambda_k]$ where $\lambda_i \geq \lambda_{i+1}$, which means that the parts of λ are $\lambda_1, \ldots, \lambda_k$. We allow zero parts for convenience.

The partial ordering on nilpotent orbits in classical types can be interpreted using partitions (Gerstenhaber [Ger61], Hesselink [Hes76]). Let $\lambda = [\lambda_1, \ldots, \lambda_n]$ and $\mu = [\mu_1, \ldots, \mu_n]$ be two partitions. (We can add trailing zeros so that the two partitions have the same number of parts.) We say $\lambda \leq \mu$ if $\sum_{1\leq i\leq k} \lambda_i \leq \sum_{1\leq i\leq k} \mu_i$ for all $1 \leq k \leq n$. Then $\lambda < \mu$ in the orbit order if and only if $\lambda < \mu$ in the partition order.

Using the above description, we can identify several nilpotent orbits in classical types. The results are listed in the following table. (Special orbits in classical types are discussed later.)

	zero	minimal	minimal special	regular	subregular
A_n	$[1^{n+1}]$	$[2, 1^{n-1}]$	$[2, 1^{n-1}]$	$[n + 1]$	$[n, 1]$
B_n	$[1^{2n+1}]$	$[2^2, 1^{2n-3}]$	$[3, 1^{2n-2}]$	$[2n + 1]$	$[2n - 1, 1^2]$
C_n	$[1^{2n}]$	$[2, 1^{2n-2}]$	$[2^2, 1^{2n-4}]$	$[2n]$	$[2n - 2, 2]$
D_n	$[1^{2n}]$	$[2^2, 1^{2n-4}]$	$[2^2, 1^{2n-4}]$	$[2n - 1, 1]$	$[2n - 3, 3]$

Table 1. Some nilpotent orbits in classical types

Now we describe the Lusztig-Spaltenstein duality in classical types using partitions. In type A, the involution is just transpose, and all orbits are special.

In other classical types, simply taking the transpose may result in non-admissible partitions. The way to resolve this is to "collapse" the partition to an admissible one.

For types B, C and D, and any partition λ (not necessarily admissible), there exists a unique largest admissible partition that is $\leq \lambda$ in the partition order. The partition is denoted λ_B, λ_C or λ_D depending on the type.

In types B, C and D, the Lusztig-Spaltenstein duality is $\lambda \mapsto \lambda^t_X$ where $X = B$, C, or D is the type. In type D, additional care is needed for nilpotent orbits corresponding to very even partitions. We omit it here.

8. Combinatorics in Type A

Work in type A_n. The Weyl group is $W = S_{n+1}$. Let $\mathcal{P}(n+1)$ be the set of partitions of $n+1$. For $\lambda \in \mathcal{P}(n+1)$, define $\mathrm{SYT}(\lambda)$ to be the set of standard Young tableaux with shape λ. The well-known Robinson-Schensted correspondence is a bijection

$$(L, R) : W \to \bigsqcup_{\lambda \in \mathcal{P}(n+1)} \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda).$$

The Robinson-Schensted correspondence has many interesting properties. We have $L(w) = R(w^{-1})$ for $w \in W$. Also, there is a pair of maps $P, Q : \mathrm{SYT}(\lambda) \to \mathrm{SYT}(\lambda^t)$ such that for all $w \in W$, we have

$$(L(w_0 w), R(w_0 w)) = (P(L(w)), Q(R(w))).$$

Therefore, $L(w_0 w)$ depends on $L(w)$ but not on $R(w)$. Actually, the map P is the transpose and the map Q is transpose composed with the so-called **Schützenberger involution**.

The combinatorics is closely related to geometry. We have seen that partitions are in bijection with nilpotent orbits.

Starting from a nilpotent element n in orbit (partition) λ and a flag $V = V_0 \subseteq \cdots \subseteq V_{n+1}$ preserved by n. The Jordan decomposition of n on V_{n+1} gives a partition of $n+1$. The Jordan decomposition of n on V_n then gives a partition of n that is one smaller than the previous partition. This means one position is removed from the Young diagram. We mark the position with $n+1$. Then we consider the Jordan decomposition on V_{n-1} and mark a position with n. Continuing this process, we finally get a marking of the initial Young diagram that is a valid Young tableau. In this way we associate a Young tableau of shape λ to a flag in the Springer fiber over the nilpotent orbit λ.

Let T be a Young tableau of shape λ. Define $\mathcal{B}_{\lambda,T}$ be the set of flags over λ that give the Young tableau T. Then the closures of $\mathcal{B}_{\lambda,T}$ are the irreducible components of $\mathcal{B}_\lambda$. Fix two Young tableaux $T_1, T_2 \in \mathrm{SYT}(\lambda)$. For two generic points $V_1 \in \mathcal{B}_{\lambda,T_1}$, $V_2 \in \mathcal{B}_{\lambda,T_2}$, their relative position is an element $w(T_1, T_2) \in W$. The relative position is consistent with the Robinson-Schensted correspondence, i.e. $(T_1, T_2) = (L(w), R(w))$ (Steinberg [Ste88]).

The Schützenberger involution can also be interpreted geometrically (van Leeuwen [vL00]). Let $V = V_0 \subseteq \cdots \subseteq V_{n+1}$ be a flag fixed by a nilpotent element n in orbit λ. We can get a Young tableau of shape λ in a different way than the above. We first consider the Jordan decomposition on $V_{n+1} = V_{n+1}/V_0$. Then we consider the Jordan decomposition on V_{n+1}/V_1 and mark the removed position with $n+1$. Then we consider V_{n+1}/V_2 and so on. In this way we can also get a standard Young tableaux of shape λ. Let T be the standard Young tableaux gotten using the first method, and T' be the standard Young tableaux gotten using the second method. Then T' is the Schützenberger involution of T.

The Kazhdan-Lusztig cells in type A can also be described using the Robinson-Schensted correspondence (Barbasch-Vogan [BV82]). For $w, w' \in W$, we have

(1) $w \sim_L w'$ if and only if $L(w) = L(w')$;
(2) $w \sim_R w'$ if and only if $R(w) = R(w')$;
(3) $w \sim_{LR} w'$ if and only if $L(w)$ has the same shape with $L(w')$.

9. Standard Domino Tableaux

Garfinkle [Gar90] [Gar92] [Gar93] [Gar] gave a combinatorial description of the cells in classical types.

A **standard domino tableaux** is a Young diagram (i.e. a partition) with every position marked with a positive integer, such that

(1) there is some integer m such that each integer from 1 to m occurs twice, and other integers do not occur;
(2) if a positive integer occurs twice, then the two positions are adjacent in the Young diagram;
(3) the numbers are non-decreasing in each row and column.

Define the **shape of a standard domino tableaux** to be the shape of the underlying Young diagram. Let $\mathrm{SDT}(n)$ be the set of standard domino tableaux with n positions. For $\lambda \in \mathcal{P}(n)$, let $\mathrm{SDT}(\lambda)$ be the set of standard domino tableaux with shape λ.

Garfinkle [Gar90] gave a correspondence for standard domino tableaux.

$$(L, R) : H_n \to \bigsqcup_{\lambda \in \mathcal{P}(n)} \mathrm{SDT}(\lambda) \times \mathrm{SDT}(\lambda).$$

Note that Garfinkle's correspondence does not depend on type.

Garfinkle's correspondence has properties similar to the Robinson-Schensted correspondence. For $w \in H_n$, we have $R(w) = L(w^{-1})$.

Garfinkle defined an operation on standard domino tableaux called "moving through". Two standard domino tableaux are called equivalent if they can be transformed into each other by moving through cycles. Then we have $w \sim_L w'$ if and only if $L(w)$ and $L(w')$ are equivalent. (The criteria for $\sim_R$ and $\sim_{LR}$ can be derived from this.) However, note that the moving through operation does not preserve shape. Therefore Kazhdan-Lusztig two-sided cells in types B, C and D are not characterized by shapes of the corresponding standard domino tableaux.

10. Signed Domino Tableaux

Van Leeuwen [vL89] used signed domino tableaux to give a combinatorial description of the relative position of two flags. In this section we introduce his work.

Fix a type B, C or D. An **admissible domino tableaux** is a Young diagram with every position marked with a non-negative integer, such that

(1) in type B, the upper-left position is marked with 0, and there is some integer m such that each integer from 1 to m occurs twice, and other positive integers do not occur;
(2) in type C or D, there is some integer m such that each integer from 1 to m occurs twice, and other integers do not occur;
(3) if a positive integer occurs twice, then the two positions are adjacent in the Young diagram;
(4) the numbers are non-decreasing in each row and column;
(5) the shape of the Young diagram is admissible;
(6) when $m \geq 1$, if we remove the positions marked with m, the remaining tableau is an admissible domino tableaux.

Note that this is a recursive definition.

A domino is a pair of positions marked with the same numbers. Define $\epsilon_j = -\epsilon(-1)^j$. (Recall that $\epsilon = 1$ in type B and D, and $\epsilon = -1$ in type C.) Define the **type of a domino** to be

(I+) if the domino is vertical and in a row j with $\epsilon_j = 1$;
(I-) if the domino is vertical and in a row j with $\epsilon_j = -1$;
(N) if the domino is horizontal.

A **signed domino tableaux** is an admissible domino tableaux where each domino of type (I+) is marked with a sign, $+$ or $-$. The set of signed domino tableaux of size n is denoted $\Sigma\mathrm{DT}(n)$. The set of signed domino tableaux of shape λ is denoted $\Sigma\mathrm{DT}(\lambda)$.

Let T be an admissible domino tableau. Van Leeuwen [vL89] defined a notion of **clusters**, which is a certain partition of the set of positions of T. Certain clusters are called **open clusters**, while others (except for one cluster in type B or C) are called **closed clusters**.

Now let T and T' be two signed domino tableaux that correspond to the same admissible domino tableaux. We say $T \sim_{\mathrm{op,cl}} T'$ if for all open clusters and closed clusters, the product of signs of type (I+) dominoes in the cluster are the same in T and T'. We say $T \sim_{\mathrm{cl}} T'$ if for all closed clusters, the product of signs of type (I+) dominoes in the cluster are the same in T and T'. Let $\Sigma\mathrm{DT}_{\mathrm{op,cl}}(n)$ (resp. $\Sigma\mathrm{DT}_{\mathrm{op,cl}}(\lambda)$) be the set of equivalence classes of $\Sigma\mathrm{DT}(n)$ (resp. $\Sigma\mathrm{DT}(\lambda)$) under the equivalence relation $\sim_{\mathrm{op,cl}}$. Let $\Sigma\mathrm{DT}_{\mathrm{cl}}(n)$ (resp. $\Sigma\mathrm{DT}_{\mathrm{cl}}(\lambda)$ be the set of equivalence classes of $\Sigma\mathrm{DT}(n)$ (resp. $\Sigma\mathrm{DT}(\lambda)$) under the equivalence relation $\sim_{\mathrm{cl}}$.

Van Leeuwen [vL89] proved that if u is a unipotent element which corresponds to partition λ, then there is a bijection between $\Sigma\mathrm{DT}_{\mathrm{op,cl}}(\lambda)$ with $\mathrm{Irr}(\mathcal{F}_u)$, the set of irreducible components of the variety of flags fixed by u.

Furthermore, he defined a map $\Sigma\mathrm{DT}_{\mathrm{op,cl}}(\lambda) \times \Sigma\mathrm{DT}_{\mathrm{op,cl}}(\lambda) \to \tilde{W}$ that gives a bijection

$$\bigsqcup_{\lambda} A_\lambda \backslash (\Sigma\mathrm{DT}_{\mathrm{op,cl}}(\lambda) \times \Sigma\mathrm{DT}_{\mathrm{op,cl}}(\lambda)) \to \tilde{W}$$

and coincides with the relative position map

$$\bigsqcup_{u} A_u \backslash (\mathrm{Irr}(F_u) \times \mathrm{Irr}(F_u)) \to \tilde{W}.$$

There is an obvious projection map $\Sigma\mathrm{DT}_{\mathrm{op,cl}} \to \Sigma\mathrm{DT}_{\mathrm{cl}}$. Let $\Sigma\mathrm{DT}_{\mathrm{cl}} \to \Sigma\mathrm{DT}_{\mathrm{op,cl}}$ be its section (right inverse) that maps an equivalence class under $\sim_{\mathrm{cl}}$ to the equivalence class under $\sim_{\mathrm{op,cl}}$ that has negative product of signs in every open cluster and the same product of signs in other clusters. In this way we can see $\Sigma\mathrm{DT}_{\mathrm{cl}}$ as a subset of $\Sigma\mathrm{DT}_{\mathrm{op,cl}}$.

Pietraho [Pie04] proved that

(1) Garfinkle's moving through map gives a bijection $\Sigma\mathrm{DT}_{\mathrm{op,cl}}(n) \to \mathrm{SDT}(n)$;
(2) restricting to $\Sigma\mathrm{DT}_{\mathrm{cl}}(n)$, the map preserves shape;
(3) for any admissible partition λ, the restriction gives a bijection $\Sigma\mathrm{DT}_{\mathrm{cl}}(\lambda) \to \mathrm{SDT}(\lambda)$.

Using known bijections $\Sigma\mathrm{DT}_{\mathrm{cl}}(\lambda) = A_\lambda \backslash \Sigma\mathrm{DT}_{\mathrm{op,cl}}(\lambda)$ and $\mathrm{Irr}(\mathcal{O} \cap \mathfrak{n}) = A_u \backslash \mathrm{Irr}(\mathcal{F}_u)$ (where $\mathcal{O}$ is a nilpotent orbit corresponding to λ and $\mathfrak{n}$ is a Borel subalgebra), we see that there is a bijection between the set of irreducible components of the orbital variety $\mathcal{O} \cap \mathfrak{n}$ and the set of standard domino tableaux of shape λ.

11. Acknowledgements

This project is supported in part by the MIT UROP+ program. The author is very thankful to his mentor Konstantin Tolmachov for helping and discussing during this project. The author thanks Prof. Roman Bezrukavnikov for suggesting this interesting project.

References

[BB81] Alexandre Beilinson and Joseph Bernstein. Localisation de g-modules. *CR Acad. Sci. Paris Sér. I Math.*, 292(1):15–18, 1981.

[BB82] Walter Borho and Jean-Luc Brylinski. Differential operators on homogeneous spaces. i. *Inventiones mathematicae*, 69(3):437–476, 1982.

[BK81] Jean-Luc Brylinski and Masaki Kashiwara. Kazhdan-Lusztig conjecture and holonomic systems. *Inventiones mathematicae*, 64(3):387–410, 1981.

[BV82] Dan Barbasch and David Vogan. Primitive ideals and orbital integrals in complex classical groups. *Mathematische Annalen*, 259(2):153–199, 1982.

[CM93] David H Collingwood and William M McGovern. *Nilpotent orbits in semisimple Lie algebra: an introduction*. CRC Press, 1993.

[Duf77] Michel Duflo. Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple. *Annals of Mathematics*, pages 107–120, 1977.

[Gar] Derva Garfinkle. On the classification of primitive ideals for complex classical Lie algebras, iv. in preparation.

[Gar90] Devra Garfinkle. On the classification of primitive ideals for complex classical Lie algebras, i. *Compositio Mathematica*, 75(2):135–169, 1990.

[Gar92] Devra Garfinkle. On the classification of primitive ideals for complex classical Lie algebras, ii. *Compositio Mathematica*, 81(3):307–336, 1992.

[Gar93] Devra Garfinkle. On the classification of primitive ideals for complex classical Lie algebras, iii. *Compositio Mathematica*, 88(2):187–234, 1993.

[Ger61] Murray Gerstenhaber. Dominance over the classical groups. *Annals of Mathematics*, pages 532–569, 1961.

[Hes76] Wim Hesselink. Singularities in the nilpotent scheme of a classical group. *Transactions of the American Mathematical Society*, 222:1–32, 1976.

[Jos85] Anthony Joseph. On the associated variety of a primitive ideal. *Journal of algebra*, 93(2):509–523, 1985.

[KL79] David Kazhdan and George Lusztig. Representations of Coxeter groups and Hecke algebras. *Inventiones mathematicae*, 53(2):165–184, 1979.

[KT84] Masaki Kashiwara and T Tanisaki. The characteristic cycles of holonomic systems on a flag manifold. *Inventiones mathematicae*, 77(1):185–198, 1984.

[Pie04] Thomas Pietraho. Components of the Springer fiber and domino tableaux. *Journal of Algebra*, 272(2):711–729, 2004.

[Spa82] Nicolas Spaltenstein. *Classes unipotentes et sous-groupes de Borel*. 1982.

[Ste76] Robert Steinberg. On the desingularization of the unipotent variety. *Inventiones mathematicae*, 36(1):209–224, 1976.

[Ste88] Robert Steinberg. An occurrence of the Robinson-Schensted correspondence. *Journal of Algebra*, 113(2):523–528, 1988.

[vL89] Marc A. A. van Leeuwen. A Robinson-Schensted algorithm in the geometry of flags for classical groups. 1989.

[vL00] Marc A. A. van Leeuwen. Flag varieties and interpretations of Young tableau algorithms. *Journal of Algebra*, 224(2):397–426, 2000.