On the Fixed Points of Unipotent Action on Flag Varieties and Its Relation to Pairs Of Young Tableaux

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1 Abstract

The aim of this paper is to first introduce some of the properties of the set of fixed points of a unipotent action on a flag variety. Then we go on to explain a correspondence from the set of irreducible components of the flag variety and a set of standard tableaux of a certain shape written about by N. Spaltenstein [1]. From this correspondence we will describe how it is used to establish an even more interesting correspondence between pairs of pairs of different fixed points in the flag variety and pairs of standard tableaux through the use of the Robinson-Schensted Correspondence as exposed in a paper by Robert Steinberg [2]. All of this will be preceded by the necessary background on the Robinson-Schensted Algorithm [3], basic representation theory of the symmetric group [4], and the flag variety and actions on it by the general linear group.

2 Introduction

In the representation theory of S_n Young tableaux arise everywhere. Not only are the irreducible representations of S_n indexed by Young diagrams of size n but the dimension of a given irreducible is equal to the number of standard tableaux of the shape of its index. Young tableaux are tied to vectors spaces and the symmetric group in many fundamental ways and this paper will look at how they relate to the flag variety through the Robinson-Schensted algorithm (Here on referred to as the RSK algorithm).The motivation for studying this relationship is to ultimately gain an understanding of the relationship between springer fiber and the RSK algorithm which is the ultimate goal of my research.

The paper will be an expository paper of previous work done by N.Spaltenstein (on the fixed point set of the flag variety under a unipotent action) and Robert Steinberg(on the relation between the RSK algorithm and flag varieties).

The first section on the general representation theory of S_n we will give a very basic description of these irreducible representations. Using the fact that for a finite group G with irreducible representations V_i it is always true that $\sum_i dim(V_i)^2 = \mid G \mid$ we will get a counting formula that will motivate looking into the RSK algorithm. The RSK algorithm gives a constructive way to go from any element of the symmetric group to a pair of filled in standard young tableaux and back again. The ultimate goal is to explore the intimate relationship between this construction and the flag variety.

After we make explicit the RSK construction we will describe flags and how they can be acted on. In this section the basics of flags, flag varieties and actions on flag varieties will be outlined. We will describe the borel subgroup of GL_n and their relation to the flag variety. In the next section we will particularly concerned with what happens when a unipotent operator acts on the flag variety. In this section we will describe the inductive argument that shows that any fixed point of the flag manifold maps to a standard young tableaux. Given this subjective map we will then investigate the fibers and make our correspondence between irreducible components of the fixed point variety exact. The next section will then tie this all back to the RSK algorithm and who the final correspondence we want to establish.

3 Preliminaries

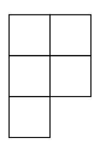
Here we will briefly define a few things we are going to use later in the paper.

P(V) denotes the projectivization of a vector space V which is the set of lines through the origin in V. If V has dimension n then the projectivization of V has dimension $n-1$.

A partition $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_p$ of n is a set of positive numbers such that $\lambda_i \geq \lambda_{i+1}$ and

$$\sum_{i=1}^{p} \lambda_i = n$$

A young diagram of shape λ is a set of left justified boxes such that row i is of length λ_i. For example when $\lambda = 2, 2, 1$ the a young diagram of shape λ is the following:



A young tableaux of shape λ is the young diagram filled in with the numbers 1 through n where $n = \sum_{i=1}^{p} \lambda_i$.

So for young diagram shown above an example of a young tableaux of that shape is:

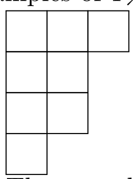
1	4
2	5
3	

A standard young tableaux is a tableaux where each row and column of the tableaux has strictly increasing values. You can verify that the above example is in fact a standard young tableaux.

4 Basic representation Theory of S_n

Here we will describe what the irreducible representations of S_n are and what the conclusions we can draw from them are.

For a given partition of size $n, \lambda = \{\lambda_1, \lambda_2, ...\lambda_p\}$ we have a Young diagram of shape λ, Y_λ and we can fill in the diagram with numbers 1 through n from left to right and top to bottom to produce a tableaux T_λ. The following are examples of Y_λ and T_λ for $\lambda = \{3, 2, 2, 1\}$:



and

1	2	3
4	5	
6	7	
8		

Then we define two subgroups as follows: P_λ is the subgroup of S_n that maps elements to other elements in the same row of T_λ and Q_λ is the subgroup of S_n that maps elements to other elements in the same column of T_λ. The intersection of these two subgroups can easily be seen to consist only of the identity element.

From these two subgroups one can define the Young projectors A_λ and B_λ as follows:

$$A_\lambda = \sum\nolimits_{g \in P_\lambda} g$$

$$B_\lambda = \sum\nolimits_{g \in Q_\lambda} (-1)^g g$$

Both of these are defined in the algebra $\mathbb{C}[S_n]$. From these two projectors we define a third operator

$$C_\lambda = A_\lambda B_\lambda$$

. Given that the only intersection of P_λ and Q_λ is the identity, the map

$$\pi : P_\lambda \times Q_\lambda \to S_n, (p, q) \to pq$$

is injective we conclude that this operator C_λ is non zero.

We then consider the space

$$\mathbb{C}[S_n]C_\lambda \subset \mathbb{C}[S_n]$$

This space is denoted V_λ and under the action of left multiplication by elements of S_n is an irreducible representation called a specht module. A proof of this fact is found in Etingofs introduction to representation theory[4]. Given that conjugacy classes of S_n can be indexed by cycle partitions of the numbers 1 through n we know that these are all the irreducible representations of S_n.

A natural question to ask is what is the dimension of V_λ. It turns out that $dimV_\lambda$ is the number of standard tableaux of shape lambda. We will not go into a proof of this here but a proof can be found in "The Symmetric Group:

Representations, Combinatorial Algorithms and Symmetric Functions" a book by Bruce E. Sagan[3].

Given that

$$\sum_\lambda dim(V_\lambda)^2 = n!$$

we can conclude that the set of pairs of standard tableaux of shape lambda has the same order as S_n. It is only natural to try to find a bijection between the two.

The RSK algorithm provides this bijection constructively.

5 The Robinson-Schensted Algorithm

We will explain the algorithm as described by Sagan. First we will provide a map ρ from the symmetric group to pairs of standard tableaux.

From a permutation σ we construct a pair of standard tableau (P, Q) through a sequence of partial tableaux pairs that converge on the final pair. A partial tableaux in this case is tableaux filled in with some subset of the numbers 1 through n that has increasing rows and columns and whose shape is a set of blocks that is a subset of the shape of P and Q. We call write the sequence as:

$$(P_0, Q_0), (P_1, Q_1), ..., (P_n, Q_n) = (P, Q)$$

P_i is defined recursively from P_{i-1} using an operation called insertion and Q_i is defined recursively from Q_{i-1} using an operation called placement. P_0 and Q_0 are empty. For a tableaux T we define shT to denote the shape of the young diagram of T. Our operations insertion and placement are defined so that $shP_i = shQ_i$ for all i.

Let us have a permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ x_1 & x_2 & x3 & \cdots & x_n - 1 & x_n \end{pmatrix}$$

We insert the x_i in order to obtain P. So P_i always contains $x_1, x_2, x_3, ...x_i$. Insertion works the following way:

When inserting x_{i+t} into P_i either x_{i+1} is greater than any element in the first row or we replace the smallest member of the first row greater than x_{i+1} with x_{i+1}. Then with the replaced element we compare it to the second row and if its greater than every element of the second row we place it at the end and if its not we replace the smallest element greater than it in the second row by it and prcede the same way starting with the third row and so on.

The following is a demonstration with

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 7 & 1 & 4 & 3 & 6 & 8 \end{pmatrix}$$

$$\begin{array}{|c|}\hline 2\\\hline\end{array} \rightarrow \begin{array}{|c|c|}\hline 2&5\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 2&5&7\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1&5&7\\\hline 2\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1&4&7\\\hline 2&5\\\cline{1-2}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1&3&7\\\hline 2&4\\\cline{1-2} 5\\\cline{1-1}\end{array} \rightarrow$$

$$\begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&4&7\\\hline 5\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 1&3&6&8\\\hline 2&4&7\\\cline{1-3} 5\\\cline{1-1}\end{array}$$

Now for Q at each step we place an element. This time we place the numbers 1 through n in chronological order. At step i we place the number i in the box at which the ith insertion terminated in the partial P tableaux. We will demonstrate the steps for constructing Q corresponding to the process for P shown above. The sequence is the following:

$$\begin{array}{|c|}\hline 1\\\hline\end{array} \rightarrow \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} \rightarrow \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array} \rightarrow$$

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&7\\\hline 6\\\cline{1-1}\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 1&2&3&8\\\hline 4&5&7\\\cline{1-3} 6\\\cline{1-1}\end{array}$$

Note that both the insert a delete process preserve the property that the tableaux at each step. This guarantees that we end up with a pair of fully filled in standard tableaux. To prove that the map $\sigma : S_n \to (P, Q)$ is indeed a bijection all we need is an inverse map from every ordered pair to an element of S_n. This inverse is very intuitive as it is literally the process of insertion and placement in reverse.

Suppose we have any ordered pair of standard tableaux (P, Q) we want to map them back to an element of S_n that would be mapped to them under the previously described insert and placement algorithm.

The process is exactly the reverse of what we just did. We look for the block where n is placed in Q and we remove n from that spot. Now we go to P and whatever number is in that corresponding spot we move it up to the row above it and replace the greatest element smaller than it by it and then take that element and have it replace the greatest element smaller than it in the row above that and so on until we remove some number from the top row. Then we go back to Q and remove $n-1$ and go to P and do the same thing as we described previously with the element in the corresponding block. At each step we remove the largest number from Q and then go to the element of P and replace the greatest element less than it in the previous row by it and so on until we remove an number from the top row of P. This is literally the reverse of insertion and placement. The ith number that is removed from the top row is $\sigma(n+1-i)$. In other words we place the numbers that leave the top in permutation in reverse order.

We will now demonstrate this by taking our pair of tableaux

$$\left(\begin{array}{|c|c|c|c|} \hline 1 & 3 & 6 & 8 \\ \hline 2 & 4 & 7 & \\ \cline{1-3} 5 & & & \\ \cline{1-1} \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 8 \\ \hline 4 & 5 & 7 & \\ \cline{1-3} 6 & & & \\ \cline{1-1} \end{array} \right)$$

and showing how to do two iterations of the process in broken down steps:

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 8 \\ \hline 4 & 5 & 7 & \\ \cline{1-3} 6 & & & \\ \cline{1-1} \end{array} \rightarrow \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 7 \\ \hline 6 & & \\ \cline{1-1} \end{array}$$

and for P we get the following backwards insertion in the following steps.

$$\begin{array}{|c|c|c|c|} \hline 1 & 3 & 6 & 8 \\ \hline 2 & 4 & 7 & \\ \cline{1-3} 5 & & & \\ \cline{1-1} \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline 1 & 3 & 6 & \\ \hline 2 & 4 & 7 & \\ \cline{1-3} 5 & & & \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|} \hline 8 \\ \hline \end{array} \rightarrow$$

The next iteration for Q goes as follows:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 7 \\ \hline 6 & & \\ \cline{1-1} \end{array} \rightarrow \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \cline{1-2} 6 & & \\ \cline{1-1} \end{array}$$

and for P we get the following backwards insertion:

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 4 & 7 \\ \hline 5 & & \\ \cline{1-1} \end{array} \rightarrow \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 4 & \\ \hline 5 & & \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|} \hline 7 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 4 & \\ \hline 5 & & \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 7 \\ \hline 2 & 4 & \\ \hline 5 & & \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|} \hline 6 \\ \hline \end{array} \rightarrow$$

So in the first iteration 8 is discarded from the first row and in the second iteration 6 is discarded from the first row. So $\sigma(8) = 8$ and $\sigma(7) = 6$. This matches up with our initial permutation.

So this process is our inverse and we have indeed established a bijection.

6 The Flag Variety and Unipotent Transformations

A flag is a sequence of vector spaces of increasing dimension incrementing by one $V = \{V_0, V_1, ...V_n\}$ such that $V_i \subset V_{i+1}$ for all i. V_0 has no dimension and V_n has dimension n. One can describe each space by an increasing set of basis vectors that spans vector spaces of increasing dimension but multiple sets of basis vectors can describe the same flag so we avoid this representation.

If we have a vector space V we can define a flag variety $F(V)$ which is the set of all flags on this vector space such that $V_n = V$.

The the general linear group clearly acts on the flag variety since it preserves dimension of vector spaces and preserves the subspace relationship between two vector spaces.

A subgroup of GL_n that is intimately connected to the flag variety is the subgroup of invertible upper triangular matrices. We will denote this subgroup by B . A Borel subgroup of of GL_n is subgroup conjugate to B.

When taking one basis representation of a flag variety to another the transformation will always be upper triangular. It turns out that the stabilizer of the flag variety under the action of GL_n is the set of upper triangular matrices since they take any set of basis for a vector space in a flag variety to another basis that spans the same space.

So if we let the flag variety be denoted by $\mathbf{F} = \mathbf{F}(V)$ then it it is clear that that F is acted on transitively by GL_n since one can construct an invertible transformation that takes any set of basis vectors to any other set of basis vectors. Thus it is clear that since $stab(\mathbf{F}) = B$

$$\mathbf{F} \simeq GL(V)/B$$

as algebraic varieties.

Now that we have some idea of the flag variety as an algebraic variety lets look at actions on it. We the fixed point set of a unipotent transformation u. We will describe how to construct correspondence as done by N.Spaltenstein in [1].

Consider standard young tableaux of size n. We let σ_i be the column number of the number i in the tableaux. Given that the tableaux is standard the sequence $\sigma = \sigma_1, \sigma_2, ..., \sigma_n$ completely determines the tableaux.

We also introduce an order on the standard tableaux as follows: If for some $j(1 < j < n)\sigma_j < \tau_j$ and $\sigma_i = \tau_i$ for all $i > j$ then we say that $\sigma < \tau$. This is a total ordering on the standard young tableaux. Let the ordered set of standard tableaux be T

First we take our unipotent matrix u and put it in jordan normal form. To u we can assign a partition of $n, \lambda = \{\lambda_1, \lambda_2, \lambda_3, ..., \lambda_p\}$ such that there are as many partitions of length i as there are jordan blocks of size i in u. Let C_i denote the length of column i.

We will denote the set of fixed points of u in the flag variety by $Y = \mathbf{F}(V)_u$.

We now have the definitions to define a surjective $\gamma : Y \to T$. It is difficult to make this map explicit straight away so we consider a flag $F = (V_0, V_1, ...V_n = V) \in Y$ and the restriction of u to $(V/V1)$ which we will denote u_1. Since $F \in Y$ we know that V_1 is generated by an eigenvector of u (obviously of eigenvalue 1) and therefore u_1 fixes $F/V_1 = (V_1/V_1, V_2/V_1, ..., V/V_1)$. Clearly the young diagram associated to u_1 has is the same as the one associated to u minus some corner corresponding to the jordan block that had a basis that spanned V_1. Then by induction on dimension we can assume that that we have a standard tableaux σ_1 of size $n-1$ associated to F/V_1. Then we know that the standard tableaux for F is the same as this one with an n in the corner that is in the young diagram of u and not in that of u_1

So this establishes that γ is indeed a surjective map.

We can look at the fibers of a given tableaux in Y. We define

$$Y_\sigma = \gamma^-1(\sigma)$$

Obviously the $Y\sigma$'s partition Y. These $Y\sigma$ as we will soon see correspond exactly to the irreducible components of Y although they aren't exactly irreducible.

To show this we define a map $p : Y \to \mathbf{P}(V)$ ($\mathbf{P}$ denoting the projectivization) such that $F \to V_1$.

We also define a flag $W = (W_0, W_1, W_2, ...W_{C_1}) \in \mathbf{F}(Ker(u-1))$. The C_1 here is indeed also referring to the number of elements in the first column which makes sense when we defines more general flag that this is one of. We let $W_{d_i} = Ker(u-1) \cap Im(u-1)^{i-1}$ for all $i > 1$. In this context our original flag $W = W_{d_1}$. This means that W_{d_i} is a vector space that is spanned by the vectors that are in the kernel of $u-1$ in jordan blocks of dimension i or greater. It makes perfect sense that its dimension is equal to the number of elements in column i of our standard tableaux.

We present the following propositions without proof. A proof of them can be found in [1].

$$1. \bigcup_{\tau \geq \sigma} Y_\tau$$

is closed in Y and Y_σ is locally closed Y.

$$2. dim Y_\sigma = \sum\nolimits_{s \geq 1} d_s(d_s - 1)/2$$

3.For every $\sigma \in S$ there is a partition of Y_σ into spaces that are isomorphic to affine spaces:

$$(X_j)_{1<j<m}$$

where

$$\bigcup_{k\leq j} X_k$$

is closed in $Y_\sigma(1<j<m)$

This shows that the Y_σ are dense open subsets in the irreducible components of Y. Due to these three propositions, in the zariski topology we have:

$\overline{\mathbb{Y}_\sigma}$ are the irreducible components.

This establishes a one to one correspondence between standard young tableaux and the irreducible components of Y.

7 Robinson-Schensted Algorithim and Flag Varieties

Here we will establish a geometric equivalence between pairs of young tableaux and pairs of flag varieties through the RSK algorithm that isn't at all obvious on the surface.

The correspondence found in the previous section is made inductively and not explicitly. Here we will give an explicit way of seeing it that is quite literally the RSK algorithm.

This will be short exposition of the ideas expressed by Robert Stienberg in [3].

We start by defining the notion of the "relative position" of two flags F and F'. The relative position of these flags is a permutation:

$$w = w(F, F')$$

This permutation has the property that if $F = (V_0, V_1, ...V_n)$ and $F' = (V'_0, V'_1, ...V'_n)$ $(V_n = V'_n = V)$ then there is a basis $v_1, ...v_n$ of V where $v_1, v_2, ...v_i$ is a basis of V_i and $v_{w_1}, v_{w_2}, ..., v_{w_j}$ is a basis of V'_j for all i and j.

This both exists and is unique but is not proved in the paper and will not be proved here.

Now there is also a permutation associated to any pair of standard tableaux (P, Q) that can be found by doing the RSK algorithm backwards as was demonstrated in the section on the RSK algorithm. we can write this permutation as $w(P, Q)$

The correspondence we want to prove is that if we have a flag manifold $\mathbf{F}(V)$ and a unipotent operator u of shape λ (this notion is defined in the previous section and is based on its jordan decomposition) and two standard tableaux P and Q corresponding to irreducible components of $\mathrm{F}(V)$ C and C' respectively by the correspondence outlined in the previous section then for any suitable (in suitable dense and open subsets of their respective irreducible components) F and F' in C and C' respectively:

$$w(F,F') = w(P,Q)$$

The exact proof of this is very clear but complicated and can be found in [3] but here we will give the intuition behind the proof and show the part where RSK shows up obviously.

Most of the desired intuition comes from the following lemma:

Let $F = (V_0, V_1, ...V_n)$ and $F' = (V'_0, V'_1, ...V'_n)$ be the two flags we were referring to earlier. We let F'_2 be a sub flag of F' terminating at V'_{n-1}. Let F_2 be the flag on V'_{n-1} such that if r is the smallest number such that V_r is not a subspace of V'_{n-1} then the subspaces in F_2 are $W_i = V_i \cap V'_{n-1}$ for all i not equal to r. We label these subspaces 1 through n with r excluded. Given these definitions the tableaux's corresponding to F_2 and F'_2 are obtained from F and F' from the first round of the RSK algorithm.

This lemma combined with induction on the number of subspaces in the flag clearly gives the intuition for our theorem since the tableaux for F' is going to lack be the tableaux for F'_2 with n placed and the tableaux for F is the tableaux for F_2 with an insertion of r corresponding to the permutation.

The proof for this in the article is literally working the RSK algorithm backwards and can be found in [3].

This is the true nature of the correspondence between pairs of flags from different irreducible components and pairs of standard tableaux corresponding to them.

8 Bibliography

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