EQUIVARIANT LINEAR ISOMETRIES OPERADS OVER NON-ABELIAN p-GROUPS

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1. Introduction

In stable homotopy theory, spaces or spectra are often equipped with a multiplication operation that satisfies commutativity or associativity only up to homotopy. These properties are best described with an *operad*. An important example is an E_∞-operad, which encodes an operation that is commutative and associative up to all higher homotopies. Blumberg and Hill have studied the generalization of E_∞-operads to the equivariant setting. They define N_∞-operads [BH15], which is the equivariant analog of E_∞-operads. We will define N_∞-operads in Section 2.1.

A fundamental result in stable homotopy theory is that all E_∞-operads are equivalent. However, this is not true for N_∞-operads. Blumberg and Hill found a map from N_∞-operads over a group G to *indexing systems* on G [BH15]. Various authors including [BBR21] and [Rub21] simplified indexing systems into a combinatorial object called *transfer systems*, and have proved that this map is in fact a correspondence[1].

There are many geometric models of the E_∞-operad, such as the little cubes operad and the linear isometries operad. It does not matter which model of E_∞-operad one chooses, as all E_∞-operads are equivalent. For N_∞-operads over a group G, we can similarly define the Steiner operad (which is a better-behaved version of the little cubes operad) and the linear isometries operad for any G-universe U. This will be defined in Section 2.2. These operads arise naturally in geometric settings. For example, similar to how the E_∞ little cubes operad acts on the infinite loop space, the Steiner operad also acts naturally on equivariant infinite loop spaces. The equivariant linear isometries operad are used to study the multiplicative structures on G-spectra [BH15].

Even though we have classified the equivalence classes of N_∞-operads with transfer systems, it is hard to tell a geometric property from the combinatorial data. For example, given a transfer system, can we identify if it comes from a Steiner operad or a linear isometries operad? For linear isometries operads, a necessary criterion called *saturation* was found in [BH15]. Rubin asked if the converse is true, i.e., whether all saturated transfer systems over a group are realized by a linear isometries operad. Rubin found that it is true for cyclic groups of order p^n and of order pq when $p, q > 3$ [Rub21], and MacBrough [Mac23] identified for which abelian groups it is true. Our goal is to investigate this question for non-abelian p-groups.

[1] technically speaking, an equivalence of the homotopy categories

2. Basic Notions

2.1. **N_∞-operads and transfer systems.** We will give some basic definitions in this section. Before defining an equivariant operad, we will start with a non-equivariant operad.

Definition 2.1. A *(symmetric) operad* $\mathcal{O}$ is a sequence of spaces $\mathcal{O}_0, \mathcal{O}_1, \dots$, where $\mathcal{O}_n$ has an action of the symmetric group Σ_n, together with the data

(i) composition functions $\mathcal{O}_k \times (\mathcal{O}_{n_1} \times \cdots \times \mathcal{O}_{n_k}) \to \mathcal{O}_{n_1+\cdots+n_k}$,
(ii) unit $* \to \mathcal{O}_1$

that satisfy certain coherence axioms.

Definition 2.2. An E_∞*-operad* is an operad $\mathcal{O}$ such that Σ_n acts freely on $\mathcal{O}_n$, and all $\mathcal{O}_n$ are contractible.

Think of $\mathcal{O}$ as acting on some space X. Each point in $\mathcal{O}_n$ is an n-ary operation $X^n \to X$, and the operations can be composed with the composition functions. The unit is the identity $\mathrm{id}_X : X \to X$. Then, in an E_∞-operad, all n-ary operations are homotopy equivalent. This is (loosely speaking) why an E_∞-operad describes an operation associative and commutative up to all higher homotopies.

We would like to generalize this to the equivariant setting. That is, adding in the action of a group G.

Definition 2.3. A G*-operad* $\mathcal{O}$ is a sequence of spaces $\mathcal{O}_n$ with a $G \times \Sigma_n$ action, together with a G-fixed unit element and G-equivariant composition functions.

The analog of an E_∞-operad is an N_∞-operad, defined by Blumberg and Hill [BH15].

Definition 2.4. An N_∞*-operad* is a G-operad $\mathcal{O}$ such that

- the action of Σ_n on $\mathcal{O}_n$ is free,
- the fixed space $\mathcal{O}_n^\Gamma$ is empty or contractible for every $\Gamma \subseteq G \times \Sigma_n$,
- $\mathcal{O}_n^G$ is non-empty for every n.

All E_∞-operads are equivalent, but Blumberg and Hill [BH15] showed that there are inequivalent N_∞-operads by defining a functor from the category of N_∞-operads to the poset of *indexing systems* on G. They conjectured that this functor is an equivalent of homotopy categories. Rubin [Rub21] defined a combinatorial object called *transfer systems*, which are equivalent to indexing systems, and proved the conjecture by Blumberg and Hill. Thus, the N_∞-operads on a group G can be understood via the transfer systems on G. We will denote by $\mathrm{Tr}(\mathcal{O})$ the transfer system corresponding to $\mathcal{O}$.

Definition 2.5. A *transfer system* on a group G is a partial relation $\to$ on the subgroups of G, such that

- $K \to H$ only if K is a subgroup of H,
- if $K \to H$, then for any $g \in G$, $gKg^{-1} \to gHg^{-1}$,
- if $K \to H$, then for any subgroup L of H, we have $L \cap K \to L$.

Definition 2.6. A transfer system is *saturated* if whenever $K \to H$ and $K \subseteq L \subseteq H$, we have $L \to H$. (We also have $K \to L$ by the third condition in Definition 2.5.)

2.2. **Linear isometries operad.** The Steiner operad $\mathcal{K}(U)$ and the linear isometries operad $\mathcal{L}(U)$ are two important examples of N_∞-operads in equivariant homotopy theory. They are defined for a G-universe U. We will only define the linear isometries operad here. Both of their definitions can be fonud in [BH15].

From now on, let G be a finite group. Let $\mathrm{Irr}(G)$ denote the set of irreducible $\mathbb{C}$-representations of G, and let 1 denote the trivial representation.

Definition 2.7. A *G-universe* U is an infinite dimensional real representation of G such that each finite dimensional subrepresentation of U appears as a direct summand infinitely many times, and U has a nonzero point fixed by every element of G.

Let $S_U \subseteq \mathrm{Irr}(G)$ be the set of irreducible representations that embed in $U \otimes_{\mathbb{R}} \mathbb{C}$. Then U is given by

$$U = \left(\bigoplus_{V \in S_U} V \right)^\infty .$$

Thus, a G-universe U corresponds to a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant subset $S_U \subseteq \mathrm{Irr}(G)$ that contains 1. Let $\mathcal{U}_G$ be the set of all such subsets, and $\mathcal{V}_G$ be the set of all subsets of $\mathrm{Irr}(G)$. If H is a subgroup of G and $S \in \mathcal{V}_G$, then

$$\mathrm{Res}^G_H \left(\bigoplus_{V \in S} V \right)^\infty = \left(\bigoplus_{W \in T} W \right)^\infty$$

for some $T \in \mathcal{V}_H$. We will denote the map $\mathcal{V}_G \to \mathcal{V}_H, S \mapsto T$ with res^G_H. Likewise, Ind^G_H denotes the usual induction of representations, while ind^G_H denotes the map $\mathcal{V}_H \to \mathcal{V}_G$.

Definition 2.8. The *linear isometries operad* $\mathcal{L}(U)$ is given by $\mathcal{L}(U)_n = \mathcal{L}(U^{\oplus n}, U)$, where $\mathcal{L}(U^{\oplus n}, U)$ is the space of (not necessarily G-equivariant) linear isometries from $U^{\oplus n}$ to U. G acts on $\mathcal{L}(U^{\oplus n}, U)$ by changing coordinates on both $U^{\oplus n}$ and U, and Σ_n acts by permuting the summands in $U^{\oplus n}$.

The transfer system that corresponds to a linear isometries operad is computed in [BH15].

Theorem 2.9. *Let U be a G-universe. Then $K \to H$ in $\mathrm{Tr}(\mathcal{L}(U))$ if and only if there is an H-equivariant embedding*

$$\mathrm{Ind}^H_K \mathrm{Res}^G_K U \to \mathrm{Res}^G_H U.$$

The existence of this embedding can be checked on each irreducible subrepresentation. In other words, this becomes a condition on the subset S_U of $\mathrm{Irr}(G)$, namely

$$\mathrm{ind}^H_K \mathrm{res}^G_K S_U \subseteq \mathrm{res}^G_H S_U.$$

2.3. **The saturation conjecture.** The homotopy equivalence classes of N_∞-operads are classified by the transfer systems. On the other hand, the N_∞-operads that appear naturally usually arise from a geometrical context, such as the Steiner operad and the linear isometries operad. It is then natural to ask: which transfer systems correspond to a Steiner operad or a linear isometries operad? The question for Steiner operads has been answered by Rubin [Rub21]. For linear isometries operads, Rubin found the following necessary condition.

Theorem 2.10. *The transfer system* $\mathrm{Tr}(\mathcal{L}(U))$ *is saturated for any universe* U.

Rubin [Rub21] found that this condition is sufficient for cyclic groups of order p^n and for cyclic groups of order pq when $p, q > 3$. They also found that there are saturated transfer systems not realized by linear isometries operads for some small non-abelian groups such as Σ_3 and Q_8. Rubin conjectured that for abelian groups whose prime divisors of the order are sufficiently large, all saturated transfer systems correspond to a linear isometries operad. More generally, we will say that a group G *satisfies the saturation conjecture* if all saturated transfer systems on G correspond to a linear isometries operad.

MacBrough [Mac23] showed that the saturation conjecture holds for cyclic groups whose order is coprime to 6 and for rank 2 abelian groups whose prime divisors of the order are very large. They also proved that the saturation conjecture is false for abelian groups of rank at least 3.

In the following sections, we will consider the case for a non-abelian p-group.

3. Non-abelian p-groups

From now on, let G be a non-abelian p-group, where p is an odd prime. First, we state a direct consequence of Theorem 2.9, which will be repeatedly throughout this section.

Lemma 3.1. *The arrow* $1 \to H$ *is in* $\mathrm{Tr}(\mathcal{L}(U))$ *if and only if* $\mathrm{res}^G_H S_U = \mathrm{Irr}(H)$.

Proof. By Theorem 2.9, if the arrow $1 \to H$ is in the transfer system, then $\mathrm{res}^G_H S_U \supseteq \mathrm{ind}^H_1 \mathrm{res}^G_1 S_U$. Clearly $\mathrm{res}^G_1 S_U = \{1\}$, and $\mathrm{ind}^H_1 \{1\} = \mathrm{Irr}(H)$, so this is true if and only if $\mathrm{res}^G_H S_U \supseteq \mathrm{Irr}(H)$. □

Our goal is to find a saturated transfer system not realized by a linear isometries operad. Let's demonstrate this with an example.

Example. Suppose that $G = \langle x, y \mid x^{p^2} = y^p = 1, yxy^{-1} = x^{p+1}\rangle$. This is one of the non-abelian groups of order p^3 known as the extraspecial group of order p^3. Its center is $Z = \langle x^p\rangle$, which is also its commutator subgroup. The center has order p, and by the class equation, there are $p^2 - 1$ conjugacy classes of size p besides the center. There are p^2 characters of G of dimension 1 and $p - 1$ characters of dimension p. The p-dimensional characters evaluate to $p\zeta_p^i$ for some integer i at $g \in Z$, and evaluate to 0 at $g \notin Z$.

Now, suppose that the transfer system of a linear isometries operad $\mathcal{L}(U)$ contains the arrow $1 \to Z$. Then, by the previous lemma, $\mathrm{res}^G_Z S_U = \mathrm{Irr}(Z)$. S_U has to contain a p-dimensional representation, since the restrictions of the 1-dimensional representations on Z are trivial. Now, consider the arrow $1 \to \langle y\rangle$. From the character table, it's not hard to see that the restriction of any of the p-dimensional representation on $\langle y\rangle$ is the regular representation. Thus, $\mathrm{res}^G_{\langle y\rangle} S_U = \mathrm{Irr}(\langle y\rangle)$, which means the arrow $1 \to \langle y\rangle$ is in the transfer system as well. This shows that the transfer system containing only the arrow $1 \to \langle x^p\rangle$ cannot be realized by a linear isometries operad.

In the general case, let G' be the commutator subgroup of G. Let $\mathrm{Irr}_1(G) \subseteq \mathrm{Irr}(G)$ be the subset consisting of the 1-dimensional representations, and $\mathrm{Irr}_{>1}(G) = \mathrm{Irr}(G) \setminus \mathrm{Irr}_1(G)$.

Lemma 3.2. *Suppose that G' is abelian. Let H be an abelian subgroup of G such that $G' \subseteq H$ and $[G : H] = p$. If $\operatorname{res}^G_H S = \operatorname{Irr}(H)$, then $S \supseteq \operatorname{Irr}_{>1}(G)$.*

Proof. Suppose $|G| = p^n$ and $|G'| = p^m$. Then $|\operatorname{Irr}_1(G)| = |G/G'| = p^{n-m}$, and

$$\sum_{V \in \operatorname{Irr}_{>1}(G)} d_V^2 = p^n - p^{n-m},$$

where d_V is the dimension of V.

Since H is abelian, the irreducible representations of H are all 1-dimensional characters, and there are $|H| = p^{n-1}$ characters of H in total. For each H-character χ, let V be an irreducible subrepresentation of $\operatorname{Ind}^G_H \chi$. Then $\langle \chi, \operatorname{Res}^G_H V\rangle = \langle \operatorname{Ind}^G_H \chi, V\rangle > 0$, so χ embeds in $\operatorname{Res}^G_H V$ for some $V \in \operatorname{Irr}(G)$.

The characters of H that embed in $\operatorname{Res}^G_H V$ for some $V \in \operatorname{Irr}_1(G)$ must be equal to $\operatorname{Res}^G_H V$. These characters are 1 on G', so they correspond to characters of H/G'. Thus, there are $|H/G'| = p^{n-1-m}$ such characters. This leaves $p^{n-1} - p^{n-1-m}$ remaining characters of H. They must all embed in $\bigoplus_{V \in \operatorname{Irr}_{>1}(G)} \operatorname{Res}^G_H V$. Comparing the dimensions gives

$$\sum_{V \in \operatorname{Irr}_{>1}(G)} d_V \geq p^{n-1} - p^{n-1-m}.$$

Notice that d_V divides $|G| = p^n$ and is greater than 1, so $d_V \geq p$. This shows that equality must be reached. Each irreducible representation $V \in \operatorname{Irr}_{>1}(G)$ must be p-dimensional, so $\operatorname{Res}^G_H V$ factors as a direct sum of p characters. Each character of H that is nontrivial on G' appear in a factorization of $\operatorname{Res}^G_H V$ for some V exactly once. Thus, since $\operatorname{res}^G_H S$ contains all these characters, we get $S \supseteq \operatorname{Irr}_{>1}(G)$. □

Lemma 3.3. *Let K be a subgroup of G such that $K \not\supseteq G'$. Then $\operatorname{res}^G_K \operatorname{Irr}_{>1}(G) = \operatorname{Irr}(K)$.*

Proof. Let χ be a character of K. We will show that $\chi \in \operatorname{res}^G_K \operatorname{Irr}_{>1}(G)$.

If χ is not 1-dimensional, or if χ is 1-dimensional but is not trivial on $K \cap G'$, then let V be any irreducible subrepresentation of $\operatorname{Ind}^G_K \chi$. By Frobenius reciprocity, $\langle \chi, \operatorname{Res}^G_K V\rangle = \langle \operatorname{Ind}^G_K \chi, V\rangle > 0$, so χ is a subrepresentation of $\operatorname{Res}^G_K V$. If V is 1-dimensional, χ would also be 1-dimensional and is equal to $\operatorname{Res}^G_K V$, so χ is trivial on G', which is a contradiction. Thus, $V \in \operatorname{Irr}_{>1}(G)$, so $\operatorname{res}^G_K \operatorname{Irr}_{>1}(G) \supseteq \operatorname{res}^G_K(\{V\}) \ni \chi$.

If χ is 1-dimensional and is trivial on $K \cap G'$, it induces a character $\overline{\chi}$ on $K/(K \cap G') = KG'/G'$. Again, if a 1-dimensional representation of G is a subrepresenation of $\operatorname{Ind}^G_K \chi$, then it has to restrict to χ on K. Let's count the number of 1-dimensional representations of G that restricts to χ on K. Since 1-dimensional representations of G are just characters of G/G', this is the same as the number of characters of G/G' that restricts to $\overline{\chi}$ on KG'/G'. This is $[G/G' : KG'/G'] = [G : KG'] < [G : K]$. On the other hand, $\dim \operatorname{Ind}^G_K \chi = [G : K]$, which shows that $\operatorname{Ind}^G_K \chi$ has a irreducible subrepresentation V that is not 1-dimensional. Again, we have $\operatorname{res}^G_K \operatorname{Irr}_{>1}(G) \supseteq \operatorname{res}^G_K(\{V\}) \ni \chi$.

Thus, we have shown that $\operatorname{res}^G_K \operatorname{Irr}_{>1}(G) \supseteq \operatorname{Irr}(K)$. Clearly the opposite inclusion holds, so the two sets are equal. □

Combining the previous lemmas, we get a criterion for which the saturation conjecture fails.

Theorem 3.4. *If there are subgroups H, K of G such that*

- *G' and H are abelian,*
- *$[G : H] = p$,*
- *$G' \subseteq H$,*
- *$G' \nsubseteq K$ and $H \nsupseteq K$,*

then there is a saturated transfer system of G not realized by any linear isometries operad.

Proof. Consider the transfer system that has the arrow $L_1 \to L_2$ if and only if $L_1 \subseteq L_2 \subseteq H$ or $L_1 = L_2$. It is not hard to check that this is a saturated transfer system (for the closure under conjugation, notice that H is normal since $[G : H] = p$). In particular, this contains the arrow $1 \to H$ and does not contain $1 \to K$. Suppose this transfer system corresponds to the linear isometries operad $\mathcal{L}(U)$. By Lemma 3.1, $\operatorname{res}^G_H S_U = \operatorname{Irr}(H)$, so by Lemma 3.2, $S_U \supseteq \operatorname{Irr}_{>1}(G)$. Then by Lemma 3.3, $\operatorname{res}^G_H S_U \supseteq \operatorname{Irr}(K)$, and by Lemma 3.1 again, $1 \to K$ is in the transfer system that corresponds to $\mathcal{L}(U)$, which is a contradiction. □

With Theorem 3.4, we have reduced disproving the saturation conjecture to finding certain subgroups. For example, we have the following result.

Corollary 3.5. *If all proper subgroups of G are abelian, then there is a saturated transfer system of G not realized by a linear isometries operad.*

Proof. We will use a proposition about the subgroup structure of abelian groups. Let A be an abelian group and B be a subgroup of A, and suppose $|A| = p^n$. If A is not cyclic, then for every $0 < m < n$, there exists a subgroup C of order p^m such that $B \nsubseteq C$ and $C \nsubseteq B$. This proposition is easy to check with the structure theorem of abelian groups.

Recall that for a p-group in general, if G/G' is cyclic, then G is cyclic. In our case, this means G/G' is not cyclic. Another known result in group theory is that if all maximal subgroups of a p-group G is cyclic, where p is an odd prime, then G is cyclic. In our case, we can pick a subgroup $G' \subsetneq A \subsetneq G$, such that A is not cyclic. Applying the proposition above to A, we can find a subgroup $K \subseteq A$ such that $G' \nsubseteq K$ and $K \nsubseteq G'$.

By picking K this way, we can ensure that K and G' do not generate G. Then, we can use the proposition again on G/G' to find an index-p subgroup that does not contain KG'/G'. This corresponds to an index p subgroup of G containing G', which can be chosen as H. □

4. Further Questions

The groups that satisfy the conditions in Theorem 3.4 and Corollary 3.5 are quite rare. However, if the answer to the following question is true, then we can show that the saturation conjecture fails for all non-abelian p-groups. This is because even though a general non-abelian p-group G does not satisfy the condition of Corollary 3.5, it has a subgroup that does.

Question 4.1. Suppose the saturation conjecture is true for a group G. Is it necessarily true for its normal subgroup H? In particular, is it true for its normal subgroups of index p?

This is in fact true when G is abelian. A prove is given in [Mac23]. In short, this is because a saturated transfer system of the subgroup H can be extend to a saturated transfer system of G by only adding the arrows $K \to K$. A G-universe whose linear isometries operad realizes this transfer system restricts to a H-universe that realizes the original transfer system. However, for non-abelian groups, the closure under conjugation becomes an extra condition on transfer systems. A transfer system on H cannot be easily extended to G while still being closed under conjugation.

The next question that I would like to investigate is

Question 4.2. Does our method carry over to non-abelian groups in general (for groups whose order is not a power of p)?

If we let p be the minimal prime that divides the order of G, most of the proof of 3.2 makes sense. However, our proof heavily depend on the commutator subgroup being much smaller than G. This becomes a serious problem for non-solvable groups.

It seems likely to me that the saturation conjecture fails for most, or even for all non-abelian groups. I have not found any case in which the saturation conjecture is true. It is proved by MacBrough [Mac23] that the saturation conjecture is false for all abelian groups of rank at least 3. Then, it is natural for one to ask,

Question 4.3. Are there other criteria that characterize a transfer system corresponding to a linear isometries operad?

Based on our results, it might be possible to find a criterion for the transfer system involving the commutator subgroup. However, this would still not explain MacBrough's result in the abelian case. According to Rubin [Rub21], the saturation condition is the "most promising criterion" for detemining whether a transfer system corresponds to a linear isometries operad. It would be very challenging to find a perfect answer to this question.

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