Computing Signatures for Representations of the Hecke and Cherednik Algebras

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Abstract

In this paper, we compute the signatures of the contravariant form on Specht modules for the cyclotomic Hecke algebra and compute the signature character of the contravariant form on the polynomial representation of the rational Cherenik algebra associated to $G(r, 1, n)$.

1 Introduction

Let A be an algebra and V a left A-module which admits a non-degenerate invariant Hermitian form. The problem of determining whether this form is positive-definite is an important one in representation theory. This problem has been explored for the rational Cherednik algebra by Etingof and Stoica in [ES], and for the cycolotomic Hecke algebra by Stoica in [S]. In this paper, we consider the more general problem of determining the signature of this form. We define the signature of a form on a finite-dimensional vector space as follows.

Definition. Let V be a finite-dimensional vector space with non-degenerate Hermitian form $\langle \cdot, \cdot \rangle$. Let $\{e_i\}$ be a basis for V which is orthogonal with respect to this form. Then the *signature* $s(V)$ of V is the number of basis elements with positive norm minus the number of basis elements with negative norm.

Now for an infinite-dimensional vector space with a natural grading, we may define the signature character as follows.

Definition. Let V be a vector space with non-degenerate Hermitian form $\langle \cdot, \cdot \rangle$. Suppose there exists a grading $V = \bigoplus_{m=0}^{\infty} V_m$ so that V_m and V_n are orthogonal when $m \neq n$. Then we may define the *signature character*

$$ch_s(V) = \sum_{w=0}^{\infty} t^w s(V_w)$$

of V with respect to this form.

In Section 2, we present the definition of the cyclotomic Hecke algebra, as well as some preliminary theorems that will state what its irreducible representations are and how to compute in them. In Section 3, we derive a formula for the signature of all the irreducible representations of the Hecke algebra. In Section 4, we present definitions and preliminaries for the rational Cherednik algebra and its polynomial representation. Finally in Section 5, we compute the signature character of this representation.

2 Preliminaries for the Hecke Algebra

Before we introduce the Hecke algebra, we first motivate its definition by introducing complex reflection groups.

Definition. Let $\mathfrak{h}$ be a finite dimensional vector space over $\mathbb{C}$. A *reflection* of $\mathfrak{h}$ is a unitary transformation s of $\mathfrak{h}$ with $\mathrm{rk}(s-1)|_{\mathfrak{h}} = 1$. If a finite group W is generated by reflections of $\mathfrak{h}$, we say that W is a *complex reflection group* acting on $\mathfrak{h}$.

Throughout this paper, we will be dealing with the complex reflection group $G(m,1,n) = S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n$. This group can also be expressed with the following generators and relations.

Theorem 1 ([AK], Proposition 2.1)**.** *The complex reflection group* $G(m,1,n)$ *is generated by* $s_0, s_1, \ldots, s_{n-1}$ *subject to*

$$s_0^r = 1$$

$$s_i^2 = 1 \textit{ for } i > 0$$

$$s_i s_j = s_j s_i \textit{ if } |i-j| > 1$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \textit{ for } i > 0$$

$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0.$$

This motivates the definition of the cyclotomic Hecke algebra as a deformation of the above complex reflection group.

Definition. Let R be a commutative domain, with k its field of fractions, and take $q, q_1, \ldots, q_m \in R^\times$. The *cyclotomic Hecke algebra* $H_{R,n} = H_{R,n}(q, q_1, \ldots, q_m)$ is defined as the unital associative R-algebra generated by the elements $T_0, \ldots, T_{n-1}$ subject to

$$(T_0 - q_1) \cdots (T_0 - q_m) = 0$$

$$(T_i - q)(T_i + 1) = 0 \text{ for } i > 0$$

$$T_i T_j = T_j T_i \text{ if } |i-j| > 1$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \text{ for } i > 0$$

$$T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0.$$

Throughout this paper, we take $R = \mathbb{C}$ and we take $q, q_1, \ldots, q_m$ to have norm 1 and be generic. (In particular, we require that $\frac{q_i}{q_j} q^a \neq 1$ for any i, j, a.) We first introduce the notion of m-partitions and m-tableaux because the irreducible representations of the Hecke algebra will be given by its actions on the standard Young m-tableaux.

Definition. An *m-partition* $\lambda = (\lambda^1, \dots, \lambda^m)$ of n is an ordered m-tuple so that each λ^i is a partition of an integer $|\lambda^i|$, with $\sum_{i=1}^m |\lambda^i| = n$. We then write $\lambda \vdash_m n$.

Definition. Let $\lambda = (\lambda^1, \dots, \lambda^m) \vdash_m n$. The *Young m-diagram* $[\lambda]$ of shape λ is the m-tuple of Young diagrams $([\lambda^1], \dots, [\lambda^m])$. We call $[\lambda^i]$ the *components* of $[\lambda]$. A *standard Young m-tableau* of shape λ is an enumeration from 1 to n of the boxes of $[\lambda]$ so that each row and each column of each component is increasing. We denote the set of all standard Young m-tableaux of shape λ by $\mathrm{Std}(\lambda)$ and its formal k-linear span by V_λ.

Finally, we define some notation which will be necessary to state our main preliminary theorem.

Definition. Let $\lambda \vdash_m n$ and $s \in \mathrm{Std}(\lambda)$. For $1 \leq i \leq n$, the *content* $c_s(i)$ of i in s is the row index of i in its component of s minus the column index of i in its component of s. For $1 \leq a, b \leq n$, we define the *axial distance* $r_s(a, b) = c_s(a) - c_s(b)$. We also denote the index of the component of i in s by $\tau_s(i)$.

Theorem 2 ([AK], Theorems 3.7 and 3.10)**.** *$H_{R,n}$ is semisimple. Its irreducible representations are exactly V_λ for all $\lambda \vdash_m n$, with the action of T_i on $s \in Std(\lambda)$ given by:*

- $T_0 s = q_{\tau_s(1)} s$.
- *For $i > 0$, if i and $i+1$ lie in the same row of the same component of s, then $T_i s = qs$.*
- *For $i > 0$, if i and $i+1$ lie in the same column of the same component of s, then $T_i s = -s$.*
- *For $i > 0$, if neither of the above hold, let $t = (i, i+1)s$ be the standard m-tableau gotten by swapping the positions of i and $i+1$ in s. Then*

$$T_i s = -\frac{1-q}{1 - \frac{q_{\tau_s(i)}}{q_{\tau_s(i+1)}} q^{r(i+1,i)}} s + \frac{q - \frac{q_{\tau_s(i)}}{q_{\tau_s(i+1)}} q^{r(i+1,i)}}{1 - \frac{q_{\tau_s(i)}}{q_{\tau_s(i+1)}} q^{r(i+1,i)}} t$$

Moreover, [S] (Proposition 3.2) defines a non-degenerate Hermitian form $\langle \cdot, \cdot \rangle$ on V_λ which is $H_{R,n}$-invariant (i.e. $\langle v, w \rangle = \langle T_i v, T_i w \rangle$ for all $0 \leq i \leq n-1$ and $v, w \in V_\lambda$), and shows that $s \in \mathrm{Std}(\lambda)$ form an orthogonal basis with respect to $\langle \cdot, \cdot \rangle$. This will be the form with respect to which we compute the signature.

3 Signature for the Representations of the Hecke Algebra

Now we move on to compute the signature of the representation. First we derive a formula for how the norm of an element changes when we switch its entries.

Proposition 1. *Let s, t be standard m-tableaux of shape λ, with $t = (i, i+1)s$. Then*

$$\langle t, t \rangle = 2 \cdot Re \left[\frac{1}{q - \frac{q_{\tau_s(i)}}{q_{\tau_s(i+1)}} q^{r_s(i+1,i)}} \right] \langle s, s \rangle.$$

Proof. For convenience, write $a = \tau_s(i)$, $b = \tau_s(i+1)$, and $r = r_s(i+1, i)$. We have

$$T_i s = -\frac{1-q}{1 - \frac{q_a}{q_b} q^r} s + \frac{q - \frac{q_a}{q_b} q^r}{1 - \frac{q_a}{q_b} q^r} t.$$

Since $\langle \cdot, \cdot \rangle$ is $H_{R,n}$-invariant, then

$$\langle s, s \rangle = \langle T_i s, T_i s \rangle = \left| \frac{1-q}{1 - \frac{q_a}{q_b} q^r} \right|^2 \langle s, s \rangle + \left| \frac{q - \frac{q_a}{q_b} q^r}{1 - \frac{q_a}{q_b} q^r} \right|^2 \langle t, t \rangle.$$

Rearranging, we get

$$\langle t, t \rangle = \frac{\left|1 - \frac{q_a}{q_b} q^r\right|^2 - |1-q|^2}{\left|q - \frac{q_a}{q_b} q^r\right|^2} \langle s, s \rangle = \frac{q + \bar{q} - \frac{q_a}{q_b} q^r - \overline{\frac{q_a}{q_b} q^r}}{\left|q - \frac{q_a}{q_b} q^r\right|^2} \langle s, s \rangle$$

$$= \left(\frac{1}{q - \frac{q_a}{q_b} q^r} + \frac{1}{\overline{q - \frac{q_a}{q_b} q^r}} \right) = 2 \cdot \mathrm{Re} \left[\frac{1}{q - \frac{q_a}{q_b} q^r} \right] \langle s, s \rangle.$$

□

We now establish a distinguished m-tableau whose norm we will use to compute the norms of all other m-tableaux. In addition we define some notation which will important for stating our result.

Definition. Consider the standard m-tableau $t_0 \in \mathrm{Std}(\lambda)$ gotten by putting the numbers $\lambda^{i-1} + 1, \ldots, \lambda^i$ in the i^{th} component (where $\lambda^0 = 0$), and arranging the numbers within each component in consecutive increasing order across rows.

Definition. For $s \in \mathrm{Std}(\lambda)$ and $1 \leq i \leq n$, let $j_s(i)$ denote the number lying in the box of t_0 corresponding to the box of s in which i lies. Then we say a and b are *inverted* in s if $(a-b)(j_s(a)-j_s(b)) < 0$, and we write $a \underset{s}{\leftrightarrow} b$.

Definition. For $z \in \mathbb{C}^\times$, let $\{z\} = \frac{\mathrm{Re}\, z}{|\mathrm{Re}\, z|}$.

Now we prove the general signature formula for irreducible representations of the Hecke algebra.

Theorem 3. *Taking the convention that* $\langle t_0, t_0 \rangle > 0$, *we have*

$$s(V_\lambda) = \sum_{s \in Std(\lambda)} \prod_{\substack{1 \leq a < b \leq n \\ a \underset{s}{\leftrightarrow} b}} \left\{ q - \frac{q_{\tau_s(a)}}{q_{\tau_s(b)}} q^{r_s(b,a)} \right\}.$$

Proof. Since $s \in \mathrm{Std}(\lambda)$ form an orthogonal basis with respect to $\langle \cdot, \cdot \rangle$, we need only verify that the above product correctly gives the sign of the norm of each s. It is clear that the product is empty for $s = t_0$ and so correctly gives positive sign. Now note that we can arrive at any standard m-tableau of shape λ by applying transpositions $(i, i+1)$ to t_0 (such that each intermediate m-tableau is also a standard). In particular, this means that we need only show that if the above product gives the correct sign for the norm of s, it does so for $(i, i+1)s$ as well. (Its validity for t_0 then shows its validity for all s.)

Now suppose the sign of $\langle s, s \rangle$ is as given by the above product. Note that each pair of numbers not including i or $i+1$ is inverted in s if and only if it is inverted in $t = (i, i+1)s$. Moreover if some j was inverted with i in s, then it will be inverted with $i+1$ in t, and we get the same factor in the product, and vice-versa for $i+1$ and i. The only factor we gain/lose is the factor from the inversion of i and $i+1$, which necessarily become inverted or become not inverted in t. Therefore the product for t evaluates to

$$\left\{ q - \frac{q_{\tau_s(i)}}{q_{\tau_s(i+1)}} q^{r_s(i+1,i)} \right\} \{\langle s, s \rangle\}.$$

(This is true whether the factor is gained or lost because $(\pm 1)^{-1} = \pm 1$). Noting that $\{z^{-1}\} = \{z\}$, we see that the product is equal to

$$\left\{ \frac{1}{q - \frac{q_{\tau_s(i)}}{q_{\tau_s(i+1)}} q^{r_s(i+1,i)}} \right\} \{\langle s, s \rangle\} = \{\langle t, t \rangle\}.$$

Therefore the product correctly gives the sign of the norm of all elements, and so the signature formula is correct. □

4 Preliminaries for the Cherednik Algebra

Now we turn our attention to the rational Cherednik algebra. First we define some notation which will be useful for the definition of the Cherednik algebra.

Definition. Let $\mathfrak{h}$ be a finite dimensional vector space over $\mathbb{C}$ with a positive definite Hermitian form $(\cdot,\cdot)$, and s a reflection of $\mathfrak{h}$. Let $\alpha_s \in \mathfrak{h}^*$ and $\alpha_s^\vee \in \mathfrak{h}$ be such that

$$sx = x - (x, \alpha_s^\vee)\alpha_s$$

for all $x \in \mathfrak{h}^*$.

Now we give the definition of the rational Cherednik algebra.

Definition. Let W be a complex reflection group acting on $\mathfrak{h}$, and S its set of reflections. For $\kappa \in \mathbb{C}$ and W-invariant function $c : S \to \mathbb{C}$ (i.e. $c(g) = c(h)$ if $h = wgw^{-1}$ for $w \in W$), the *rational Cherednik algebra* $H_{\kappa,c} = H_{\kappa,c}(W, \mathfrak{h})$ is the quotient of the algebra $\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations

$$[x, x'] = 0, \quad [y, y'] = 0, \quad [y, x] = \kappa(y, x) - \sum_{s \in S} c_s(y, \alpha_s)(x, \alpha_s^\vee)s$$

for all $x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}$.

Throughout this paper, we will only consider the case $W = G(m, 1, n)$. Fix some orthonormal basis $x_1, \ldots, x_n$ of $\mathfrak{h}^*$. Then we have m conjugacy classes of reflections – one containing all reflections which switch two basis elements (whose corresponding value we denote by c_0), and then for each $1 \le i \le m-1$, one conjugacy class containing all reflections which are diagonal and multiply only one basis element by ζ_m^i (whose corresponding value we denote by c_i). We take $\kappa, c_0, c_1, \ldots, c_{m-1}$ to be generic.

In this paper, we will consider only the polynomial representation V of $H_{\kappa,c}$ given by its action on $T(\mathfrak{h}^*)$, where $w \in W, x \in \mathfrak{h}^*, y \in \mathfrak{h}$ act by their usual multiplication in $H_{\kappa,c}$ but $y \in \mathfrak{h}$ acts as 0 at the end of a word and $w \in W$ acts as 1 at the end of a word. Then a basis for V is given by all monomials in $x_1, \ldots, x_n$ (as such, we sometimes write $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$ for $\mu \in \mathbb{Z}_{\ge 0}^n$).

If $y_1, \ldots, y_n \in \mathfrak{h}$ is the dual basis of $x_1, \ldots, x_n$, we can define a contravariant Hermitian form $\langle \cdot, \cdot \rangle$ by requiring that x_i and y_i are adjoint and the adjoint of $w \in W$ is w^{-1} (along with the requirement that the form evaluated on degree zero monomials is just their product). Now we will present a theorem which introduces operators and a basis which make computations with this form simpler. First, we introduce some notation to simplify our formulae.

Definition. For $\mu \in \mathbb{Z}^n_{\geq 0}$ and $1 \leq i \leq n$, let

$$v_\mu(i) = |\{j < i \mid \mu_j < \mu_i\}| + |\{j > i \mid \mu_j \leq \mu_i\}| + 1.$$

Definition. Let $d_i = \sum_{j=1}^{m-1} \zeta_m^{ij} c_j$.

Definition. Let $\alpha_{a,b} = a\kappa - brc_0$.

Theorem 4 ([G1], Theorem 6.1 and Corollary 6.2)**.** *There exist operators* $\sigma_i, \Phi, \Psi \in H_{\kappa,c}$ *for* $1 \leq i \leq n-1$ *and polynomials* $f_\mu = x^\mu + o(x^\mu) \in V$ *for all* $\mu \in \mathbb{Z}^n_{\geq 0}$ *which satisfy*

$$\sigma_i f_\mu = \begin{cases} f_{s_i\mu}, & \text{if } \mu_i \neq \mu_{i+1} \pmod r \text{ or } \mu_i < \mu_{i+1} \\ 0, & \text{if } \mu_i = \mu_{i+1} \\ C f_{s_i\mu}, & \text{if } \mu_i = \mu_{i+1} \pmod r \text{ and } \mu_i > \mu_{i+1}, \end{cases}$$

where

$$C = \frac{\alpha_{\kappa(\mu_i-\mu_{i+1}),v_\mu(i)-v_\mu(i+1)-1} \cdot \alpha_{\kappa(\mu_i-\mu_{i+1}),v_\mu(i)-v_\mu(i+1)+1}}{(\alpha_{\kappa(\mu_i-\mu_{i+1}),v_\mu(i)-v_\mu(i+1)})^2},$$

$$\Phi f_\mu = f_{\phi\mu},$$

and

$$\Psi f_\mu = \begin{cases} 0, & \text{if } \mu_n = 0 \\ (\alpha_{\mu_n,v_\mu(n)-1} - d_0 + d_{-\mu_n}) f_{\psi\mu}, & \text{if } \mu_n \neq 0. \end{cases}$$

Here s_i *acts on an* n*-tuple by exchanging the* i^{th} *and* $(i+1)^{th}$ *entries,* ϕ *acts by*

$$\phi(\mu_1, \ldots, \mu_n) = (\mu_2, \ldots, \mu_n, \mu_1 + 1),$$

and ψ *acts by the inverse of* ϕ.

Moreover in [G2] (Section 6), it is shown that the f_μ's form an orthogonal basis with respect to $\langle \cdot, \cdot \rangle$, the σ_i's are self-adjoint, and Φ is adjoint to Ψ. This will allow us to compute the signature character of V.

5 Signature Character for the Polynomial Representation

Now we present the formula for the norm of f_μ, and as a corollary we derive the formula for the signature character of V.

Theorem 5. *Let $\mu \in \mathbb{Z}^n_{\geq 0}$. Fix some nondecreasing reordering of the entries of μ. Let $g(i)$ denote the index of μ_i in this reordering. Also let $p(i,x)$ denote the number of entries of μ which are greater than $\mu_i + x$ or which are equal to $\mu_i + x$ and have index less than i. Then*

$$\langle f_\mu, f_\mu \rangle = \left(\prod_{i=1}^{n} \prod_{j=1}^{\mu_i} (\alpha_{j,g(i)-1} - d_0 + d_{-j}) \right) \cdot \left(\prod_{j=1}^{\infty} \frac{\alpha_{jm,n}}{\alpha_{jm,0}} \prod_{i=1}^{n} \frac{\alpha_{jm,n-g(i)-p(i,jm)}}{\alpha_{jm,n-g(i)-p(i,jm)+1}} \right).$$

Proof. Because the operators σ_i are self-adjoint, then when $\lambda_i \not\equiv \lambda_{i+1} \pmod m$, we have

$$\langle f_\lambda, f_\lambda \rangle = \langle \sigma_i f_{s_i\lambda}, f_\lambda \rangle = \langle f_{s_i\lambda}, \sigma_i f_\lambda \rangle = \langle f_{s_i\lambda}, f_{s_i\lambda} \rangle,$$

and when $\lambda_i \equiv \lambda_{i+1} \pmod m$ with $\lambda_i > \lambda_{i+1}$

$$\begin{aligned}\langle f_\lambda, f_\lambda \rangle &= \langle \sigma_i f_{s_i\lambda}, f_\lambda \rangle = \langle f_{s_i\lambda}, \sigma_i f_\lambda \rangle \\ &= \frac{\alpha_{\lambda_i-\lambda_{i+1}, v_\lambda(i)-v_\lambda(i+1)+1} \cdot \alpha_{\lambda_i-\lambda_{i+1}, v_\lambda(i)-v_\lambda(i+1)-1}}{(\alpha_{\lambda_i-\lambda_{i+1}, v_\lambda(i)-v_\lambda(i+1)})^2} \langle f_{s_i\lambda}, f_{s_i\lambda} \rangle.\end{aligned}$$

Likewise since Φ and Ψ are adjoint, we have (for $\lambda_n \neq 0$)

$$\langle f_\lambda, f_\lambda \rangle = \langle \Phi f_{\psi\lambda}, f_\lambda \rangle = \langle f_{\psi\lambda}, \Psi f_\lambda \rangle = (\alpha_{\lambda_n, v_\lambda(n)-1} - d_0 + d_{-\lambda_n}) \langle f_{\psi\lambda}, f_{\psi\lambda} \rangle.$$

Now we describe a sequence of s_i and ϕ operations which will lead us from the zero string $0^n = (0, \dots, 0)$ to μ, and we will calculate the norm of f_μ by multiplying the factors we acquire when traversing the same sequence with σ_i and Φ operations acting on f_{0^n}. We begin by applying ϕ^n to 0^n, and repeat this $\mu_{h(1)}$ times so that we are left with the string $\mu^n_{h(1)} = (\mu_{h(1)}, \dots, \mu_{h(1)})$ (where h denotes the inverse of g). Now we apply ϕ^{n-1} and then $s_{n-1}s_{n-2}\dots s_1$ so that we are left with the string $(\mu_{h(1)} + 1, \dots, \mu_{h(1)} + 1, \mu_{h(1)})$. We repeat this $\mu_{h(2)} - \mu_{h(1)}$ times to leave us with $(\mu_{h(2)}, \dots, \mu_{h(2)}, \mu_{h(1)})$. Next we will apply ϕ^{n-2} followed by $s_{n-2}\dots s_1 s_{n-1}\dots s_2$, and repeat this $\mu_{h(3)} - \mu_{h(2)}$ times to get the string $(\mu_{h(3)}, \dots, \mu_{h(3)}, \mu_{h(2)}, \mu_{h(1)})$. We continue in this way, incrementing $n - i$ of the entries and passing the other i entries back to the front so that they are not incremented, until the value of the $n-i$ entries is $\mu_{h(i)}$, at which point we repeat for $i+1$. The one exception is that for each pair i,j such that $\mu_i \equiv \mu_j \pmod m$ with $\mu_i > \mu_j$ and $i > j$, we do not pass one entry of value μ_j through one entry of value μ_i. Thus at the end of the process we have a string with all the entries of μ, so that all entries with the same value modulo m appear in the string in the same order as they appear in μ. We may then freely reorder the entries using the s_i operations, without switching any entries $\mu_i \equiv \mu_j \pmod m$, to arrive at the string μ. The corresponding actions of σ_i will therefore contribute no factors, so this final step does not change the norm.

Now we first analyze the factors acquired from the actions of Φ. The first n times we apply it, we acquire the factors $\prod_{i=1}^{n}(\alpha_{1,i-1} - d_0 + d_{-1})$ because $\lambda_n = 1$ (after the action) and while the first time, all $j-1$ other entries in λ are less than 1 (after the action), each successive time, one less entry is strictly less than 1. Likewise the j^{th} time we increment the entries, the factors we acquire are $\prod_{i=n-k+1}^{n}(\alpha_{j,i-1} - d_0 + d_{-j})$, where k is the number of entries we increment. Note that the number of times $i-1$ appears as the second argument of α is $\mu_{h(i)}$ (in particular it appears for $j = 1, \ldots, \mu_{h(i)}$). Replacing the indexing variable i by $g(i)$, we find that the total factor acquired is

$$\prod_{i=1}^{n}\prod_{j=1}^{\mu_i}(\alpha_{j,g(i)-1} - d_0 + d_{-j}).$$

Now we analyze the factors acquired from the actions of σ_i. Unless $\lambda_i \equiv \lambda_{i+1} \pmod m$, we acquire no factor. Therefore we only acquire factors when we pass an entry of value μ_i to the front at the $(\mu_i + jm)^{\text{th}}$ step. In particular, we acquire a factor for each (i,j,k) such that $\mu_k - \mu_i \geq jm$ (equality only contributes a factor if $k < i$). For fixed i,j, the factors we acquire are

$$\prod_{k=1}^{p(i,jm)} \frac{\alpha_{jm,n-g(i)-k}\alpha_{jm,n-g(i)-k+2}}{(\alpha_{jm,n-g(i)-k+1})^2} = \frac{\alpha_{jm,n-g(i)-p(i,jm)}\alpha_{jm,n-g(i)+1}}{\alpha_{jm,n-g(i)-p(i,jm)+1}\alpha_{jm,n-g(i)}}.$$

Taking the product over all i,j we get

$$\prod_{j=1}^{\infty}\prod_{i=1}^{n} \frac{\alpha_{jm,n-g(i)-p(i,jm)}\alpha_{jm,n-g(i)+1}}{\alpha_{jm,n-g(i)-p(i,jm)+1}\alpha_{jm,n-g(i)}} = \prod_{j=1}^{\infty}\frac{\alpha_{jm,n}}{\alpha_{jm,0}}\prod_{i=1}^{n}\frac{\alpha_{jm,n-g(i)-p(i,jm)}}{\alpha_{jm,n-g(i)-p(i,jm)+1}}.$$

Combining these two factors and taking $\langle f_{0^n}, f_{0^n}\rangle = 1$ proves the formula. (Note that if $\mu_k - \mu_i < jr$ for all i,k, then $p(i,jm) = 0$ for all i, and the j^{th}

term in this product will cancel out to leave 1. Since this is true for all j sufficiently large, then all but finitely many of the terms in this product are in fact 1.) □

Corollary 1. *The signature character of the polynomial representation is*

$$\sum_{w=0}^{\infty} t^w \sum_{\substack{\mu \in \mathbb{Z}_{\geq 0}^n \\ |\mu| = w}} \left\{ \prod_{i=1}^{n} \prod_{j=1}^{\mu_i} (\alpha_{j,g(i)-1} - d_0 + d_{-j}) \cdot \prod_{j=1}^{\infty} \frac{\alpha_{jm,n}}{\alpha_{jm,0}} \prod_{i=1}^{n} \frac{\alpha_{jm,n-g(i)-p(i,jm)}}{\alpha_{jm,n-g(i)-p(i,jm)+1}} \right\}.$$

6 Conclusion and Future Research

In this paper, we employed computational results from other works to calculate the signature of all finite-dimensional irreducible representations of the cyclotomic Hecke algebra and the signature character of the polynomial representation of the rational Cherednik algebra. As future research, we would certainly like to calculate the signature character of all irreducible representations of the rational Cherednik algebra, for which there already exist similar computational tools. We would also like to see how our results behave when we take assymptotic limits of the parameters. Finally, we would consider exploring how we might be able to undergo this Cherednik algebra calculation in a basis which is preserved by the action of S_n (unlike the f_μ's that we used).

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