MAXIMAL PRODUCTS OF SU(2) IN COMPACT LIE GROUPS
UROP+ FINAL PAPER SUMMER 2015

Lara Booth
Mentor: Daniel Thompson
Project Suggested by:
Professor David Vogan

Abstract. — We consider maximal products of SU(2) as subgroups of a compact Lie group G. We describe the adjoint representations of these subgroups in U(n). Maximal products of SU(2) are the smallest non-abelian analogues to maximal tori of G. We aim to study the structure of subgroups of G that are not described by the existing decomposition of maximal tori using their associated root data.

Contents

1. Introduction

Élie Cartan and Hermann Weyl introduced a powerful mathematical theory which allows us to determine a compact Lie group by its root systems and additional data. Existing structure theory relates compact Lie groups to maximal tori and their corresponding root systems. This theory provides a powerful mathematical tool for dealing with the structure of subgroups of compact Lie groups. This paper explores a parallel structure in which maximal tori are replaced by maximal products of SU(2) and U(1) in a compact Lie group.

Let G be a compact Lie group. Then G contains a maximal subgroup S that is locally isomorphic to instances of SU(2) and U(1). These subgroups S are the smallest non-abelian analogues of maximal tori of G. Explicit examples of these subgroups S in U(n) can be found in *Section 4*.

This paper works to classify these maximal subgroups of G and analyze their adjoint representations on the complexified Lie algebra $\mathfrak{g}$ of G. We want an abstract description of the subgroup and adjoint representation pair analogous to root data. This work aims to find a collection of axioms or some combinatorial structure on a set of representations of S that allow us to determine the adjoint representation of such a subgroup on $\mathfrak{gl}$. We hope that this analysis will reveal and give us some structural insight on subgroups that do not correspond to sub-root systems of G.

2. Background

Throughout, we will make frequent use of the compact Lie group SU(2). In this section we will lay out the relevant foundations.

2.1. Complexified Lie Algebras

Let $\mathfrak{g}$ be the Lie algebra of any real Lie subgroup of GL(n, $\mathbb{C}$).

Definition 2.1.1. — the *complexification* of $\mathfrak{g}$ is defined as $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$. We extend the Lie bracket on $\mathfrak{g}$ to $\mathfrak{g}_{\mathbb{C}}$ by $\mathbb{C}$-linearity.

Observation. — We can consider any n×n complex matrix in terms of its Hermitian and skew-Hermitian components $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) \oplus i\mathfrak{u}(n)$.Thus, for any G $\subseteq$ GL(n,$\mathbb{C}$) its complexified Lie algebra is $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$.

Automatically, this gives the relations $\mathfrak{su}(n)_{\mathbb{C}} = \mathfrak{su}(n) \oplus i\mathfrak{su}(n) = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{u}(n)_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$. We will use these equalities frequently.

Lemma 2.1.2. — *Let (π, V) be a representation of* $\mathfrak{g}$. *V is irreducible under* $\mathfrak{g}$ *if and only if it is irreducible under* $\mathfrak{g}_{\mathbb{C}}$.

Proof. — Observe that V must be a complex vector space. Then it is immediate that V is $\pi(\mathfrak{g})$-invariant if and only if V is $\pi(\mathfrak{g}_{\mathbb{C}})$-invariant. □

2.2. Representation Theory of $\mathfrak{sl}(2)$

It is well-known that for any compact Lie group G one can find a well-defined G-invariant positive-definite Hermitian form on any representation of G. Thus every representation of G is unitary. We can use this and induction to show that every finite-dimensional representation of a compact Lie group G is completely reducible.

Weyl used the following fact to develop his unitary trick, which tells us that any finite-dimensional representation of a complex semisimple Lie algebra is completely reducible. For any complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ there exists a real Lie algebra $\mathfrak{g}$ such that when complexified, $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}_{\mathbb{C}}$, whose simply connected form is in correspondence with the compact Lie group G. Given a representation V of $\mathfrak{g}_{\mathbb{C}}$, we can restrict V to $\mathfrak{g}$ and exponentiate to find a representation of G.

We've already seen that $\mathfrak{sl}(2)$ is the complexified Lie algebra of SU(2). We can see from the fact that SU(2) is compact (alternatively, from $\mathfrak{sl}(2)$ being semisimple) that every finite-dimensional representation of $\mathfrak{sl}(2)$ is completely reducible. We'll now look at the beautifully simple properties of the irreducible representations of $\mathfrak{sl}(2)$.

Let V be an irreducible representation of $\mathfrak{sl}(2)$.

Choose the standard basis for the lie algebra $\mathfrak{sl}(2)$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

which satisfies the relations

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H \tag{2.2.1}$$

For any semisimple Lie algebra, its representations preserve Jordan decomposition. Hence the action of the semisimple element H on V is diagonalizable. This gives us the decomposition

$$V = \bigoplus_{\alpha} V_{\alpha}$$

where the $\alpha \in \mathbb{C}$ are values such that $\forall$ v $\in$ V, H(v) $=\alpha$v for some α.

Let v $\in$ V be an eigenvector of H with eigenvalue α, so that H(v) $=\alpha$v. Then

$$\begin{aligned} H(F(v)) &= F(H(v)) + [H, F](v) \\ &= F(\alpha v) - 2F(v) \\ &= (\alpha - 2) \cdot F(v) \end{aligned}$$

so that F(v) is also an eigenvector of H with eigenvalue (α - 2). Similarly, we find E(v) is an eigenvector of H with eigenvalue (α + 2).

V is irreducible, hence we have an unbroken string of eigenvalues α_i that are all congruent mod 2: $\alpha_0, \alpha_0 + 2, \alpha_0 + 4, \ldots, \alpha_0 + 2k$.

Let n be the last element in this string, which must exist because V is finite-dimensional, and choose v $\in V_n$. That means $V_{n+2} = 0$, hence E(v) $= 0$.

Claim. — {v, F(v), F^2(v), ...} span V.

Proof. — Fulton and Harris provide a nice proof of this fact in Chapter 11[**1**]. Since V is irreducible, we need only check that the action of H, E, and F carry the subspace W generated by the span of these vectors into itself. For F and H this calculation is obvious.

E needs a little work. By working a few examples and using induction, we end up with the result

$$E(F^m(v)) = m(n - m + 1)F^{m-1}(v) \tag{2.2.2}$$

□

Corollary 2.2.3. — *All eigenspaces V_α are one-dimensional.*

We understand the action of H, E, and F on any vector in V, meaning we can completely determine V based on the number n. We use the finite-dimensionality of V to prove the existence of both upper and lower bounds on α for which $V_\alpha \neq 0$.

Let m be the smallest power of F such that v $\in V_n$ vanishes. Then, by (2.2.2),

$$\begin{aligned} & 0 = E(F^m(v)) = m(n - m + 1)F^{m-1} \\ & 0 \neq F^{m-1} \\ \implies \quad & 0 = m(n - m + 1) \end{aligned}$$

and then m $\in \mathbb{N}$ tells us that n is a non-negative integer.

We have now determined all irreducible representations of $\mathfrak{sl}(2)$. These are the unique (n+1)-dimensional representations V_n which exist for all non-negative integers n and having eigenvalues under H: $-n, 2-n, \dots, n-4, n-2, n$.

The finite-dimensional irreducible representations of products of compact Lie groups are (external) tensor products of the irreducible representations of their factors. Thus this description of finite-dimensional representations of $\mathfrak{sl}(2)$ can be extended to a direct sum of copies of $\mathfrak{sl}(2)$. The representations will be the same for a direct sum of copies of $\mathfrak{sl}(2)$ and its Lie group, a product of copies of SU(2).

2.3. Nilpotent Orbits for $\mathfrak{sl}$(n)

In tackling the classification of maximal products of SU(2), it's important to have a way of identifying such subgroups up to conjugacy. In this section we will present a method of defining conjugacy classes of SU(2) within U(n).

For any n $\in \mathbb{Z}$, $\mathcal{P}$(n) will be the set of partitions of n, up to permutation.

Example 2.3.1. — n=4, $\mathcal{P}(4)$ = {[4], [1,3], [2,2], [1,1,2], [1,1,1,1]}.

Recall that an elementary Jordan block of type k is a nilpotent endomorphism of $\mathbb{C}^k$ given by

$$J_k = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{2.3.2}$$

Given any partition $[d_1, d_2, \dots, d_k] \in \mathcal{P}(\mathrm{n})$, we can form a nilpotent matrix from the diagonal sum of elementary Jordan blocks

$$E_{[d_1,d_2,\dots,d_k]} = \begin{pmatrix} J_{d_1} & 0 & \dots & \dots & 0 \\ 0 & J_{d_2} & \dots & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & J_{d_k} \end{pmatrix} \tag{2.3.3}$$

This is a nilpotent endomorphism of $\mathbb{C}^{d_1+d_2+\dots+d_k} = \mathbb{C}^n$. Indeed $E_{[d_1,d_2,\dots,d_k]}$ is a nilpotent element of $\mathfrak{sl}(\mathrm{n})$. PSL(n) is the adjoint group for $\mathfrak{sl}(\mathrm{n})$, or the image under its adjoint representation. So we obtain the nilpotent orbit

$$\mathcal{O}_{[d_1,d_2,\dots,d_k]} = PSL_n \cdot E_{[d_1,d_2,\dots,d_k]} \tag{2.3.4}$$

Proposition 2.3.5. — *The set of nilpotent orbits of $\mathfrak{sl}(n)$ are in one-to-one correspondence with the set $|\mathcal{P}(n)|$ of partitions of n, up to permutation. The correspondence sends a nilpotent element E to the partition determined by the block size of its Jordan normal form as in (2.3.3).*

Let $\mathfrak{g}$ be any complex semisimple Lie algebra. Any semisimple subalgebra of $\mathfrak{g}$ that is isomorphic to $\mathfrak{sl}(2)$ is spanned by a standard triple {H,E,F} that satisfies the relations (2.2.1). We call these the neutral, nilpositive, and nilnegative elements of the subalgebra, respectively. H acts semisimply and both E and F are nilpotent.

We define the linear map $\rho_r : \mathfrak{sl}(2) \to \mathfrak{sl}(r+1)$

$$\rho_r(H) = \begin{pmatrix} r & 0 & 0 & \dots & 0 \\ 0 & r-2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 2-r & 0 \\ 0 & 0 & \dots & 0 & -r \end{pmatrix}$$

$$\rho_r(E) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$\rho_r(F) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \mu_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & \mu_r & 0 \end{pmatrix}$$

with $r \geq 0$, and $1 \leq i \leq r$, $\mu_r = i(r+1-i)$

The following lemma (stated without proof) is relevant.

Lemma 2.3.6. — *The map ρ_r is an irreducible representation of $\mathfrak{sl}(2)$ of dimension r+1. Indeed, for any finite-dimensional representation π of $\mathfrak{sl}(2)$ there exists some non-negative integer r such that $\rho_r \cong \pi$.*

You may notice immediately the connection between ρ_r and the irreducible representations V_n of $\mathfrak{sl}(2)$ that we encountered previously. We'll tie these theories concretely in Section 3.1.

Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{sl}$(n). By Proposition 2.3.5, there exists a partition such that $\mathcal{O} = \mathcal{O}_{[d_1,d_2,\ldots,d_k]}$.

Define another homomorphism: $\phi_{\mathcal{O}}$: $\mathfrak{sl}(2) \to \mathfrak{sl}(n)$ by $\phi_{\mathcal{O}} = \bigoplus\limits_{1\leq i\leq k} \rho_{d_{i-1}}$.

By Schur's Lemma, the image of $\phi_{\mathcal{O}}$ is either zero or isomorphic to $\mathfrak{sl}(2)$. For any $\phi_{\mathcal{O}}$, $\phi_{\mathcal{O}}$(E) = $\mathrm{E}_{[d_1,\ldots,d_k]}$. The map $\phi_{\mathcal{O}}$ is the zero-map when its image corresponds to the trivial partition [1, …,1].

Indeed, the image of a standard triple {H,E,F} under nonzero $\phi_{\mathcal{O}}$ is given by $\{\mathrm{H}_{[d_1,\ldots,d_k]}, \mathrm{E}_{[d_1,\ldots,d_k]}, \mathrm{F}_{[d_1,\ldots,d_k]}\}$.

***Theorem 2.3.7* (Jacobson-Morozov).** — *Let $\mathfrak{g}$ be any complex semisimple Lie algebra. If E is a nonzero nilpotent element of $\mathfrak{g}$ then E is the nilpositive element of a standard triple in $\mathfrak{g}$.*

Proof. — for general proof, see Sepanski Chapter 3 [**4**]. Observe this follows from Jacob normal form if $\mathfrak{g} = \mathfrak{sl}(2)$. □

***Theorem 2.3.8* (Kostant).** — *Let $\mathfrak{g}$ be any complex semisimple Lie algebra. Any two standard triples {H,E,F} and {H',E,F'} with the same nilpositive element E are conjugate under an element of the adjoint group of $\mathfrak{g}$.*

We have outlined the correspondence between distinct partitions of n and conjugacy classes of $\mathfrak{sl}(2)$ for SU(2) in U(n). We see that the conjugacy class of a particular $\mathfrak{sl}(2)$ is completely determined by its partition, since each partition corresponds to an E with a unique nilpotent orbit.

3. Methods

3.1. Determining Centralizers of $SU(2)$

Let G be a compact Lie group.

Definition 3.1.1. — A subgroup S = SU(2) $\times \ldots \times$ SU(2) if SU(2) $\not\subseteq$ G^S, the centralizer of S in G does not contain another SU(2).

Let H $\subseteq$ End(V).

Recall that the centralizer of H in End(V)^H is defined as

$$End(V)^H = \{g \in End(V) | gh = hg, \forall h \in H\}.$$

For any g $\in$ Hom$_H$(V,V), h $\in$ H we have g(hv) = h(gv) for all v $\in$ V. Hence, by definition, the centralizer of H in End(V) is

$$End(V)^H = Hom_H(V, V).$$

If instead we take H $\subseteq$ GL(V), then GL(V)^H = End(V,V)$^H\cap$ GL(V). The centralizer of H in this case is precisely the matrices in the centralizer of H for the group of endomorphisms of V that are invertible. Similarly, in the case where H $\subseteq$ U(V), then U(V)^H = End(V,V)$^H\cap$ U(V).

In Section 2.2 we identified that the irreducible representations of $\mathfrak{sl}(2)$ are exactly V$_n$. Let H = SU(2) $\subset$ U(n).

Take a representation V of H, and decompose it into isotypic irreducible components with highest weights n_i and multiplicities $r_i \in \mathbb{Z}^+$

$$V = V_{n_1-1}^{\oplus r_1} \oplus V_{n_2-1}^{\oplus r_2} \oplus \ldots \oplus V_{n_k-1}^{\oplus r_k}.$$

By Schur's Lemma, all H-linear homomorphisms $V_{n_i}^{\oplus r_i} \to V_{n_i}^{\oplus r_i}$ must be either isomorphisms or the zero map. The nonzero isomorphisms will be invertible square matrices with dimension equal to the multiplicities of the isotypic components

$$Hom_H(V,V) = Mat_{r_1 \times r_1} \times Mat_{r_2 \times r_2} \times \ldots \times Mat_{r_k \times r_k}.$$

Since H $\subset$ U(n), we take the intersection with the unitary matrices of each dimension, so the centralizer of H in U(n) is

$$U(n)^H = U(r_1) \times U(r_2) \times \ldots \times U(r_k).$$

For any subgroup of U(n) we have a direct correspondence between $\mathfrak{sl}(2)$, and thus between its Lie group SU(2), and any partition of n. We can then read off the centralizer of this SU(2) as simply a product of unitary groups with dimensions equal to the multiplicities of distinct elements in the partition of n.

Example 3.1.2. — Let n = 19, and consider the partition [2,2,3,4,4,4]. Then the corresponding SU(2) will have centralizer $U(19)^{SU(2)} = U(2) \times U(1) \times U(3)$.

3.2. Adjoint Representations of $\mathfrak{sl}(2)$ in $\mathfrak{gl}(n)$

Recall that the complexified Lie algebra of U(n) is $\mathfrak{gl}$(n), or the set of all n×n complex matrices.

Observation. — the adjoint representation $\mathfrak{gl}_n$(V) = $\mathrm{End}_{\mathbb{C}}$(V) = V $\otimes$ V*.

Consider $\mathfrak{sl}(2)$. Each finite-dimensional representation contains corresponding positive and negative one-dimensional weight spaces for each weight, determined by a part in the partition. By this natural symmetry, one can define a $\mathfrak{sl}(2)$-invariant, symmetric, non-degenerate bilinear form for $\mathfrak{sl}(2) \otimes \mathfrak{sl}(2)$. Hence the finite-dimensional representations V of $\mathfrak{sl}(2)$ are isomorphic to their dual representations V*.

Thus for a finite-dimensional representation V of $\mathfrak{sl}(2)$

$$\mathfrak{gl}_n(V) = V \otimes V.$$

Example 3.2.1. — Let n=4 and consider the partition [1,3]. Each part in the partition corresponds to a (d_i-1)-dimensional representation with highest weight d_i-1. Hence the adjoint representation of SU(2) is $V = V_2 \oplus V_0$. Then $\mathfrak{gl}_n(V) = (V_2 \oplus V_0) \otimes (V_2 \oplus V_0)$. By the Clebsch-Gordan formula (see Chapter 11 in Fulton and Harris [**3**]), this gives

$$\mathfrak{gl}_n(V) = V_4 \oplus V_2^{\oplus 3} \oplus V_0^{\oplus 2}.$$

4. Findings

Mostly using the tools that we've laid out, we have developed a method for finding the adjoint representation of maximal products of SU(2) contained within U(n). We have also made some headway in finding shortcuts that will lead us to read off the adjoint representations of SU(2) subgroups based on symmetry attached to certain automorphisms and counting dimensions.

The first step in classifying representations is to analyze the centralizers of each partition. Let Z = U(n_1)$\times \ldots \times$U(n_r) be the centralizer of a product SU(2)$\times \ldots \times$SU(2) $\subset$ U(n). Any U(n_i) in Z will contain an SU(2) with corresponding partition $[d_1, \ldots, d_k]$ with $d_j \leq n_i$ for all $j \in \{1, \ldots, k\}$. Each nontrivial partition of some n_i will form another factor of SU(2) in the product.

Carrying on in this manner, we pick out another SU(2) from Z until the centralizer of the product in U(n) no longer contains a nontrivial SU(2). So the centralizer of a maximal SU(2) subgroup is some torus U(1)$\times \ldots \times$U(1). At the end of this section, we see representations of a product of a maximal SU(2) subgroup and its centralizer. The next step is to find representatives for H,E of the standard $\mathfrak{sl}(2)$ triple corresponding to each partition. With this information, it is possible to make explicit calculations.

4.1. U(4)

TABLE 4.1.1. Maximal Products of SU(2) in U(4)

Partition(s)	Adjoint Representation
$[2,1,1] \times [1,1,2]$	$(V_2 \otimes V_0) \oplus (V_0 \otimes V_2) \oplus (V_1 \otimes V_1)^{\oplus 2} \oplus (V_0 \otimes V_0)^{\oplus 2}$
$[2,2] \times [2,2]$	$(V_2 \otimes V_2) \oplus (V_2 \otimes V_0) \oplus (V_0 \otimes V_2) \oplus (V_0 \otimes V_0)$
$[1,3]$	$V_4 \oplus V_2^{\oplus 3} \oplus V_0^{\oplus 2}$

For the rest of this section, we describe the methods used to construct this table. Use the procedures of Section 3 to obtain Table 4.1.2 for U(4).

Table 4.1.2. Representations of SU(2) in U(4)

Partition	Adjoint Representation	Centralizer
$[1,1,2]$	$V_2 \oplus V_1^{\oplus 4} \oplus V_0^{\oplus 5}$	U(2)×U(1)
$[2,2]$	$V_2^{\oplus 4} \oplus V_0^{\oplus 4}$	U(2)
[1,3]	$V_4 \oplus V_2^{\oplus 3} \oplus V_0^{\oplus 2}$	U(1)×U(1)

(*) bold indicates that this SU(2) is **maximal** in U(4).

We can immediately read off that the partitions [1,1,2] and [2,2] contain only one other SU(2) in their centralizers, which also has 2 as the highest part in its partition. The challenge now is to find representatives with which to compute the adjoint representations of the products of SU(2).

Observation. — The isomorphism formed $\mathbb{C}_2 \otimes \mathbb{C}_2 \cong \mathbb{C}_4$ induces a Kronecker product map U(2)×U(2) → U(4)

We can draw an analogy between the adjoint action of a product of instances of SU(2) on U(4) and the Kronecker product of copies of $\mathbb{C}_2$. By the symmetry of the tensor product, both instances of SU(2) must correspond to the partition [2,2] in this product.

Example 4.1.1. — Consider U(2)×U(2) → U(4).

Now let's examine the tensor product of the Lie algebra inside each factor.

Find a representative for the nilpositive element E of each SU(2) by taking the tensor product of the standard representative for $\mathfrak{sl}(2)$ corresponding to the nontrivial partition [2] with the identity in the other $\mathfrak{sl}(2)$.

$$E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Similarly, we can find H for each factor.

$$H_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$H_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We find matrices A so that $[E_1, A] = 0$. Let $a_{ij} \in \mathbb{C}$.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{11} & 0 & a_{13} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & a_{31} & 0 & a_{33} \end{pmatrix}$$

Apply H_1 to find the weights corresponding to each space (represented by some a_{ij}).

$$[H_1, A] = \begin{pmatrix} 0 & 2a_{12} & 0 & 2a_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 2a_{32} & 0 & 2a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now we find matrices A' so that both $[E_1, A'] = 0$ and $[E_2, A'] = 0$.

$$A' = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{11} & 0 & a_{13} \\ 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{11} \end{pmatrix}$$

Apply H_2 to find the weights corresponding to each remaining space.

$$[H_2, A'] = \begin{pmatrix} 0 & 2a_{12} & 2a_{13} & 0 \\ 0 & 0 & 0 & -2a_{13} \\ 0 & 0 & 0 & -2a_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To calculate the adjoint rep of this product SU(2)×SU(2) we take a direct sum of the tensor product of the corresponding weight spaces for $a_{11}, a_{12}, a_{13}, a_{14}$ in [2,2] and [2,2]

$$(V_2 \otimes V_2) \oplus (V_2 \otimes V_0) \oplus (V_0 \otimes V_2) \oplus (V_0 \otimes V_0).$$

4.1.1. Symmetry of Representations

Observe that the result of Example 4.1.1 is a representation that is symmetric about tensor products, i.e., that $\mathrm{V}_i \otimes \mathrm{V}_k = \mathrm{V}_k \otimes \mathrm{V}_i$ for representations V_i, V_k. In fact, this occurs generally in the representation of a product of conjugate copies of SU(2) (having the same nilpotent orbit).

Let G be any Lie group. Let ϕ be an automorphism G $\to$ G. If V is the representation of G then we can form the pullback representation ϕ^*V.

$$\begin{tikzcd} G \arrow[r, "\phi"] & G \\ \phi^*(V_1 \otimes V_2) \arrow[u, hook] & V_1 \otimes V_2 \arrow[u, hook] \arrow[l, dotted] \end{tikzcd}$$

As vector spaces, we have equality V $=$ ϕ^*V. However, the action of the representation is different between V and ϕ^*V. If g acts on V, $\forall v \in$ V, we have

$$g \cdot v = \phi(g)v.$$

Let ψ be an automorphism G$\times$G $\to$ G$\times$G which switches the two factors G.

Then say that $V_1 \otimes V_2$ is a representation of G$\times$G

$$h \times g \xrightarrow{\psi} g \times h \circlearrowright_{V_1 \otimes V_2}$$

Claim. — The pullback isomorphism sends $\psi^*(V_1 \otimes V_2) = V_2 \otimes V_1$.

Proof. — To prove this claim, we must investigate the action of the representation and the pullback representation given by ψ on $V_1 \otimes V_2$.

Consider the standard action

$$(h \times g) \cdot (v_1 \otimes v_2) = (hv_1 \otimes gv_2).$$

By definition

$$\psi(h \times g) \cdot (v_1 \otimes v_2) = (gv_1 \otimes hv_2).$$

There is a natural isomorphism relating $V_1 \otimes V_2 \cong V_2 \otimes V_1$. Hence

$$(gv_1 \otimes hv_2) \cong (hv_2 \otimes gv_1),$$

which gives the action of the pullback representation ψ^*

$$\psi^*(v_1 \otimes v_2) = (v_2 \otimes v_1).$$

$\square$

We now understand how the automorphism ψ (that directly switches the factors of a group product) acts on a representation of G. In general, the group of automorphisms of G acts on the set of representations of G

$$Aut(G) \hookrightarrow \{G - representations\}.$$

The reason we consider the adjoint representation in particular is that the adjoint representation $\mathfrak{g} \in$ G-reps is naturally associated to G so that it is fixed under the action of Aut(G).

We can now apply this to the case of conjugate copies of SU(2) in U(n).

Suppose we have SU(2)×SU(2) $\subset$ G, with both copies of SU(2) corresponding to the same partition.

Both SU(2) factors are conjugate, and are contained in each other's centralizers for which we have an explicit description. Therefore, there exists a natural automorphism ψ which directly switches the two SU(2) factors

$$SU(2)' \times SU(2) \xrightarrow{\psi} SU(2) \times SU(2)'.$$

The adjoint representation is fixed under the action of Aut(G), hence its restriction to the subgroup SU(2)×SU(2) is also fixed under Aut(G).

Thus, when we apply the pullback automorphism that switches the factors SU(2) in the product, the pullback representation ψ^*V must equal the adjoint representation V.

4.1.2. Adjoint Representations of Products by Dimension

Sometimes it's simple to extrapolate the adjoint representation for a maximal product of copies of SU(2) simply by using their adjoint representations and counting dimension. Generally, this method works best on copies of SU(2) with adjoint representations that have a small number of weight spaces and relatively low multiplicities.

Starting with two copies of SU(2) or products of SU(2) that correspond with each other's centralizers, write out their adjoint representations. Then begin to construct tensor products of the summands within the adjoint representation. When constructing these products, we think of the dimension of the summand on one side of the tensor product as the multiplicity of the summand on the other. The multiplicity of each summand must not exceed its multiplicity in the original adjoint representation.

Carry on in this way constructing tensor products of these summands until the direct sum of these has the correct dimension. The following is an example of this method.

Example 4.1.2. — Consider [2,1,1] and [1,1,2].

These SU(2)s are conjugate, so the representation of their product must be symmetric. Both have the same representation

$$V_2 \oplus V_1^{\oplus 4} \oplus V_0^{\oplus 5}.$$

We know immediately that the representation of the product contains at least one $V_0 \otimes V_0$ summand since we are working with the adjoint representation of the complexified Lie algebra of U(4). The adjoint representation of SU(2) for the complexified Lie algebra $\mathfrak{gl}$(n) of U(n) must contain one summand corresponding to the one-dimensional space $\mathbb{C}$ since $\mathfrak{gl}(n) = \mathfrak{sl}(n) \oplus \mathbb{C}$.

We want the dimensions of the summands in our adjoint representation to add up to 16. So far we definitely have one summand of dimension 1

$$\underbrace{V_0 \otimes V_0}_{1}.$$

Now we are left with 4 V_0, 4 V_1, and 1 V_2 on each side of the tensor product. We also know that the representation is symmetric about tensor products. Since we have only a single three-dimensional representation, this expands our representation to

$$\underbrace{V_2 \otimes V_0}_{3} \oplus \underbrace{V_0 \otimes V_2}_{3} \oplus \underbrace{V_0 \otimes V_0}_{1}.$$

We are now left with 1 V_0 and 4 V_1. The only option is

$$\underbrace{V_2 \otimes V_0}_{3} \oplus \underbrace{V_0 \otimes V_2}_{3} \oplus \underbrace{(V_1 \otimes V_1)^{\oplus 2}}_{8} \oplus \underbrace{(V_0 \otimes V_0)^{\oplus 2}}_{2}.$$

Notice this adjoint representation has the correct dimension and is symmetric about tensor products. From Table 4.1.1, we see this is in fact the correct representation for [2,1,1]×[1,1,2].

4.2. U(5)

Table 4.2.1. Maximal Products of SU(2) in U(5)

Partition(s)	Adjoint Representation
$[1,4]$	$V_6 \oplus V_4 \oplus V_3^{\oplus 2} \oplus V_2 \oplus V_0^{\oplus 2}$
$[2,3]$	$V_4 \oplus V_3^{\oplus 2} \oplus V_2^{\oplus 2} \oplus V_1^{\oplus 2} \oplus V_0^{\oplus 2}$
$[1,1,3] \times [2,1,1,1]$	$(V_4 \otimes V_0) \oplus (V_2 \otimes V_0) \oplus (V_0 \otimes V_2) \oplus (V_2 \otimes V_1)^{\oplus 2}$ $\oplus (V_0 \otimes V_0)^{\oplus 2}$
$[2,2,1] \times [1,2,2]$	$(V_2 \otimes V_2) \oplus (V_2 \otimes V_0) \oplus (V_0 \otimes V_2) \oplus (V_1 \otimes V_1)^{\oplus 2}$ $\oplus (V_0 \otimes V_0)^{\oplus 2}$
$[1,1,1,2] \times [1,2,1,1]$	$(V_2 \otimes V_0) \oplus (V_0 \otimes V_2) \oplus (V_1 \otimes V_1)^{\oplus 2} \oplus (V_1 \otimes V_0)^{\oplus 2}$ $\oplus (V_0 \otimes V_1)^{\oplus 2} \oplus (V_0 \otimes V_0)^{\oplus 3}$

The following is a table of adjoint representations of the complexified Lie algebras $\mathfrak{sl}(2)$ in complexified $\mathfrak{gl}(5)$ corresponding to the partitions of 5, using the methods of Table 4.1.2.

TABLE 4.2.2. Representations of SU(2) in U(5)

Partition	Adjoint Representation	Centralizer
$[1,1,1,2]$	$V_2 \oplus V_1^{\oplus 6} \oplus V_0^{\oplus 10}$	U(3)×U(1)
$[1,1,3]$	$V_4 \oplus V_2^{\oplus 5} \oplus V_0^{\oplus 5}$	U(2)×U(1)
$[1,2,2]$	$V_2^{\oplus 4} \oplus V_1^{\oplus 4} \oplus V_0^{\oplus 5}$	U(1)×U(2)
[1,4]	$V_6 \oplus V_4 \oplus V_3^{\oplus 2} \oplus V_2 \oplus V_0^{\oplus 2}$	U(1)×U(1)
[2,3]	$V_4 \oplus V_3^{\oplus 2} \oplus V_2^{\oplus 2} \oplus V_1^{\oplus 2} \oplus V_0^{\oplus 2}$	U(1)×U(1)

(*) bold indicates that this SU(2) is **maximal** in U(5).

In certain cases, the representatives for each $\mathfrak{sl}(2)$ triple in the product are clear. These are the cases in which the SU(2)s in our product have only one nontrivial (>1) part in their partition. For instance, the SU(2) corresponding to [3,1,1]. We can consider these cases as blocks with the block containing the nontrivial part by itself: [3] and [1,1]. These instances of SU(2) will form natural products with other SU(2)s whose nontrivial blocks replace part or all of the trivial block of the original SU(2) and replace the nontrivial block with a trivial block.

Example 4.2.1. — Consider the partitions [1,1,3] and [2,1,1,1].

Notice we can maintain the block size in both partitions, so we have blocks given by [1,1]×[2] and [3]×[1,1,1] in the product.

Starting with [1,1,3], let

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Find which matrices A satisfy $[E, A] = 0$

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & a_{15} \\ a_{21} & a_{22} & 0 & 0 & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 & a_{33} \end{pmatrix}$$

with $a_{ij} \in \mathbb{C}$

Each entry a_{ij} represents a highest weight space in the representation of this $\mathfrak{sl}(2)$. Find the corresponding weight for each entry by calculating $[H, A]$

$$[H,A] = \begin{pmatrix} 0 & 0 & 0 & 0 & 2a_{15} \\ 0 & 0 & 0 & 0 & 2a_{25} \\ 2a_{31} & 2a_{32} & 0 & 2a_{34} & 4a_{35} \\ 0 & 0 & 0 & 0 & 2a_{34} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now find which entries in A are also weight spaces for the SU(2) corresponding to [2,1,1,1] by finding which A' satisfy $[E', A'] = 0$

$$E' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ for [2,1,1,1]} \qquad A' = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & a_{15} \\ 0 & a_{11} & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & 0 & a_{33} \end{pmatrix}$$

Now

$$H' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad [H', A'] = \begin{pmatrix} 0 & 2a_{12} & 0 & 0 & a_{15} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & a_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

To calculate the adjoint rep of this product SU(2)×SU(2) we take a direct sum of the tensor product of the corresponding weight spaces for $a_{11}, a_{12}, a_{15}, a_{32}, a_{33}, a_{34}, a_{35}$ in [1,1,3] and [2,1,1,1]

$$(V_4 \otimes V_0) \oplus (V_2 \otimes V_0) \oplus (V_0 \otimes V_2) \oplus (V_2 \otimes V_1)^{\oplus 2} \oplus (V_0 \otimes V_0)^{\oplus 2}$$

4.3. U(6)

Table 4.3.1. Maximal Products of SU(2) in U(6)

Partition(s)	Adjoint Representation
$[1,5]$	$V_8 \oplus V_6 \oplus V_4^{\oplus 3} \oplus V_2 \oplus V_0^{\oplus 2}$
$[1,2,3]$	$V_4 \oplus V_3^{\oplus 2} \oplus V_2^{\oplus 4} \oplus V_1^{\oplus 4} \oplus V_0^{\oplus 3}$
$[2,4]$	$V_6 \oplus V_4^{\oplus 3} \oplus V_2^{\oplus 4} \oplus V_0^{\oplus 2}$
$[3,3] \times [2,2,2]$	$(V_4 \otimes V_2) \oplus (V_4 \otimes V_0) \oplus (V_2 \otimes V_2) \oplus (V_0 \otimes V_2) \oplus (V_0 \otimes V_0)$
$[2,1,2,1] \times [2,2,2]$	$(V_2 \otimes V_2) \oplus (V_0 \otimes V_2)^{\oplus 2} \oplus (V_1 \otimes V_2)^{\oplus 2} \oplus (V_2 \otimes V_0)$ $\oplus (V_1 \otimes V_0)^{\oplus 2} \oplus (V_0 \otimes V_0)^{\oplus 2}$
$[3,1,1,1] \times [1,1,1,3]$	$(V_4 \otimes V_0) \oplus (V_0 \otimes V_4) \oplus (V_2 \otimes V_2)^{\oplus 2} \oplus (V_2 \otimes V_0)$ $\oplus (V_0 \otimes V_2) \oplus (V_0 \otimes V_0)^{\oplus 2}$
$[1,1,1,1,2] \times [4,1,1]$	$(V_6 \otimes V_0) \oplus (V_4 \otimes V_0) \oplus (V_3 \otimes V_1)^{\oplus 2} \oplus (V_2 \otimes V_0)$ $\oplus (V_0 \otimes V_2) \oplus (V_0 \otimes V_0)^{\oplus 2}$
$[1,1,1,1,2] \times [1,1,2,1,1]$ $\times [2,1,1,1,1]$	$(V_2 \otimes V_0 \otimes V_0) \oplus (V_0 \otimes V_2 \otimes V_0) \oplus (V_0 \otimes V_0 \otimes V_2)$ $\oplus (V_1 \otimes V_1 \otimes V_0)^{\oplus 2} \oplus (V_1 \otimes V_1 \otimes V_1)^{\oplus 2} \oplus (V_0 \otimes V_1 \otimes V_1)^{\oplus 2}$ $\oplus (V_0 \otimes V_0 \otimes V_0)^{\oplus 3}$
$[2,2,1,1] \times [1,1,2,2]$ $\times [2,1,1,1,1]$	$(V_2 \otimes V_2 \otimes V_0) \oplus (V_0 \otimes V_0 \otimes V_2) \oplus (V_2 \otimes V_0 \otimes V_0)$ $\oplus (V_0 \otimes V_2 \otimes V_0) \oplus (V_1 \otimes V_1 \otimes V_1)^{\oplus 2} \oplus (V_0 \otimes V_0 \otimes V_0)^{\oplus 2}$

The following is a table of adjoint representations of the complexified Lie algebras $\mathfrak{sl}(2)$ in complexified $\mathfrak{gl}(6)$ corresponding to the partitions of 6, using the methods of Table 4.1.2.

Table 4.3.2. Representations of SU(2) in U(6)

Partition	Adjoint Representation	Centralizer
$[1,1,1,1,2]$	$V_2 \oplus V_1^{\oplus 8} \oplus V_0^{\oplus 17}$	U(4)×U(1)
$[1,1,1,3]$	$V_4 \oplus V_2^{\oplus 7} \oplus V_0^{\oplus 10}$	U(3)×U(1)
$[1,1,4]$	$V_6 \oplus V_4 \oplus V_3^{\oplus 4} \oplus V_2 \oplus V_0^{\oplus 5}$	U(2)×U(1)
[1,5]	$V_8 \oplus V_6 \oplus V_4^{\oplus 3} \oplus V_2 \oplus V_0^{\oplus 2}$	U(1)×U(1)
$[1,1,2,2]$	$V_2^{\oplus 4} \oplus V_1^{\oplus 8} \oplus V_0^{\oplus 8}$	U(2)×U(2)
[1,2,3]	$V_4 \oplus V_3^{\oplus 2} \oplus V_2^{\oplus 4} \oplus V_1^{\oplus 4} \oplus V_0^{\oplus 3}$	U(1)×U(1)×U(1)
$[2,2,2]$	$V_2^{\oplus 9} \oplus V_0^{\oplus 9}$	U(3)
[2,4]	$V_6 \oplus V_4^{\oplus 3} \oplus V_2^{\oplus 4} \oplus V_0^{\oplus 2}$	U(1)×U(1)
$[3,3]$	$V_4^{\oplus 4} \oplus V_2^{\oplus 4} \oplus V_0^{\oplus 4}$	U(2)

(*) bold indicates that this SU(2) is **maximal** in U(5).

4.4. Adjoint Representations of Products of SU(2) with U(1)

Let G = U(n). A maximal product S = SU(2)× ... ×SU(2) ⊂ G will have some torus as its centralizer

$$G^S = U(1) \times \ldots \times U(1).$$

We may want to find the adjoint representation of products of S with the torus G^S. Many products of SU(2) have a trivial centralizer G^S = U(1) which is just n×n scalar matrices. However, some will have a centralizer that is a product U(1)× ... ×U(1). In these cases, the U(1) are blockwise scalar matrices with each U(1) corresponding to a part in the associated partitions of S.

Example 4.4.1. — Consider the partition [1,2,3].

The centralizer of the corresponding SU(2) in U(6) is U(1)×U(1)×U(1).

The matrix of the centralizer is $Z = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{pmatrix}$

To find the adjoint representation of S×G^S, we can make explicit calculations like those used to find the adjoint representations of products of SU(2). We write the Lie algebra of G^S explicitly exactly as we describe its torus in matrix form since each U(1) is just described by some scalar.

Then we examine the adjoint action of the Lie algebra of the centralizer on the weight spaces of the Lie algebra of S. Tori are abelian, hence the representations of each U(1) will be one-dimensional.

The resulting adjoint representation of S×G^S will be an (external) tensor product of the isotypic component of the weight spaces which are acted on by the centralizer with some one-dimensional representation $\mathbb{C}_{(w_1,\ldots,w_k)}$. Each factor U(1)× ... ×U(1) acts on $\mathbb{C}_{(w_1,\ldots,w_k)}$ with weight $w_i \in \{-1, 0, 1\}$, respectively.

We use the method from Example 4.4.2, which follows, to construct the table below. The table shows the representations of products of several maximal SU(2) subgroups S in G = U(n) and their centralizers. We let n be the sum of the parts in the respective partitions.

TABLE 4.4.1. Some Representations of S$\times$G^S in G

Partition	Adjoint Representation
$[1,3]$	$V_4 \oplus V_2 \oplus (V_2 \otimes \mathbb{C}_{(1,-1)}) \oplus (V_2 \otimes \mathbb{C}_{(-1,1)}) \oplus V_0^{\oplus 2}$
$[1,4]$	$V_6 \oplus V_4 \oplus (V_3 \otimes \mathbb{C}_{(1,-1)}) \oplus (V_3 \otimes \mathbb{C}_{(-1,1)}) \oplus V_2 \oplus V_0^{\oplus 2}$
$[2,3]$	$V_4 \oplus (V_3 \otimes \mathbb{C}_{(1,-1)}) \oplus (V_3 \otimes \mathbb{C}_{(-1,1)}) \oplus V_2^{\oplus 2} \oplus (V_1 \otimes \mathbb{C}_{(1,-1)})$ $\oplus (V_1 \otimes \mathbb{C}_{(-1,1)}) \oplus V_0^{\oplus 2}$
$[1,5]$	$V_8 \oplus V_6 \oplus V_4 \oplus (V_4 \otimes \mathbb{C}_{(1,-1)}) \oplus (V_4 \otimes \mathbb{C}_{(-1,1)}) \oplus V_2 \oplus V_0^{\oplus 2}$
$[2,4]$	$V_6 \oplus (V_4 \otimes \mathbb{C}_{(1,-1)}) \oplus (V_4 \otimes \mathbb{C}_{(-1,1)}) \oplus V_4 \oplus V_2^{\oplus 2} \oplus (V_2 \otimes \mathbb{C}_{(1,-1)})$ $\oplus (V_2 \otimes \mathbb{C}_{(-1,1)}) \oplus V_0^{\oplus 2}$
$[1,2,3]$	$V_4 \oplus (V_3 \otimes \mathbb{C}_{(0,1,-1)}) \oplus (V_3 \otimes \mathbb{C}_{(0,-1,1)}) \oplus V_2^{\oplus 2} \oplus (V_2 \otimes \mathbb{C}_{(1,0,-1)})$ $\oplus (V_2 \otimes \mathbb{C}_{(-1,0,1)}) \oplus (V_1 \otimes \mathbb{C}_{(1,-1,0)}) \oplus (V_1 \otimes \mathbb{C}_{(-1,1,0)})$ $\oplus (V_1 \otimes \mathbb{C}_{(0,1,-1)}) \oplus (V_1 \otimes \mathbb{C}_{(0,-1,1)}) \oplus V_0^{\oplus 3}$

Example 4.4.2. — Consider the partition [1,2,3] from Example 4.4.1.

The elements of the following matrix, A, display the weight spaces of the corresponding $\mathfrak{sl}(2)$, with the weights of each space given as a superscript

$$\begin{pmatrix} a_{11}^0 & 0 & a_{13}^1 & 0 & 0 & a_{16}^2 \\ a_{21}^1 & a_{22}^0 & a_{23}^2 & 0 & a_{25}^1 & a_{26}^3 \\ 0 & 0 & a_{22}^0 & 0 & 0 & a_{25}^1 \\ a_{41}^2 & a_{42}^1 & a_{43}^3 & a_{44}^0 & a_{45}^2 & a_{46}^4 \\ 0 & 0 & a_{42}^1 & 0 & a_{44}^0 & a_{45}^2 \\ 0 & 0 & 0 & 0 & 0 & a_{44}^0 \end{pmatrix}$$

Then the action of the centralizer on A, [Z,A], gives

$$\begin{pmatrix} 0 & 0 & (\lambda_1-\lambda_2)a_{13}^1 & 0 & 0 & (\lambda_1-\lambda_3)a_{16}^2 \\ (\lambda_2-\lambda_1)a_{21}^1 & 0 & 0 & 0 & (\lambda_2-\lambda_3)a_{25}^1 & (\lambda_2-\lambda_3)a_{26}^3 \\ 0 & 0 & 0 & 0 & 0 & (\lambda_2-\lambda_3)a_{25}^1 \\ (\lambda_3-\lambda_1)a_{41}^2 & (\lambda_3-\lambda_2)a_{42}^1 & (\lambda_3-\lambda_2)a_{43}^3 & 0 & 0 & 0 \\ 0 & 0 & (\lambda_3-\lambda_2)a_{42}^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Z acts on the weight spaces $a_{13}, a_{16}, a_{21}, a_{25}, a_{26}, a_{41}, a_{42}, a_{43}$, resulting in the adjoint representation

$$\mathrm{V}_4 \oplus (V_3 \otimes \mathbb{C}_{(0,1,-1)}) \oplus (V_3 \otimes \mathbb{C}_{(0,-1,1)}) \oplus V_2^{\oplus 2} \oplus (V_2 \otimes \mathbb{C}_{(1,0,-1)}) \oplus (V_2 \otimes \mathbb{C}_{(-1,0,1)})$$
$$\oplus (V_1 \otimes \mathbb{C}_{(1,-1,0)}) \oplus (V_1 \otimes \mathbb{C}_{(-1,1,0)}) \oplus (V_1 \otimes \mathbb{C}_{(0,1,-1)}) \oplus (V_1 \otimes \mathbb{C}_{(0,-1,1)}) \oplus V_0^{\oplus 3}$$

Compare this result to the representation for the partition [1,2,3] in Table 4.3.2. Notice that the addition of the torus into the adjoint representation breaks up the multiplicities of some of its isotypic components. Also note that the addition of the centralizer cannot alter the representations of the trivial weight spaces along the diagonal.

5. Further Cases

Similar classifications can be made for all of the classical Lie groups - the orthogonal, unitary, and quaternion groups and their compact subgroups.

Let G be any compact classical Lie group, with semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}} \subseteq \mathfrak{gl}(V)$.

Then $\mathfrak{sl}(2) \subset \mathfrak{g}_{\mathbb{C}}$. This $\mathfrak{sl}(2)$ is attached to a set of partitions that are in correspondence with nilpotent orbits. These partitions differ somewhat from the partitions in the case of U(n), but are not of considerably greater complexity to handle. See Collingwood-McGovern[**1**] Chapter 5 for a breakdown of the partitions for the classical Lie groups.

The combinatorial structure attached to a partition will tell us how to decompose a representation V of $\mathfrak{sl}(2)$ into its isotypic components. We then automatically have the adjoint representation of $\mathfrak{gl}$(V).

Since G is contained in GL(V), there exists some involution τ such that $\mathfrak{g} = (\mathfrak{gl})^{\tau}$.

We have the isotypic decomposition for the adjoint representation of $\mathfrak{gl}$(V)

$$\mathfrak{gl}(V) = \bigoplus_{i \geq 0} V_{n_i}^{r_i}.$$

We can then explicitly check which V$_{n_i}$ are fixed by τ, giving us the adjoint representation of $\mathfrak{sl}(2)$ on G. Theoretically, we should be able to classify these subgroups for all classical compact Lie groups and their products in this manner.

We mentioned earlier that products of SU(2) are the smallest nonabelian analogue of maximal tori in a compact Lie group. The paper "Finite Maximal Tori" by Han and Vogan[**2**] provides a concise explanation of the existing structure theory as well as their own interesting developments on the theory of finite maximal tori.

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August 2015

Lara Booth, Massachusetts Institute of Technology • *E-mail :* lcbooth@mit.edu