ITERATED LUSZTIG–VOGAN BIJECTION AND DISTINGUISHED WEIGHTS

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Abstract. The Lusztig–Vogan bijection studies the dominant weights of a reductive algebraic group G over characteristic 0. In positive characteristic, we modify the Lusztig–Vogan bijection to a map which we call LV_p, and the modular representation theory of G is related to certain distinguished weights which are defined in terms of iterating LV_p; specifically, these distinguished weights are related to p-cells. Using the explicit algorithm for the Lusztig–Vogan bijection for GL_n described by Rush, we can compute the distinguished weights purely combinatorially. We first explicitly classify all distinguished weights for $n \leq 4$. We also prove that all distinguished weights are anti-symmetric, and we show that the distribution of these weights follows a polynomial asymptotic with a leading coefficient relating to the telephone numbers. Finally, we discuss an algebraic construction of ideals in certain Laurent polynomial rings which should be an algebraic analog of distinguished weights.

Contents

1. Introduction

The representation theory of $GL_n(\mathbb{C})$ has been well-studied and is well-known. In particular, the irreducible finite-dimensional representations can be classified by their highest weight, which is always a dominant weight. On the other hand, the representation theory of $GL_n(\overline{\mathbb{F}}_p)$ is much more unknown, being now in the realm of modular representation theory.

One tool to study this representation theory is the Lusztig–Vogan bijection, which was originally conjectured by both Lusztig [5] and Vogan [8] independently. In generality, it states the following: let G be a connected reductive algebraic group. Then this bijection is between Λ^+, the set of dominant weights of G, and Ω, the set of pairs (C, V) such that $C \subset G$ is a unipotent conjugacy class and V is an irreducible representation of the centralizer $Z_G(g)$ for some $g \in C$. In the case of $G = GL_n(\mathbb{C})$, this is a bijection between Λ_n^+, the set of weakly decreasing (non-increasing) integer sequences of length n, and Ω_n, the set of pairs (α, ν) where α is a partition of n and ν is an integer sequence that is dominant with respect to α, i.e. if $\alpha = (\alpha_1, \dots, \alpha_s)$ and $\nu = (\nu_1, \dots, \nu_t)$ where $\alpha_1 \geq \dots \geq \alpha_s$, then $s = t$ and anywhere that $\alpha_i = \alpha_{i+1}$ then $\nu_i \geq \nu_{i+1}$.

In [2], Bezrukavnikov proved this bijection in full generality with novel and non-constructive methods. Prior to this, Achar gave an algorithm in [1] for the case $G = GL_n$ to construct this bijection explicitly; the direction $\Lambda_n^+ \to \Omega_n$ is quite uninvolved, but the opposite direction is sophisticated (in fact, Achar proved this for GL_n before the bijection was established to be true in general.) In [7], Rush describes a simplified version of Achar's algorithm, which can be stated completely combinatorially. We will use Rush's algorithm as it is very concrete and we are only concerned with $G = GL_n$.

The Lusztig–Vogan bijection therefore gives us a deeper understanding of the representation theory of reductive groups over characteristic 0. To better understand the representation theory of GL_n over a field of positive characteristic p, we consider another object of study: the set of distinguished weights, a subset of the dominant weights. The distinguished weights are related to p-cells [3] and hence the modular representation theory of GL_n, and they can be computed combinatorially using the algorithm for the Lusztig–Vogan bijection. Specifically, the distinguished weights are exactly the dominant weights for which, after a finite number of iterations of the algorithm and division by p, only zeroes remain. In this paper, we study these distinguished weights obtained in this fashion.

1.1. **Main results and conjectures.** We state some of the main results of this paper. Let $\Lambda^+_{n,\mathrm{dist}}$ be the set of distinguished weights of length n. Then we have the following anti-symmetry, a simple result with a lengthy proof.

Theorem 1.1.1 (Corollary 4.2.3)**.** *If $\lambda \in \Lambda^+_{n,\mathrm{dist}}$ where $\lambda = (\lambda_1, \dots, \lambda_n)$, then $\lambda_i = -\lambda_{n+1-i}$.*

Next, if we define $\Lambda^+_{n,k}$ to be the set of distinguished weights which map to all zeroes in at most k iterations of the algorithm. We find a recursive formula for $|\Lambda^+_{n,k}|$ in Theorem 4.3.1, which leads to the following asymptotic formula:

Theorem 1.1.2 (Theorem 4.3.4)**.** *We have*

$$|\Lambda^+_{n,k}| \sim \frac{a_{\lfloor \frac{n+1}{2} \rfloor}}{\lfloor \frac{n}{2} \rfloor !} k^{\lfloor \frac{n}{2} \rfloor},$$

where a_i is a sequence defined by $a_0 = a_1 = 1$ and $a_i = a_{i-1} + (i-1)a_{i-2}$.

This sequence of a_i is commonly known as the telephone numbers and appears in more natural contexts, such as counting the number of involutions of S_n, the number of matchings on n nodes, or the total number of standard Young tableaux on n cells (OEIS [6, A000085]). We also study the distribution of these distinguished weights, which have a regular pattern outside of the diagonals in $\mathbb{Z}^{\lfloor \frac{n}{2} \rfloor}$.

Finally, we also define some filtration on a polynomial ring that is an algebraic analog of the algorithm. This allows for a construction of ideals (we call these distinguished ideals) which are indexed by the same set as distinguished weights, and we conjecture that these ideals are closely related to distinguished weights. Specifically, we conjecture that these ideals are free modules over certain Laurent polynomial rings (see Conjecture 5.2.11), and that a certain algorithm should give a concrete relationship between the distinguished weights and the distinguished ideals, see Section 5.3.

1.2. **Structure of the paper.** The paper is structured as follows. In Section 2, we review the theory of the Lusztig–Vogan bijection, including the combinatorial description of the algorithm due to Rush [7] (see Section 2.2). In Section 2.3 we also describe a modification LV_p of the Lusztig–Vogan bijection adapted to the modular setting, and describe one of the main objects of study in this paper, the distinguished weights, in terms of LV_p. In Section 3, we fully classify the distinguished weights for the cases $n = 2, 3$, and 4. In Section 4, we prove some general facts about the algorithm and distinguished weights, leading up to one of the main results, which gives a recursive formula and asymptotics for $|\Lambda^+_{n,k}|$, the number of distinguished weights of length n which take at most k iterations to become zero (Theorem 4.3.4). Furthermore, we state and prove another main result that all distinguished weights have anti-symmetry (Corollary 4.2.3). Finally, in Section 5, we define certain ideals in Laurent polynomial rings which should be an algebraic analog of

LV_p, and describe some properties of them. We give a conjectural relationship of these ideals with the distinguished weights from before, suggesting an algebraic viewpoint towards the distinguished weights.

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2. Background

2.1. **Notation.** We introduce notation that is used throughout the paper. Let $\alpha \vdash n$ denote that α is a partition of n. We may write $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s)$ to denote that $\alpha_i \in \mathbb{Z}$, $\sum_{i=1}^{s} \alpha_i = n$, and $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s \geq 1$. Then we can write $|\alpha| := s$. We may also write $\alpha = (a_1^{m_1}, a_2^{m_2}, \dots, a_t^{m_t})$ where $a_1 > a_2 > \cdots > a_t \geq 1$ and $\sum_{i=1}^{t} a_i m_i = n$, which emphasizes the multiplicity of each part.

We denote V_α to be the irreducible representation of S_n, also known as a Specht module, corresponding to the partition α.

We denote $\operatorname{dom}(\cdot)$ to be the function mapping a finite integer sequence (or set) to the ordered permutation of itself which is sorted in weakly decreasing (i.e. non-increasing) order.

Throughout this paper we fix p to be a sufficiently large prime. We expect that this bound need not be particularly large, for example we expect that $p > 2h$ for h the Coxeter number should be sufficient, but we will not concern ourselves with precise lower bounds. Of course, the "distinguished" weights and ideals depend on p, but we will suppress this from our notation.

2.2. **Algorithm for Lusztig–Vogan bijection.** We first introduce the integer sequences algorithm for computing the Lusztig–Vogan bijection, keeping the notation in [9]. The algorithm for the Lusztig–Vogan bijection is the composition of three maps: ϕ, E^{-1}, and κ. First, we describe the spaces that these maps are between.

Definition 2.2.1. We define Λ_n^+ as the set of all weakly decreasing integer sequences of length n, i.e. $\lambda \in \Lambda_n^+$ if $\lambda = (\lambda_1, \dots, \lambda_n)$ where $\lambda_i \in \mathbb{Z}$ for all $1 \leq i \leq n$ and $\lambda_1 \geq \cdots \geq \lambda_n$.

Definition 2.2.2. Ω_n is the set of pairs (α, ν), where $\alpha = (\alpha_1, \dots, \alpha_l)$ is a partition of n and $\nu = (\nu_1, \dots, \nu_l)$ is a dominant integer sequence with respect to α, i.e. if $\alpha_i = \alpha_{i+1}$, then $\nu_i \geq \nu_{i+1}$.

For the algorithm, we can also define Ω_n to be the set of tuples $(\mu_1, \dots, \mu_s)$ such that μ_i are weakly decreasing integer sequences, $\mu_s \neq \emptyset$, and $\sum_{i=1}^{s} i|\mu_i| = n$. These definitions are equivalent: given a pair (α, ν) such that ν is a dominant integer sequence with respect to α, then let $\mu_i = \operatorname{dom}(\{\nu_j : \alpha_j = i\})$ for $1 \leq i \leq |\alpha|$. On the other hand, given a tuple $(\mu_1, \dots, \mu_s)$ satisfying the above conditions, then let $\alpha = (s^{|\mu_s|}, (s-1)^{|\mu_{s-1}|}, \cdots, 1^{|\mu_1|})$ and ν to be the concatenation of $\operatorname{dom}(\mu_s), \operatorname{dom}(\mu_{s-1}), \dots, \operatorname{dom}(\mu_1)$, in that order.

Definition 2.2.3. We call X a *weighted diagram of size* n if $X = (X_1, \dots, X_l)$ is a tuple of weakly decreasing sequences of integers, where $n = \sum_{i=1}^{l} |X_i|$. Denote $X_{i,j}$ as the jth element of X_i, for $1 \leq j \leq |X_i|$. Denote the ith row of X as X_i, and the jth column of X as $\{X_{i,j} : 1 \leq i \leq l, 1 \leq j \leq |X_i|\}$. Let D_n be the set of all weighted diagrams of size n.

We can begin defining the maps between these spaces that are part of the algorithm. In order to introduce the first map, we give the following definition, as appears in [9].

Definition 2.2.4. Let $\sigma \in \Lambda_n^+$. The *maximal clumping of* σ is the unique partition of σ into sequences $A_1, \dots, A_l$, where l is minimal such that the concatenation of the $A_1, \dots, A_l$, in that order, is σ, and the underlying set of A_i consists of consecutive numbers. Call each A_i a *maximal clump* or just *clump*.

Definition 2.2.5 (Construction of ϕ)**.** Define $\phi : \Lambda_n^+ \to D_n$ by the following iterative construction:

Let $\sigma \in \Lambda_n^+$. We build the image $X = \phi(\sigma)$ column-by-column. To start, let $\sigma_1 = \sigma$. To build the rth column, suppose we are given σ_r, a weakly decreasing integer sequence. Let $\sigma_r = A_1 \cup \cdots \cup A_l$ be its maximal clumping, and let B_i be the sequence A_i with repetitions removed. We construct a set Z_r as follows:

For $1 \leq i \leq l$, let $B_i = (b_1, b_2, \dots)$. If both r and $|B_i|$ are even, then append to Z_r the elements $b_2, b_4, b_6, \dots$. Otherwise, append to Z_r the elements $b_1, b_3, b_5, \dots$.

If $r = 1$, then the first column of X is Z_r, arranged in decreasing order from top to bottom and in rows 1 to $|Z_r|$. Otherwise, the rth column of X is still Z_r, arranged in decreasing order from top to bottom, with the additional condition that for every $z \in Z_r$, it must be placed in row i such that $X_{i,r-1}$ is z or $z + (-1)^r$. Then remove the elements of Z_r from σ_r to create σ_{r+1} (for repetitions, only remove from σ_r the multiplicity of the element in Z_r).

Repeat this procedure until all elements are removed. The validity of this algorithm is proved in [9].

Remark. There is clearly an inverse map ϕ^{-1}, which sorts the elements of $X \in D_n$ in weakly decreasing order.

Definition 2.2.6 (Construction of E)**.** Define $E : D_n \to D_n$ by the following elementwise operation:

Let $X \in D_n$ with $X = (X_1, \dots, X_l)$. We construct $E(X) = (X_1', \dots, X_l')$ satisfying the following. First, $|X_i| = |X_i'|$. Second, for every $X_{i,j} \in X$, we have $X'_{i,j} = X_{i,j} + 2m_{i,j} - (c_j - 1)$, where $c_j = |\{r : 1 \leq r \leq l, 1 \leq j \leq |X_r|\}|$ is the total number of elements in column j and $m_{i,j}$ is the number of r such that $X_{r,j} \in X$ and $(X_{r,j}, -r)$ is lexicographically before $(X_{i,j}, -i)$.

Remark (Construction of E^{-1}). If X is in the image of ϕ, then each column of $X \in D_n$ is strictly decreasing with consecutive differences at least 2 by construction. In this case, there is an inverse map E^{-1}, constructed as follows:

Let $X \in D_n$ with $X = (X_1, \dots, X_l)$ satisfying the above properties, i.e. $X_{r_1,j} - X_{r_2,j} \geq 2$ if $r_1 < r_2$. Then $E^{-1}(X) = (X_1', \dots, X_l')$, satisfying the following. First, $|X_i| = |X_i'|$. Second, for every $X_{i,j} \in X$, we have $X'_{i,j} = X_{i,j} + (c_j - 1) - 2m_{i,j}$, where c_j and $m_{i,j}$ are defined the same as the above definition. Since the numbers in each column of X are already sorted in decreasing order from top to bottom, this is the same as adding $(-(c_j - 1), -(c_j - 3), \dots, c_j - 1)$ to the jth column, from top to bottom.

Definition 2.2.7 (Construction of κ)**.** Define $\kappa : D_n \to \Omega_n$ as follows:

Let $X \in D_n$ with $X = (X_1, \dots, X_l)$. Then $\kappa(X) = (\mu_1, \mu_2, \dots, \mu_s)$ where $s = \max_{1 \leq i \leq l}\{|X_i|\}$ and $\mu_i = \operatorname{dom}(\{\sum_{j=1}^{|X_r|} X_{r,j} : |X_r| = i\})$, i.e. the weakly decreasing sequence of row sums of rows with length i.

Remark. The simplified construction for κ^{-1} is the algorithm $\mathfrak{A}(\alpha, \nu)$ given in [7]. It is omitted here due to its length.

With these maps, we can define the map $LV : \Lambda_n^+ \to \Omega_n$ by $LV = \kappa \circ E^{-1} \circ \phi$. This is the algorithm for one direction of the Lusztig–Vogan bijection.

Example 2.2.8. Let $n = 8$ and let $\sigma = (46, 46, 45, 1, -1, -45, -46, -46) \in \Lambda_n^+$. Then

$$(46, 46, 45, 1, -1, -45, -46, -46) \overset{\phi}{\longmapsto} \begin{array}{|c|c|c|} \hline 46 & 45 & 46 \\ \hline 1 \\ \cline{1-1} -1 \\ \hline -45 & -46 & -46 \\ \hline \end{array} \xmapsto{E^{-1}} \begin{array}{|c|c|c|} \hline 43 & 44 & 45 \\ \hline 0 \\ \cline{1-1} 0 \\ \hline -42 & -45 & -45 \\ \hline \end{array} \overset{\kappa}{\longmapsto} ((0,0), (), (132, -132))$$

Hence, $LV(\sigma) = ((0,0), (), (132, -132))$.

2.3. **Iterations of the algorithm.** Fix a prime $p > n$. Define $LV_p = \frac{1}{p} \circ LV$, i.e. if $LV(\sigma) = (\mu_1, \dots, \mu_s)$ for $\sigma \in \Lambda_n^+$, then $LV_p(\sigma) = (\frac{\mu_1}{p}, \dots, \frac{\mu_s}{p})$. Note that LV_p may not always be defined, because division by p may not always become integers. However, in this paper, we are only concerned with the subset of Λ_n^+ which this map is defined, i.e. is integral after this division by p.

The output of LV_p, assuming it is integral, is an element of Ω_n, which by Definition 2.2.2 can also be written as $(\mu_1, \dots, \mu_s)$, where each μ_i is a weakly decreasing sequence. This means that $\mu_i \in \Lambda_{n_i}^+$ for some n_i, so we can apply LV_p again to each μ_i, only if the result is still integral and if $n_i \geq 2$. We have a tuple of tuples of weakly decreasing sequences, so we can iterate again. In this way, we can keep iterating LV_p until every sequence is either of length 0, length 1, or another iteration would make the numbers non-integral.

Definition 2.3.1. Let $\Lambda^+_{n,\mathrm{dist}} \subset \Lambda^+_n$ be the set of *distinguished weights* of GL_n; in regards with the above algorithm, these are exactly the elements in Λ^+_n such that upon iteration of LV_p, every remaining number is a zero. Note that

$$LV : (0, \ldots, 0) \mapsto ((), \ldots, (), (0)),$$

so we can stop iterating on a sequence once it has become all zeroes.

Furthermore, define $\Lambda^+_{n,k} \subset \Lambda^+_{n,\mathrm{dist}}$ as the weakly decreasing sequences of length n which become 0 after at most k iterations of the algorithm. It is clear that $\Lambda^+_{n,0} = \{(0, \ldots, 0)\}$ and $\Lambda^+_{n,0} \subset \Lambda^+_{n,1} \subset \Lambda^+_{n,2} \subset \cdots$.

The following example gives an example of a distinguished weight.

Example 2.3.2. Continuing Example 2.2.8 for $n = 8$ and $p = 11$, we find that

$$\begin{aligned}\sigma = (46, 46, 45, 1, -1, -45, -46, -46) &\xmapsto{LV_p} ((0, 0), (), (12, -12))\\ &\xmapsto{LV_p} ((0, 0), (), (1, -1))\\ &\xmapsto{LV_p} ((0, 0), (), (0, 0)).\end{aligned}$$

Since three iterations of LV_p becomes all zeroes, then $\sigma \in \Lambda^+_{n,\mathrm{dist}}$. In particular, we have that $\sigma \in \Lambda^+_{n,k}$ for $k \geq 3$.

The map LV_p returns an element in Ω_n, which can be written as $(\alpha^{(1)}, \nu)$, where $\alpha^{(1)}$ is a partition of n and ν is an integer sequence dominant with respect to $\alpha^{(1)}$. Let $\alpha^{(1)} = (a_1^{m_1}, \ldots, a_s^{m_s})$. Then when we iterate by applying LV_p again, we obtain further partitions $\alpha_1^{(2)} \vdash m_1$, $\alpha_2^{(2)} \vdash m_2$, and so on. This tuple of partitions $(\alpha_1^{(2)}, \ldots, \alpha_s^{(2)})$ is a refinement of the partition $\alpha^{(1)}$. When we iterate further, we recursively obtain further refinements of these partitions. In this manner, each distinguished weight corresponds to some sequence $(\alpha^{(1)}, \alpha^{(2)}, \ldots)$ where $\alpha^{(1)}$ is a partition of n and each successive element is a refinement of the previous.

3. Characterization of $\Lambda^+_{n,\mathrm{dist}}$ for small n

Example 3.0.1. Let $n = 2$. Then $\Lambda^+_{n,\mathrm{dist}} = \{(\frac{p^m-1}{p-1}, -\frac{p^m-1}{p-1}) : m \geq 0\}$. When iterating, we find that

$$\left(\frac{p^m-1}{p-1}, -\frac{p^m-1}{p-1}\right) \xmapsto{m \text{ iterations}} (\underbrace{\cdots(}_{m+1} 0, 0) \cdots).$$

The algorithm works as follows:

$$(1 + p + \cdots + p^{m-1}, -(1 + p + \cdots + p^{m-1})) \xmapsto{\phi} \boxed{\begin{array}{c} 1 + p + \cdots + p^{m-1} \\ \hline -(1 + p + \cdots + p^{m-1}) \end{array}} \xmapsto{E^{-1}} \boxed{\begin{array}{c} p + \cdots + p^{m-1} \\ \hline -(p + \cdots + p^{m-1}) \end{array}}$$

$$\xmapsto{\kappa} ((p + \cdots + p^{m-1}, -(p + \cdots + p^{m-1}))) \xmapsto{\frac{1}{p}} ((1 + p + \cdots + p^{m-2}, -(1 + p + \cdots + p^{m-2}))),$$

which arrives to zeroes recursively. In particular, we have $|\Lambda^+_{2,k}| = k + 1$.

We claim that these are the only distinguished weights for $n = 2$. This is because a distinguished weight must end with all zeroes, and if $n = 2$, it must end at $(\cdots(0, 0)\cdots)$ or $(\cdots((), (0))\cdots)$, which correspond to the partitions $(1, 1)$ and (2) of 2, respectively. The second case comes directly from the first after one iteration, since $LV_p : (0, 0) \mapsto ((), (0))$. By the bijectivity of the algorithm, the characterization above consists of all possible distinguished weights for $n = 2$.

Example 3.0.2. Let $n = 3$. Then

$$\Lambda^+_{n,\mathrm{dist}} = \left\{\left(2 \cdot \frac{p^m-1}{p-1}, 0, -2 \cdot \frac{p^m-1}{p-1}\right) : m \geq 0\right\} \cup \left\{\left(\frac{p^{m+1}+p^m-2}{p-1}, 0, -\frac{p^{m+1}+p^m-2}{p-1}\right) : m \geq 0\right\}.$$

When iterating, we find that

$$\left(2 \cdot \frac{p^m-1}{p-1}, 0, -2 \cdot \frac{p^m-1}{p-1}\right) \xmapsto{m \text{ iterations}} (\underbrace{\cdots(}_{m+1} 0, 0, 0) \cdots),$$

and

$$\left(\frac{(p^{m+1}-1)+(p^m-1)}{p-1}, 0, -\frac{(p^{m+1}-1)+(p^m-1)}{p-1}\right) \xmapsto{m+1 \text{ iterations}} \underbrace{\Big(\cdots\Big(}_{m+1}(0),(0)\Big)\cdots\Big).$$

We have that $|\Lambda^+_{3,k}| = 2k+1$.

We show that these are all of the distinguished weights for $n = 3$ by a similar argument as the previous example. In particular, a distinguished weight must end in $(\cdots(0,0,0)\cdots)$, $(\cdots((0),(0))\cdots)$, or $(\cdots((),(),(0))\cdots)$, which correspond to the partitions $(1,1,1),(1,2)$, and (3), respectively. The third case comes directly from the first one after one more iteration, and by the bijectivity of the algorithm, the characterization above for $\Lambda^+_{n,\mathrm{dist}}$ covers all distinguished weights for $n = 3$.

Example 3.0.3. Let $n = 4$. We have the following families of distinguished weights:

$$\frac{p^m-1}{p-1}\cdot(3,1,-1,-3) \xmapsto{m \text{ iterations}} \underbrace{(\cdots(}_{m+1}0,0,0,0)\cdots),$$

and

$$\left(\frac{p^{m+1}+2p^m-3}{p-1}, \frac{p^m-1}{p-1}, -\frac{p^m-1}{p-1}, -\frac{p^{m+1}+2p^m-3}{p-1}\right) \xmapsto{m+1 \text{ iterations}} \underbrace{\Big(\cdots\Big(}_{m+1}(0),(),(0)\Big)\cdots\Big).$$

In addition, there are more complicated closed forms. One of them is:

$$\begin{aligned}
&\left(\frac{p^{m+k+1}+p^{k+1}+p^k-3}{p-1}, \frac{p^k-1}{p-1}, -\frac{p^k-1}{p-1}, -\frac{p^{m+k+1}+p^{k+1}+p^k-3}{p-1}\right)\\
&\xmapsto{k \text{ iterations}} \underbrace{\Big(\cdots\Big(}_{k+1}\frac{p^{m+1}+p-2}{p-1}, 0, 0, -\frac{p^{m+1}+p-2}{p-1}\Big)\cdots\Big)\\
&\xmapsto{1 \text{ iteration}} \underbrace{\Big(\cdots\Big(}_{k+1}\left(\frac{p^m-1}{p-1}, -\frac{p^m-1}{p-1}\right),(0)\Big)\cdots\Big)\\
&\xmapsto{m \text{ iterations}} \underbrace{\Big(\cdots\Big(}_{k+1}\underbrace{(\cdots(}_{m+1}0,0)\cdots),(0)\Big)\cdots\Big).
\end{aligned}$$

The next closed form is split into two cases by parity. If m is even, then we have

$$\begin{aligned}
&\left(\frac{p^{m+k}+2p^{k+1}+3p^k-6}{2(p-1)}, \frac{p^{m+k}+p^k-2}{2(p-1)}, -\frac{p^{m+k}+p^k-2}{2(p-1)}, -\frac{p^{m+k}+2p^{k+1}+3p^k-6}{2(p-1)}\right)\\
&\xmapsto{k \text{ iterations}} \underbrace{\Big(\cdots\Big(}_{k+1}\frac{p^m+2p-3}{2(p-1)}, \frac{p^m-1}{2(p-1)}, -\frac{p^m-1}{2(p-1)}, -\frac{p^m+2p-3}{2(p-1)}\Big)\cdots\Big)\\
&\xmapsto{1 \text{ iteration}} \underbrace{\Big(\cdots\Big(}_{k+1}(), \left(\frac{p^{m-1}-1}{p-1}, -\frac{p^{m-1}-1}{p-1}\right)\Big)\cdots\Big)\\
&\xmapsto{m-1 \text{ iterations}} \underbrace{\Big(\cdots\Big(}_{k+1}(), \underbrace{(\cdots(}_{m}0,0)\cdots)\Big)\cdots\Big).
\end{aligned}$$

If m is odd, then we have

$$\begin{aligned}
&\Big(\frac{p^{m+k}+p^{k+1}+4p^k-6}{2(p-1)}, \frac{p^{m+k}+p^{k+1}-2}{2(p-1)}, -\frac{p^{m+k}+p^{k+1}-2}{2(p-1)}, -\frac{p^{m+k}+p^{k+1}+4p^k-6}{2(p-1)}\Big)\\
&\xmapsto{k \text{ iterations}} \underbrace{\Big(\cdots\Big(}_{k+1}\frac{p^m+p-2}{2(p-1)}, \frac{p^m+p-2}{2(p-1)}, -\frac{p^m+p-2}{2(p-1)}, -\frac{p^m+p-2}{2(p-1)}\Big)\cdots\Big)\\
&\xmapsto{1 \text{ iteration}} \underbrace{\Big(\cdots\Big(}_{k+1}(), \Big(\frac{p^{m-1}-1}{p-1}, -\frac{p^{m-1}-1}{p-1}\Big)\Big)\cdots\Big)\\
&\xmapsto{m-1 \text{ iterations}} \underbrace{\Big(\cdots\Big(}_{k+1}(), \underbrace{(\cdots(}_{m}0,0)\cdots)\Big)\cdots\Big).
\end{aligned}$$

Carefully counting, we find that $|\Lambda^+_{4,k}| = k^2+3k+1$.

We can show that these are all of the distinguished weights for $n=4$ by a similar argument as the previous two examples.

4. Main results on distinguished weights

4.1. **Basic results.** When computing the distinguished weights, we could start at the end sequence of zeroes in many nested parentheses and use LV_p^{-1} to obtain the distinguished weight which has that particular end sequence. Since we know this algorithm is a bijection, then each end sequence unique corresponds to a distinguished weight. The following proposition gives an exact formula for iterating LV_p^{-1} on a single weakly decreasing sequence which is nested in multiple parentheses.

Proposition 4.1.1. *If $\lambda \in \Lambda_n^+$, then*

$$\lambda p^k + \rho_{(1,\ldots,1)} \cdot \frac{p^k-1}{p-1} \xmapsto{k \text{ iterations}} \underbrace{(\cdots(}_{k}\lambda)\cdots),$$

where $\rho_{(1,\ldots,1)} = (n-1, n-3, \ldots, 3-n, 1-n)$.

Remark. It should be noted that ρ_α is more generally defined for partitions $\alpha \vdash n$, but this is not needed in this paper.

Proof. Let $\rho = \rho_{(1,\ldots,1)}$. One iteration of LV_p^{-1} starting from λ would return $\lambda p + \rho$, where the ρ comes from the map E. Repeating this recursively, we obtain

$$(\cdots(((\lambda p+\rho)p+\rho)p+\rho)\cdots)p+\rho = \lambda p^k + \rho(1+p+\cdots+p^{k-1}) = \lambda p^k + \rho\cdot\frac{p^k-1}{p-1},$$

as desired. □

This proposition allows us to always give one family of distinguished weights for any n.

Corollary 4.1.2. *Let $\alpha = (1,\ldots,1)$ denote a partition of n. Then $\rho_\alpha \cdot \frac{p^m-1}{p-1} \in \Lambda^+_{n,\mathrm{dist}}$ for $m \geq 0$.*

Proof. We know $(0,\ldots,0) \in \Lambda^+_{n,\mathrm{dist}}$, then apply Proposition 4.1.1. □

This family of distinguished weights corresponds to the partition $(1,\ldots,1)$ of n. As we can already see in the $n=4$ example in Section 3, the other families of distinguished weights can become very complicated.

4.2. **Anti-symmetry of distinguished weights.** One clear observation about the distinguished weights so far is about their anti-symmetry. In particular, if $\lambda \in \Lambda^+_{n,\mathrm{dist}}$ with $\lambda = (\lambda_1,\ldots,\lambda_n)$ then $\lambda_i = -\lambda_{n+1-i}$. We prove this statement in this section, by first defining the map R for more concise notation and then proving a more general statement.

Definition 4.2.1. Define the map $R: \Lambda_m^+ \to \Lambda_m^+$ be defined as $(\lambda_1,\ldots,\lambda_m) \mapsto (-\lambda_m,\ldots,-\lambda_1)$ for all m. Overloading the notation, define the map $R: \Omega_m \to \Omega_m$ be defined as $(\mu_1,\ldots,\mu_s) \mapsto (R\mu_1,\ldots,R\mu_s)$ for m. In other words, the map R negates and reverses a weakly decreasing sequence.

The fact that all distinguished weights are anti-symmetric follows directly from the following theorem, and is presented as a corollary after the theorem's proof. The proof of the following theorem requires Lemma 4.2.4, which is stated and proved at the end of this section, having been moved due to length.

Theorem 4.2.2. *If $\lambda \in \Lambda_n^+$, then $LV(R(\lambda)) = R(LV(\lambda))$.*

Proof. Consider $\phi(\lambda)$ for an arbitrary λ. The elements in each row have consecutive differences with magnitude at most 1, so the elements must begin in the same maximal clump of λ. This means that each maximal clump corresponds to some set of rows in $\phi(\lambda)$, and the rows corresponding to different maximal clumps are disjoint.

Consider the map ϕ on λ and $R(\lambda)$. Let the maximal clumps of λ, ordered by largest numbers first, be $A_1, \dots, A_c$, and let A_1 correspond to the first a_1 rows of $\phi(\lambda)$, A_2 correspond to the next a_2 rows, etc. Then the maximal clumps of $R(\lambda)$, ordered by largest numbers first, are $R(A_c), \dots, R(A_1)$, with $R(A_1)$ corresponding to the last a_1 rows of $\phi(R(\lambda))$, $R(A_2)$ corresponding to the a_2 rows before that, and so on.

Consider clump A_i, and consider $\kappa \circ E^{-1}$ on $\phi(A_i)$ and $\phi(R(A_i))$. First, E^{-1} adds constants to the elements by column, then κ takes row sums. Lemma 4.2.4, whose proof is at the end of this section due to its length, states that this theorem's statement is true for one clump, so we know that $R(LV(A_i)) = LV(R(A_i))$; then for every row in $E^{-1}(\phi(A_i))$, there is a row of the same length with the opposite row sum in $E^{-1}(\phi(R(A_i)))$ and vice versa. For each column of $\phi(A_i)$, the map E^{-1} adds $(1-l, 3-l, \dots, l-1)$ to the column, where l is the number of elements in the column; for the corresponding column of $\phi(R(A_i))$, the map E^{-1} adds the same constant. Let t be an arbitrary scalar. If, instead, E^{-1} added $(1-l+t, 3-l+t, \dots, l-1+t)$ to the column in $\phi(A_i)$ and added $(1-l-t, 3-l-t, \dots, l-1-t)$ to the column in $\phi(R(A_i))$, then the property that every row in $E^{-1}(\phi(A_i))$ has a corresponding row of the same length with the opposite row sum in $E^{-1}(\phi(R(A_i)))$ (and vice versa) still holds, since all row sums are translated by $+t$ in $E^{-1}(\phi(A_i))$ and by $-t$ in $E^{-1}(\phi(R(A_i)))$.

The map E^{-1} adds an anti-symmetric constant to every column, in the form $(1-L, 3-L, \dots, L-1)$, where L is the total number of elements in the column. In $\phi(\lambda)$, clump A_i is represented by the rows $a_1 + \cdots + a_{i-1} + 1$ to $a_1 + \cdots + a_{i-1} + a_i$ from the top. This means that under the map E^{-1}, the constant that is added to the jth column for the rows corresponding to clump A_i in $\phi(\lambda)$ is in the form $(1-l+t, 3-l+t, \dots, l-1+t)$ for some t, where l is the number of elements in the jth column in clump A_i. On the other hand, in $\phi(R(\lambda))$, clump $R(A_i)$ is represented by the rows $a_1 + \cdots + a_{i-1} + 1$ to $a_1 + \cdots + a_{i-1} + a_i$ from the bottom. Since the corresponding rows indices for clump $R(A_i)$ in $\phi(R(\lambda))$ are opposite to clump A_i in $\phi(\lambda)$, the constant added to the jth column for the rows corresponding to clump $R(A_i)$ in $\phi(R(\lambda))$ is exactly in the form $(1-l-t, 3-l-t, \dots, l-1-t)$ for the same t. Therefore, for every row in $E^{-1}(\phi(\lambda))$, there is a corresponding row in $E^{-1}(\phi(R(\lambda)))$ of the same length with an opposite row sum and vice versa.

This means that if $\kappa(E^{-1}(\phi(\lambda))) = (\mu_1, \dots, \mu_s)$, then $\kappa(E^{-1}(\phi(R(\lambda))))$ has all row sums negated, which is exactly $(R\mu_1, \dots, R\mu_s)$. Hence, $LV(R(\lambda)) = R(LV(\lambda))$, as desired. □

Corollary 4.2.3. *If $\lambda \in \Lambda_{n,\mathrm{dist}}^+$, then $R(\lambda) = \lambda$, i.e. if $\lambda = (\lambda_1, \dots, \lambda_n)$ then $\lambda_i = -\lambda_{n+1-i}$.*

Proof. There exists some $m \geq 0$ such that $LV_p^m(\lambda)$ is all zeroes, which we denote simply as $LV_p^m(\lambda) = 0$. By the above theorem (Theorem 4.2.2), and since dividing by p clearly commutes with R, we have

$$LV_p^m(\lambda) = 0 = R(0) = LV_p^m(R(\lambda)).$$

In other words, since any sequence of all zeroes is invariant under R, then applying LV_p^{-1} preserves this. Since LV_p is bijective, then $R(\lambda) = \lambda$, as desired. □

Remark. The converse of this theorem is false, i.e. not all anti-symmetric sequences are distinguished weights. This can be quickly seen in the examples given in Section 3.

Finally, to finish the proof of the theorem, we must prove the following lemma. This proof includes somewhat lengthy casework, and examples of each case are given after the proof for concreteness.

Define ϕ', a map similar to ϕ with the only difference being we index the columns of the image starting at column 0. The rest of the algorithm for constructing the map is the same, and this new indexing entails a change in parity, which leads to a slight change in the resulting image. Accordingly, define $LV' = \kappa \circ E^{-1} \circ \phi'$. This definition allows us to have a slightly stronger statement for the lemma, which is needed for induction.

Lemma 4.2.4. *Let $\lambda \in \Lambda_n^+$ be a maximal clump, i.e. the underlying set of λ consists of consecutive numbers. Then $R(LV(\lambda)) = LV(R(\lambda))$ and $R(LV'(\lambda)) = LV'(R(\lambda))$.*

Proof. Let the underlying set of λ be B. We proceed by induction on $|B|$.

If $|B| = 1$, then $\lambda = (\lambda_1, \dots, \lambda_1)$, where λ_1 appears m_1 times. Then it is clear that $LV'(\lambda) = LV(\lambda) = ((), \dots, (), (m_1\lambda_1))$ and $LV'(R(\lambda)) = LV(R(\lambda)) = ((), \dots, (), (-m_1\lambda_1))$.

Suppose that $|B| = k$, and that the statement is true for smaller values of $|B|$. Let $B = \{\lambda_1, \dots, \lambda_k\}$, where $\lambda_i = \lambda_{i+1} + 1$ for $1 \le i \le k-1$ and λ_i occurs with multiplicity m_i in λ.

We first show that $R(LV(\lambda)) = LV(R(\lambda))$. Since κ takes row sums, then it suffices to show that for every row of $E^{-1}(\phi(\lambda))$, there is a corresponding row in $E^{-1}(\phi(R(\lambda)))$ with the same length but opposite row sum. Call this the row-length-sum property of λ and $R(\lambda)$.

We split into cases:

Case 1: k is odd. Let $k = 2t+1$. We construct the columns of $E^{-1}(\phi(\lambda))$ one by one (instead of constructing $\phi(\lambda)$ first then applying E^{-1}). Each of the first $\min\{m_i : i \text{ odd}\}$ columns of $\phi(\lambda)$ are $\lambda_{\text{odd}} := (\lambda_1, \lambda_3, \dots, \lambda_{2t+1})$, and applying E^{-1} makes the column $(\lambda_1 - t, \lambda_3 - (t-2), \dots, \lambda_{2t+1} + t)$. Since the λ_i's are decreasing consecutive numbers, then

$$s := \lambda_1 - t = \lambda_3 - (t-2) = \cdots = \lambda_{2t+1} + t,$$

so the column is just copies of s, as shown in the diagram below.

$$\begin{array}{cccc} | & | & & | \\ \lambda_{\text{odd}} & \lambda_{\text{odd}} & \cdots & \lambda_{\text{odd}} \\ | & | & & | \end{array} \xmapsto{E^{-1}} \begin{array}{cccc} s & s & \cdots & s \\ \vdots & \vdots & \cdots & \vdots \\ s & s & \cdots & s \end{array}$$

At this point in the algorithm of ϕ, one of the numbers $\{\lambda_i : i \text{ odd}\}$ has no more copies left. This means that we have at least two maximal clumps now, each with underlying set size smaller than k. Let the new clumps be $A_1, \dots, A_c$, ordered by largest numbers first.

Similarly, the first $\min\{m_i : i \text{ odd}\}$ columns of $\phi(R(\lambda))$ are $R(\lambda_{\text{odd}}) := (-\lambda_{2t+1}, -\lambda_{2t-1}, \dots, -\lambda_1)$, and applying E^{-1} makes the column $(-\lambda_{2t+1} - t, -\lambda_{2t-1} - (t-2), \dots, -\lambda_1 + t)$, which are exactly copies of $-s$, as shown in the diagram below. The maximal clumps at this point in the algorithm are $R(A_c), \dots, R(A_1)$, ordered by largest numbers first.

$$\begin{array}{cccc} | & | & & | \\ R(\lambda_{\text{odd}}) & R(\lambda_{\text{odd}}) & \cdots & R(\lambda_{\text{odd}}) \\ | & | & & | \end{array} \xmapsto{E^{-1}} \begin{array}{cccc} -s & -s & \cdots & -s \\ \vdots & \vdots & \cdots & \vdots \\ -s & -s & \cdots & -s \end{array}$$

Let $r = \min\{m_i : i \text{ odd}\} + 1$ be the next column to construct in $E^{-1}(\phi(\lambda))$. We have two further subcases:

Subcase 1a: r is odd. Then r is the same parity as 1, so constructing the next column of $E^{-1}(\phi(\lambda))$ is the same as constructing the first column of $E^{-1}(\phi(A_i))$, for the clumps A_i, and appending them in the correct places. This is the same for $E^{-1}(\phi(R(A_i)))$. We apply the inductive hypothesis to each clump, which has underlying set size less than k. In particular, for each clump A_i, we know that for each row in $E^{-1}(\phi(A_i))$, there is a row of equal length and opposite row sum in $E^{-1}(\phi(R(A_i)))$. However, there are two differences between $E^{-1}(\phi(A_i))$ and when the same clump is constructed in $E^{-1}(\phi(\lambda))$: first, we have copies of s in front in $E^{-1}(\phi(\lambda))$, and second, E^{-1} may add different constants. We claim that these differences do not change the row-length-sum property of A_i and $R(A_i)$:

First, adding copies of s in front of every row of $E^{-1}(\phi(A_i))$ and adding copies of $-s$ in front of every row of $E^{-1}(\phi(A_i))$ does not change the row-length-sum property, since row lengths are all increased by the same number and row sums are increased in $E^{-1}(\phi(A_i))$ the same amount as they are decreased in $E^{-1}(\phi(R(A_i)))$.

Second, for each column j, E^{-1} applied to $\phi(A_i)$ adds the same constants as E^{-1} applied to clump A_i in $\phi(\lambda)$ but shifted by a constant c_j. Similarly, for each column j, E^{-1} applied to $\phi(R(A_i))$ adds the same constants as E^{-1} applied to clump $R(A_i)$ in $\phi(R(\lambda))$ but shifted by $-c_j$, since the order of the clumps in $R(\lambda)$ are the reverse of those in λ. However, adding a constant to the rows in $E^{-1}(\phi(A_i))$ and subtracting the same constant to the corresponding rows in $E^{-1}(\phi(R(A_i)))$ does not change the row-length-sum property either.

Therefore, for every clump A_i, the row-length-sum property is preserved when it is constructed in $E^{-1}(\phi(\lambda))$ and $E^{-1}(\phi(R(\lambda)))$. The rows in $E^{-1}(\phi(\lambda))$ which stop after column $r-1$ clearly have their corresponding row in $E^{-1}(\phi(R(\lambda)))$ which is also length $r-1$, and their sums are $(r-1)s$ and $(r-1)(-s)$, respectively.

Subcase 1b: r is even. The same proof as Subcase 1a works, replacing ϕ with ϕ'.

Case 2: k is even. Let $k = 2t$. The first column of $\phi(\lambda)$ is $\lambda_{\text{odd}} := (\lambda_1, \lambda_3, \ldots, \lambda_{2t-1})$. The second column is $\lambda_{\text{even}} := (\lambda_2, \lambda_4, \ldots, \lambda_{2t})$. The third column is λ_{odd}, and in general, the columns alternate until some λ_i has no more copies left.

Similarly, the first column in $\phi(R(\lambda))$ is $(-\lambda_{2t}, -\lambda_{2t-2}, \ldots, -\lambda_2) = R(\lambda_{\text{even}})$; the second column is $(-\lambda_{2t-1}, -\lambda_{2t-3}, \ldots, -\lambda_1) = R(\lambda_{\text{odd}})$; the third column is $R(\lambda_{\text{even}})$. The columns alternate until some number has no copies left.

Once some λ_i has no more copies left in $\phi(\lambda)$, then to construct the next column, the remaining numbers split into more than one maximal clump. Let the $(r-1)$th column be the column after which, the remaining numbers are no longer one clump. We split into the following subcases:

Subcase 2a: r is even. This means that $r-1$ is odd, so there are no more copies of some λ_i with i odd. Let C be the set of these indices which have no more copies, i.e. $C = \{i : i \text{ odd}, m_i = \frac{r}{2}\} = \{c_1 \leq \cdots \leq c_d\}$. We claim that the rth column of $\phi(\lambda)$ is λ_{even}. The underlying sets of the maximal clumps are

$$\{\lambda_1, \ldots, \lambda_{c_1-1}\}, \{\lambda_{c_1+1}, \ldots, \lambda_{c_2-1}\}, \ldots, \{\lambda_{c_d+1}, \ldots, \lambda_{2t}\}.$$

The first clump's underlying set has an even size. Since r is even, the selected elements are $\lambda_2, \lambda_4, \ldots, \lambda_{c_1-1}$. The ith clump, for $2 \leq i \leq d$, has underlying set $\{\lambda_{c_{i-1}+1}, \ldots, \lambda_{c_i-1}\}$, which has odd size, so the selected elements are $\lambda_{c_{i-1}+1}, \lambda_{c_{i-1}+3}, \ldots, \lambda_{c_i-1}$. The $(d+1)$th clump has underlying set $\{\lambda_{c_d+1}, \ldots, \lambda_{2t}\}$, which has odd size, so the selected elements are $\lambda_{c_d+1}, \lambda_{c_d+3}, \ldots, \lambda_{2t}$. Therefore, to construct the rth column, all of the even indices are selected, so the rth column is $(\lambda_2, \lambda_4, \ldots, \lambda_{2t}) = \lambda_{\text{even}}$.

Therefore, the first r columns of $\phi(\lambda)$ are $\lambda_{\text{odd}}, \lambda_{\text{even}}, \lambda_{\text{odd}}, \ldots, \lambda_{\text{even}}$. Applying E^{-1} makes (λ_{odd} to $\lambda_1 - (t-1), \lambda_3 - (t-3), \ldots, \lambda_{2t-1} + (t-1)$) and ($\lambda_{\text{even}}$ to $\lambda_2 - (t-1), \lambda_4 - (t-3), \ldots, \lambda_{2t} + (t-1)$). Since the λ_i's are decreasing consecutive numbers, we have that

$$s := \lambda_1 - (t-1) = \lambda_3 - (t-3) = \cdots = \lambda_{2t-1} + (t-1)$$

and

$$s - 1 = \lambda_2 - (t-1) = \lambda_4 - (t-3) = \cdots = \lambda_{2t} + (t-1).$$

Hence, the columns alternate between all s's and all $(s-1)$'s, as shown in the diagram below.

$$\begin{array}{ccccc} | & | & | & & | \\ \lambda_{\text{odd}} & \lambda_{\text{even}} & \lambda_{\text{odd}} & \cdots & \lambda_{\text{even}} \\ | & | & | & & | \end{array} \xmapsto{E^{-1}} \begin{array}{ccccc} s & s-1 & s & \cdots & s-1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ s & s-1 & s & \cdots & s-1 \end{array}$$

Now, we examine what occurs in $\phi(R(\lambda))$. The $(r-2)$th column is $R(\lambda_{\text{odd}})$, and by assumption, there are still copies of all λ_i remaining. The $(r-1)$th column is $R(\lambda_{\text{even}})$. If there are still copies of all λ_i remaining, then the rth column is $R(\lambda_{\text{odd}})$. Otherwise, after the $(r-1)$th column, at least one even index i has no more copies of λ_i. Let C be the set of these indices which have no more copies, i.e. $C = \{i : i \text{ even}, m_i = \frac{r}{2}\} = \{c_1 \geq \cdots \geq c_d\}$. We claim that the rth column is still $R(\lambda_{\text{odd}})$. The underlying sets of the maximal clumps are

$$\{-\lambda_{2t}, \ldots, -\lambda_{c_d+1}\}, \{-\lambda_{c_d-1}, \ldots, -\lambda_{c_{d-1}+1}\}, \ldots, \{-\lambda_{c_1-1}, \ldots, -\lambda_1\}.$$

The first clump's underlying set has an even size. Since r is even, the selected elements are $-\lambda_{2t-1}, -\lambda_{2t-3}, \ldots, -\lambda_{c_1+1}$. The ith clump, for $2 \leq i \leq d$, has underlying set $\{-\lambda_{c_{d-i+2}-1}, \ldots, -\lambda_{c_{d-i+1}+1}\}$, which has odd size, so the selected elements are $-\lambda_{c_{d-i+2}-1}, -\lambda_{c_{d-i+2}-3}, \ldots, -\lambda_{c_{d-i+1}+1}$. The $(d+1)$th clump has underlying set $\{-\lambda_{c_d-1}, \ldots, -\lambda_1\}$, which has odd size, so the selected elements are $-\lambda_{c_d-1}, -\lambda_{c_d-3}, \ldots, -\lambda_1$. Therefore, to construct the rth column, all of the odd indices are selected, so the rth column is still $(-\lambda_{2t-1}, -\lambda_{2t-3}, \ldots, -\lambda_1) = R(\lambda_{\text{odd}})$.

In both cases, the rth column of $\phi(R(\lambda))$ is $R(\lambda_{\text{odd}})$. Therefore, the first r columns of $\phi(R(\lambda))$ are $R(\lambda_{\text{even}}), R(\lambda_{\text{odd}}), R(\lambda_{\text{even}}), \ldots, R(\lambda_{\text{odd}})$. Applying E^{-1} makes $R(\lambda_{\text{even}}$ to $(-\lambda_{2t} - (t-1), -\lambda_{2t-2} - (t-3), \ldots, -\lambda_2 + (t-1))$ and $R(\lambda_{\text{odd}}$ to $(-\lambda_{2t-1} - (t-1), -\lambda_{2t-3} - (t-3), \ldots, -\lambda_1 + (t-1))$, which satisfies

$$-(s-1) = -\lambda_{2t} - (t-1) = -\lambda_{2t-2} - (t-3) = \cdots = -\lambda_2 + (t-1)$$

and

$$-s = -\lambda_{2t-1} - (t-1) = -\lambda_{2t-3} - (t-3) = \cdots = -\lambda_1 + (t-1).$$

Hence, the columns alternate between all $-(s-1)$'s and all $-s$'s, as shown in the diagram below.

$$\begin{array}{ccccc} \Big| & \Big| & \Big| & & \Big| \\ R(\lambda_{\text{even}}) & R(\lambda_{\text{odd}}) & R(\lambda_{\text{even}}) & \cdots & R(\lambda_{\text{odd}}) \\ \Big| & \Big| & \Big| & & \Big| \end{array} \xmapsto{E^{-1}} \begin{array}{ccccc} -(s-1) & -s & -(s-1) & \cdots & -s \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -(s-1) & -s & -(s-1) & \cdots & -s \end{array}$$

Therefore, the first r columns of $E^{-1}(\phi(\lambda))$ alternate between all s's and all $s-1$'s, while the first r columns of $E^{-1}(\phi(R(\lambda)))$ alternate between all $-(s-1)$'s and all $-s$'s. In any case, the elements in a given column are the same, and since the clumps to construct the $(r+1)$th column have underlying set size less than k, we apply the inductive hypothesis in the same way as Subcase 1a, since $r+1$ is odd.

Subcase 2b: r is odd. This means that $r-1$ is even, so there are no more copies of some λ_i with i even. Let C be the set of these indices which have no more copies, i.e. $C = \{i : i \text{ even}, m_i = \frac{r-1}{2}\} = \{c_1 \geq \cdots \geq c_d\}$.

Consider what occurs in $\phi(R(\lambda))$. The $(r-2)$th column is $R(\lambda_{\text{even}})$, and by assumption, there are some λ_i with no more copies remaining, so we have more than one clump when constructing column $r-1$. We claim that the $(r-1)$th column is $R(\lambda_{\text{odd}})$. The underlying sets of the maximal clumps are

$$\{-\lambda_{2t}, \ldots, -\lambda_{c_d+1}\}, \{-\lambda_{c_d-1}, \ldots, -\lambda_{c_{d-1}+1}\}, \ldots, \{-\lambda_{c_1-1}, \ldots, -\lambda_1\}.$$

The first clump's underlying set has even size. Since $r-1$ is even, the selected elements are $-\lambda_{2t-1}, -\lambda_{2t-3}, \ldots, -\lambda_{c_1+1}$. The ith clump, for $2 \leq i \leq d$, has underlying set $\{-\lambda_{c_{d-i+2}-1}, \ldots, -\lambda_{c_{d-i+1}+1}\}$, which has odd size, so the selected elements are $-\lambda_{c_{i-1}-1}, -\lambda_{c_{i-1}-3}, \ldots, -\lambda_{c_i+1}$. The $(d+1)$th clump has underlying set $\{-\lambda_{c_d-1}, \ldots, -\lambda_1\}$, which has odd size, so the selected elements are $-\lambda_{c_d-1}, -\lambda_{c_d-3}, \ldots, -\lambda_1$. Therefore, in construction of the $(r-1)$th column, all of the odd indices are selected, so the $(r-1)$th column is $(-\lambda_{2t-1}, -\lambda_{2t-3}, \ldots, -\lambda_1) = R(\lambda_{\text{odd}})$, as desired.

Therefore, the first $r-1$ columns of $\phi(\lambda)$ are $\lambda_{\text{odd}}, \lambda_{\text{even}}, \lambda_{\text{odd}}, \ldots, \lambda_{\text{even}}$, and the first $r-1$ columns of $\phi(R(\lambda))$ are $R(\lambda_{\text{even}}), R(\lambda_{\text{odd}}), R(\lambda_{\text{even}}), \ldots, R(\lambda_{\text{odd}})$. As in Subcase 2a, the first $r-1$ columns of $E^{-1}(\phi(\lambda))$ alternates between all s's and all $(s-1)$'s, and the first $r-1$ columns of $E^{-1}(\phi(R(\lambda)))$ alternates between all $-(s-1)$'s and all $-s$'s, which is enough to apply the inductive hypothesis in the same way as Subcase 1a, since all clumps have underlying set of size less than k and r is odd.

Finally, we want to show $R(LV'(\lambda)) = LV'(R(\lambda))$. This is essentially the same casework as above, replacing ϕ with ϕ'. By induction, we are done. $\square$

Example 4.2.5. Here are examples for all of the subcases that appear above. It should be noted that the numbers themselves do not affect the shape of the image, but rather the multiplicities of the numbers do.

Subcase 1a: Let $\lambda = (9, 9, 9, 8, 8, 7, 7, 6, 6, 5, 5, 5, 5, 4, 4, 4, 3, 3)$. Then $r = 3$, and we have

$$\lambda \xmapsto{\phi} \begin{ytableau} 9 & 9 & 9 & 8 & 8 \\ 7 & 7 \\ 5 & 5 & 6 & 6 \\ 3 & 3 & 4 & 4 & 5 & 4 & 5 \end{ytableau} \xmapsto{E^{-1}} \begin{ytableau} 6 & 6 & 7 & 6 & 7 \\ 6 & 6 \\ 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 4 & 5 \end{ytableau} \xmapsto{\kappa} ((), (12), (), (24), (32), (), (39))$$

$\min\{m_i : i \text{ odd}\}$

$$R(\lambda) \xmapsto{\phi} \begin{ytableau} {-3} & {-3} \\ {-5} & {-5} & {-4} & {-4} & {-4} & {-5} & {-5} \\ {-7} & {-7} & {-6} & {-6} \\ {-9} & {-9} & {-8} & {-9} & {-8} \end{ytableau} \xmapsto{E^{-1}} \begin{ytableau} {-6} & {-6} \\ {-6} & {-6} & {-6} & {-6} & {-5} & {-5} & {-5} \\ {-6} & {-6} & {-6} & {-6} \\ {-6} & {-6} & {-6} & {-7} & {-7} \end{ytableau} \xmapsto{\kappa} ((), (-12), (), (-24), (-32), (), (-39))$$

Subcase 1b: Let $\lambda = (9, 8, 8, 8, 7, 7, 6, 6, 5, 4, 3, 3)$. Then $r = 2$, and we have

$$\lambda \xmapsto{\phi} \begin{ytableau} 9 & 8 & 8 & 7 & 8 \\ 7 & 6 & 6 \\ 5 \\ 3 & 3 & 4 \end{ytableau} \xmapsto{E^{-1}} \begin{ytableau} 6 & 6 & 6 & 7 & 8 \\ 6 & 6 & 6 \\ 6 \\ 6 & 5 & 6 \end{ytableau} \xmapsto{\kappa} ((6), (), (18, 17), (), (33))$$

$\min\{m_i : i \text{ odd}\}$

$$R(\lambda) \xmapsto{\phi} \begin{ytableau} {-3} & {-4} & {-3} \\ {-5} & {-6} & {-6} \\ {-7} & {-8} & {-8} & {-8} & {-7} \\ {-9} \end{ytableau} \xmapsto{E^{-1}} \begin{ytableau} {-6} & {-6} & {-5} \\ {-6} & {-6} & {-6} \\ {-6} & {-6} & {-6} & {-8} & {-7} \\ {-6} \end{ytableau} \xmapsto{\kappa} ((-6), (), (-17, -18), (), (-33))$$

Subcase 2a: Let $\lambda = (9, 9, 8, 8, 7, 6, 6, 5, 5, 4, 4, 4)$. Then $r = 2$, and we have

$$\lambda \xmapsto{\phi} \begin{ytableau} 9 & 8 & 9 & 8 \\ 7 & 6 & 6 \\ 5 & 4 & 4 & 4 & 5 \end{ytableau} \xmapsto{E^{-1}} \begin{ytableau} 7 & 6 & 7 & 7 \\ 7 & 6 & 6 \\ 7 & 6 & 6 & 5 & 5 \end{ytableau} \xmapsto{\kappa} ((), (), (19), (27), (29))$$

$$R(\lambda) \xmapsto{\phi} \begin{ytableau} {-4} & {-5} & {-4} & {-5} & {-4} \\ {-6} & {-7} & {-6} \\ {-8} & {-9} & {-8} & {-9} \end{ytableau} \xmapsto{E^{-1}} \begin{ytableau} {-6} & {-7} & {-6} & {-6} & {-4} \\ {-6} & {-7} & {-6} \\ {-6} & {-7} & {-6} & {-8} \end{ytableau} \xmapsto{\kappa} ((), (), (-19), (-27), (-29))$$

Subcase 2b: Let $\lambda = (9, 9, 8, 8, 7, 7, 7, 6, 5, 5, 4, 4)$. Then $r = 3$, and we have

$$\lambda \overset{\phi}{\mapsto} \begin{array}{ccccc} & & \overset{r}{\downarrow} & & \\ 9 & 8 & 9 & & \\ 7 & 6 & 7 & 7 & 8 \\ 5 & 4 & 5 & 4 & \end{array} \xrightarrow{E^{-1}} \begin{array}{ccccc} 7 & 6 & 7 & & \\ 7 & 6 & 7 & 6 & 8 \\ 7 & 6 & 7 & 5 & \end{array} \overset{\kappa}{\mapsto} ((),(),(20),(25),(34))$$

$$R(\lambda) \overset{\phi}{\mapsto} \begin{array}{ccccc} & \overset{r-1}{\downarrow} & \overset{r}{\downarrow} & & \\ -4 & -5 & -4 & -5 & \\ -6 & -7 & -7 & -8 & -7 \\ -8 & -9 & -9 & & \end{array} \xrightarrow{E^{-1}} \begin{array}{ccccc} -6 & -7 & -6 & -6 & \\ -6 & -7 & -7 & -7 & -7 \\ -6 & -7 & -7 & & \end{array} \overset{\kappa}{\mapsto} ((),(),(-20),(-25),(-34))$$

This anti-symmetry property implies that each distinguished weight is characterized by the first $\lfloor \frac{n}{2} \rfloor$ elements. This is important when we find the asymptotics regarding $|\Lambda^+_{n,k}|$ next.

4.3. **Sizes of $\Lambda^+_{n,k}$.** Recall that $\Lambda^+_{n,k}$ is the subset of distinguished weights that go to all zeroes with at most k iterations of the algorithm.

Theorem 4.3.1. *We have the following recursive formula*

$$|\Lambda^+_{n,k}| = 1 + k + \sum_{\substack{\alpha \vdash n \\ \alpha=(a^{\ell_a},\ldots,1^{\ell_1}) \\ \alpha \neq (n),(1,\ldots,1)}} \sum_{m=0}^{k-1} \prod_{i=1}^{a} |\Lambda^+_{\ell_i,m}|,$$

where $|\Lambda^+_{0,k}| = 1$ *and* $|\Lambda^+_{1,k}| = 1$ *for* $k \geq 0$.

Proof. Suppose we had a sequence $\lambda \in \Lambda^+_{n,k}$. In the sequences of outputs of repeated iterations of LV_p, there exists m such that after $k-m$ iterations it is in the form

$$\underbrace{\Big(\cdots\Big(}_{k-m}\underbrace{(*,\ldots,*)}_{\ell_1},\underbrace{(*,\ldots,*)}_{\ell_2},\ldots,\underbrace{(*,\ldots,*)}_{\ell_a}\Big)\cdots\Big),$$

where the $*$ are integers and there exists $\alpha \vdash n$ such that $\alpha = (a^{\ell_a},\ldots,1^{\ell_1})$. If $\alpha \neq (n),(1,\ldots,1)$, then the number of such sequences, for a fixed m, is $\prod_{i=1}^{a} |\Lambda^+_{\ell_i,m}|$, because recursively there are $|\Lambda^+_{\ell_i,m}|$ sequences for $\underbrace{(*,\ldots,*)}_{\ell_i}$ to go to all zeroes after at most m iterations of the algorithm.

In the case of $\alpha = (1,\ldots,1)$, this m is not unique, but we know that the number of sequences that go to all zeroes after at most k iterations is $k+1$, given by

$$\underbrace{(\cdots(}_{m}0,\ldots,0)\cdots)$$

for $1 \leq m \leq k+1$. In the case of $\alpha = (n)$, we know that LV_p sends $(0,\ldots,0) \mapsto ((),\ldots,(),(0))$, so this does not generate any additional sequences.

Therefore, the count of $|\Lambda^+_{n,k}|$ is equal to the sum

$$(k+1) + \sum_{\substack{\alpha \vdash n \\ \alpha=(a^{\ell_a},\ldots,1^{\ell_1}) \\ \alpha \neq (n),(1,\ldots,1)}} \sum_{m=0}^{k-1} \prod_{i=1}^{a} |\Lambda^+_{\ell_i,m}|,$$

as desired. $\square$

Example 4.3.2. Using this recursive formula, we can compute that

$$\begin{aligned}
|\Lambda^+_{1,k}| &= 1,\\
|\Lambda^+_{2,k}| &= k+1,\\
|\Lambda^+_{3,k}| &= 2k+1,\\
|\Lambda^+_{4,k}| &= k^2+3k+1,\\
|\Lambda^+_{5,k}| &= 2k^2+4k+1,\\
|\Lambda^+_{6,k}| &= \frac{4k^3+27k^2+29k+6}{6}.
\end{aligned}$$

From these smaller examples, we can observe that the leading power is always $\lfloor \frac{n}{2} \rfloor$.

Proposition 4.3.3. *For a fixed n, we have*

$$|\Lambda^+_{n,k}| = \Theta(k^{\lfloor \frac{n}{2} \rfloor}).$$

Proof. We proceed by induction. The cases $k = 1, \dots, 6$ are shown in the above example. Now, suppose that it is true for $k = 1, \dots, j-1$.

Let $\alpha \vdash n$ such that $\alpha \neq (n), (1, \dots, 1)$ and $\alpha = (a^{\ell_a}, \dots, 1^{\ell_1})$. Then by casework on the parity of n, it is clear that

$$\left\lfloor \frac{\ell_1}{2} \right\rfloor + \cdots + \left\lfloor \frac{\ell_a}{2} \right\rfloor \leq \left\lfloor \frac{n-2}{2} \right\rfloor.$$

This means that

$$\sum_{m=0}^{j-1} \prod_{i=1}^{a} |\Lambda^+_{\ell_i,m}| = \sum_{m=0}^{j-1} \prod_{i=1}^{a} \Theta(m^{\lfloor \frac{\ell_i}{2} \rfloor}) \leq \sum_{m=0}^{j-1} \Theta(m^{\lfloor \frac{n-2}{2} \rfloor}) = \Theta(j^{\lfloor \frac{n}{2} \rfloor}).$$

Therefore, $|\Lambda^+_{n,j}| \leq \Theta(j^{\lfloor \frac{n}{2} \rfloor})$. We claim that equality is achieved. Consider $\alpha = (2, 1, \dots, 1)$. Then $\ell_1 = n-2$ and $\ell_2 = 1$, so $\lfloor \frac{\ell_1}{2} \rfloor + \lfloor \frac{\ell_2}{2} \rfloor = \lfloor \frac{n-2}{2} \rfloor + \lfloor \frac{1}{2} \rfloor = \lfloor \frac{n-2}{2} \rfloor$, achieving equality. Therefore, $|\Lambda^+_{n,j}| = \Theta(j^{\lfloor \frac{n}{2} \rfloor})$, concluding the induction. □

We can further refine this asymptotic by computing the leading coefficient in the following theorem.

Theorem 4.3.4. *We have*

$$|\Lambda^+_{n,k}| \sim \frac{a_{\lfloor \frac{n+1}{2} \rfloor}}{\lfloor \frac{n}{2} \rfloor !} k^{\lfloor \frac{n}{2} \rfloor},$$

where a_i is a sequence defined by $a_0 = a_1 = 1$ and $a_i = a_{i-1} + (i-1)a_{i-2}$.

Remark. The numbers a_i are sometimes called the telephone numbers (OEIS [6, A000085]). Some examples of its appearances include:

(1) The number of self-inverse permutations (involutions) on n letters,
(2) The number of standard Young tableaux with n cells,
(3) The sum of the degrees of the irreducible representations of the symmetric group S_n.

Proof. From Proposition 4.3.3, we know that $|\Lambda^+_{n,k}| = \Theta(k^{\lfloor \frac{n}{2} \rfloor})$. It remains to find the coefficient of the leading term in the recursion given by Theorem 4.3.1. Let b_n be the leading coefficient, i.e. $|\Lambda^+_{n,k}| \sim b_n k^{\lfloor \frac{n}{2} \rfloor}$. We know that $b_0, \dots, b_4$ are $1, 1, 1, 2, 1$, respectively.

As in the proof of Proposition 4.3.3, the only terms that contribute to this leading term correspond to partitions $\alpha \vdash n$ such that $\alpha = (a^{\ell_a}, \dots, 1^{\ell_1})$ such that

$$\left\lfloor \frac{\ell_1}{2} \right\rfloor + \cdots + \left\lfloor \frac{\ell_a}{2} \right\rfloor = \left\lfloor \frac{n-2}{2} \right\rfloor.$$

If n is even, the only partitions that satisfy this are $(2, 1, \dots, 1)$ and $(2, 2, 1, \dots, 1)$. This means that

$$|\Lambda^+_{n,k}| \sim \sum_{m=0}^{k-1} b_1 \cdot b_{n-2} k^{\lfloor \frac{n}{2} \rfloor - 1} + \sum_{m=0}^{k-1} b_2 \cdot b_{n-4} k^{\lfloor \frac{n}{2} \rfloor - 1} \sim \frac{b_{n-2} + b_{n-4}}{\lfloor \frac{n}{2} \rfloor} k^{\lfloor \frac{n}{2} \rfloor},$$

where we use Faulhaber's formula for the sum of powers. Therefore, if n is even, $b_n = \frac{2(b_{n-2}+b_{n-4})}{n}$.

On the other hand, if n is odd, the only partitions that satisfy this are $(2,1,\dots,1),(3,1,\dots,1)$, and $(2,2,1,\dots,1)$. This means that

$$|\Lambda^+_{n,k}| \sim \sum_{m=0}^{k-1}(b_1b_{n-2}+b_1b_{n-3}+b_2b_{n-4})k^{\lfloor \frac{n}{2}\rfloor -1} \sim \frac{b_{n-2}+b_{n-3}+b_{n-4}}{\lfloor \frac{n}{2}\rfloor}k^{\lfloor \frac{n}{2}\rfloor}.$$

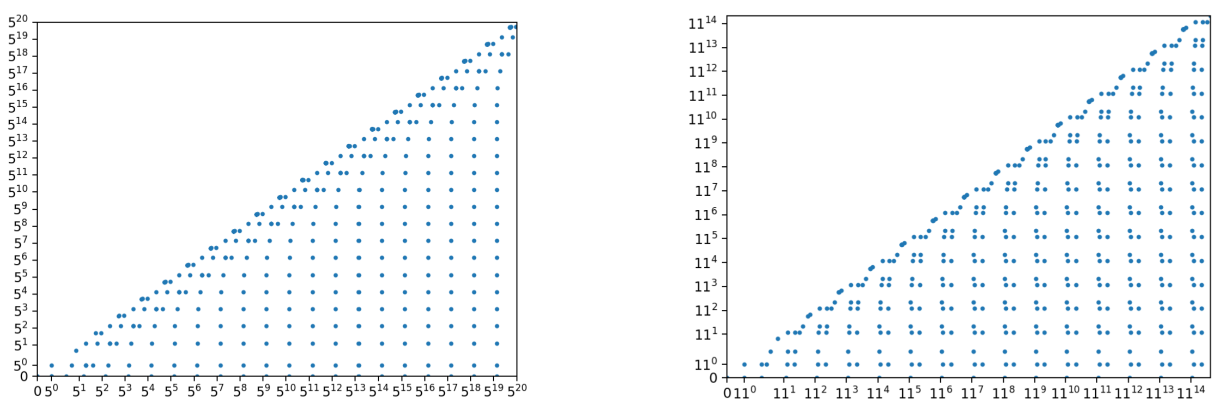
Therefore, if n is odd, then $b_n = \frac{2(b_{n-2}+b_{n-3}+b_{n-4})}{n-1}$.

Now, we claim that $b_n = \frac{a_{\lfloor \frac{n+1}{2}\rfloor}}{\lfloor \frac{n}{2}\rfloor !}$ where a_i is the sequence defined in the statement. This can be checked to work for the base cases. Using induction and by direct substitution, we could show that this also satisfies the two recursive equations we found above for b_n for both even and odd n. □

4.4. **Understanding the asymptotics.** The appearance of these numbers is quite interesting, especially with its natural connections to permutations and partitions. In an effort to explain this relationship, we note the following. First, each iteration brings the magnitude of the numbers down by a factor of p, so if we end up at zeroes after at most k iterations, then the initial numbers must be at most on the order of p^k. Also, this means that each number is going to be on the order of a power of p, plus some smaller order terms. Furthermore, in Section 4.2, we show that each distinguished weight is anti-symmetric, so each sequence is identified by the first $\lfloor \frac{n}{2}\rfloor$ numbers in the sequence. Putting these facts together, we find that each distinguished weight can be approximated by an anti-symmetric, weakly decreasing sequence of powers of p. Also, this implies our asymptotic also approximates the number of distinguished weights in the cube $[-p^k,p^k]^n$.

Furthermore, this intuition aligns with the asymptotics found in the above theorem; we can count the number of weakly decreasing sequences of length $\lfloor \frac{n}{2}\rfloor$ such that each can be drawn from the set $\{p^0,\dots,p^{k-1}\}$. By stars and bars, this is $\binom{\lfloor \frac{n}{2}\rfloor + k-1}{k-1} = \frac{(\lfloor \frac{n}{2}\rfloor + k-1)\cdots(k)}{\lfloor \frac{n}{2}\rfloor !}$. When $k >> n$, this is asymptotically $\frac{k^{\lfloor \frac{n}{2}\rfloor}}{\lfloor \frac{n}{2}\rfloor !}$, which is exactly the form in the theorem above without the telephone number coefficient.

For $n=4$, all distinguished weights are in the form $(x,y,-y,-x)$. Graphing the points (x,y) for $p=5$ and at most 20 iterations, we get the scatter plot shown below in Figure 1a. Similarly, for $n=5$, $p=11$, and at most 15 iterations, all distinguished weights are in the form $(x,y,0,-y,-x)$, and the graph that arises is shown below in Figure 1b. First, we can observe a very regular pattern outside of the diagonals. To be precise, we can observe that for (k_1,k_2) where $k_1 > k_2+2$, then there is 1 distinguished weight $(x,y,-y,-x)$ and 3 distinguished weights $(x,y,0,-y,-x)$ such that $p^{k_1} \le x \le p^{k_1+1}$ and $p^{k_2} \le y \le p^{k_2+1}$. Second, we can observe the quadratic growth that we proved in Proposition 4.3.3 through the "interior" points. It should be noted that if we zoom in, there is also quadratic growth of the points along the diagonal $y=x$, which appear in very small and tight clusters and are almost unnoticeable in the figures below.



(A) Scatter plot of (x,y) for all distinguished weights $(x,y,-y,-x) \in \Lambda_4^+$ where $p=5$. Axes are log-scaled.

(B) Scatter plot of (x,y) for all distinguished weights $(x,y,0,-y,-x) \in \Lambda_5^+$ where $p=11$. Axes are log-scaled.

FIGURE 1

This leads us to the following observation: suppose we have nonnegative integers $k_1, \ldots, k_{\lfloor \frac{n}{2} \rfloor}$ such that $k_i \geq k_{i+1} + \epsilon$ for $1 \leq i < \lfloor \frac{n}{2} \rfloor$ for some $\epsilon > 0$ large enough. Then the number of distinguished weights $\lambda = (\lambda_1, \lambda_2, \ldots, -\lambda_1)$ such that $p^{k_i} \leq \lambda_i < p^{k_i+1}$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ is 1 if n is even and $\frac{n+1}{2}$ if n is odd. This can be unrigorously seen recursively:

If n is even, then, as in the proof of Theorem 4.3.4, the leading term of the asymptotic comes recursively from the partitions $(2, 1, \ldots, 1)$ and $(2, 2, 1, \ldots, 1)$. We can explicitly describe the inverse of

$$((\lambda_1, \lambda_2, \ldots, \lambda_{\frac{n}{2}-1}, -\lambda_{\frac{n}{2}-1}, \ldots, -\lambda_1), (0))$$

and

$$((\lambda_1, \lambda_2, \ldots, \lambda_{\frac{n}{2}-2}, -\lambda_{\frac{n}{2}-2}, \ldots, -\lambda_1), (0, 0))$$

through iterations of LV_p^{-1} and keep track of just the largest order of power of p in each term of the sequence. Then we could see inductively that if the $n-2$ case only has 1 distinguished weight with a particular decreasing sequence of powers of p with large enough consecutive differences, then it is the same in the case of n. Since there is only 1 for $n = 2$, then the count always is the same for even n.

If n is odd, then similar analysis shows that there is always 1 more than the $n-2$ case, so the number of these points grow linearly with n as $\frac{n+1}{2}$.

However, this discussion does not explain combinatorially the appearance of the telephone numbers in the asymptotics of $|\Lambda^+_{n,k}|$. We would like to have a more direct way of interpreting these distinguished weights and the leading term of this asymptotic than through a recursive equation.

5. Distinguished weights in ideals

In this section, we introduce a more algebraic way that these distinguished weights should appear, by constructing particular ideals of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. This construction of these ideals is motivated from representation theory and has very natural similarities with the algorithm, which suggests a relationship with the distinguished weights that we have seen above.

5.1. **Basic definitions.** Let $R_n := \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, on which S_n acts by permuting the x_i. As a S_n-representation, we can write $R_n = \bigoplus_{\lambda \vdash n} M_\lambda \otimes V_\lambda$, where the M_λ is for multiplicity. The next few definitions define ideals $I_{\leq \lambda}$ and $I_{<\lambda}$, which in turn allows us to define I_λ, a quotient ideal that we study further in later sections.

Definition 5.1.1. Let $\lambda \vdash n$ be a partition of n. Define $(R_n)_\lambda := M_\lambda \otimes V_\lambda$ be the λ-isotypic component of R_n as an S_n-representation.

Example 5.1.2. $(R_n)_{(n)} = R_n^{S_n}$ and $(R_n)_{(1,\ldots,1)} = \left(\prod_{1 \leq i < j \leq n} (x_i - x_j)\right) R_n^{S_n}$.

Definition 5.1.3. Define the ideal $I_{\leq \lambda} = R_n \cdot (R_n)_\lambda \subset R_n$.

Example 5.1.4. $I_{\leq (n)} = R_n R_n^{S_n} = R_n$ and $I_{\leq (1,\ldots,1)} = R_n \left(\prod_{1 \leq i < j \leq n} (x_i - x_j)\right) R_n^{S_n} = \left(\prod_{1 \leq i < j \leq n} (x_i - x_j)\right)$.

Theorem 5.1.5 (From [4]). *The ideal $I_{\leq \lambda}$ is generated by $(R_n)_\mu$ for all $\mu \leq \lambda$.*

Let $\leq$ denote the partial ordering of partitions of n by dominance, i.e. $\lambda \geq \mu$, where $\lambda = (\lambda_1, \ldots, \lambda_s)$ and $\mu = (\mu_1, \ldots, \mu_t)$, if and only if

$$\begin{gathered} \lambda_1 \geq \mu_1 \\ \lambda_1 + \lambda_2 \geq \mu_1 + \mu_2 \\ \lambda_1 + \lambda_2 + \lambda_3 \geq \mu_1 + \mu_2 + \mu_3 \\ \vdots \end{gathered}$$

We have the following proposition.

Proposition 5.1.6 (From [4]). *$I_{\leq \mu} \subset I_{\leq \lambda}$ if and only if $\mu \leq \lambda$.*

Finally, we define $I_{<\lambda}$ and I_λ.

Definition 5.1.7. Define the ideal $I_{<\lambda} = \bigcup_{\mu < \lambda} I_{\leq \mu}$. Finally, define the quotient ideal $I_\lambda = I_{\leq \lambda} / I_{<\lambda}$.

In the following definition, we define a special element in $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, which lets us write the generators of I_λ.

Definition 5.1.8. Let $\lambda = (\lambda_1, \dots, \lambda_s)$. This naturally defines a partition of $\{1, \dots, n\}$ into s sets $J_1, \dots, J_s$ such that $J_k = \{\lambda_1 + \cdots + \lambda_{k-1} + 1, \dots, \lambda_1 + \cdots + \lambda_k\}$. Define

$$\Pi_\lambda := \prod_{k=1}^{s} \prod_{i,j \in J_k, i<j} (x_i - x_j).$$

We may also use the notation Π_λ but over different variables and different partitions of the corresponding index set $\{1, \dots, n\}$.

Proposition 5.1.9 (From [4])**.** *The $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$-module I_λ is generated by $\{S_n \cdot \Pi_{\lambda^*}\}$, where λ^* is the conjugate partition.*

5.2. **Properties of I_λ.** We now prove some properties of these modules that we have just constructed. We begin by defining a class of points in $\mathbb{C}^n$ which form an important subvariety that appears throughout this section.

Definition 5.2.1. Let $\alpha \vdash n$ be a partition of n with $\alpha = (\alpha_1, \dots, \alpha_l)$. Call $z \in \mathbb{C}^n$ with $z = (z_1, \dots, z_n)$ a *point of type* α if we can partition $\{z_i\}$ into subsets $P_1, \dots, P_l$ such that $|P_i| = \alpha_i$ for all i and if $z_i, z_j \in P_k$, then $z_i = z_j$ for any i, j, k. Let

$$H_\alpha := \{\text{all points of type } \alpha\} \subset \mathbb{C}^n.$$

Further define

$$H_{>\alpha} := \bigcup_{\mu > \alpha} H_\mu$$

and

$$H_{\geq\alpha} := \bigcup_{\mu \geq \alpha} H_\mu,$$

where the order on partitions is given by dominance order. It is clear that if $\mu < \lambda$, then $H_{>\lambda} \subset H_{>\mu}$.

Proposition 5.2.2. *The points in $H_{>\lambda}$ are exactly the ones on which Π_{λ^*} vanishes.*

Proof. Suppose Π_{λ^*} vanishes on some H_μ, where $\mu = (\mu_1, \dots, \mu_l)$. This is equivalent to saying that if we fill a Young diagram of shape λ with the numbers

$$\{\underbrace{y_1, \dots, y_1}_{\mu_1}, \underbrace{y_2, \dots, y_2}_{\mu_2}, \dots, \underbrace{y_l, \dots, y_l}_{\mu_l}\},$$

then no matter the filling, there always exists a column with two of the same element.

First, we show Π_{λ^*} vanishes on $H_{>\lambda}$. Choose any $\mu \vdash n$ such that $\mu > \lambda$. Let $\mu = (\mu_1, \dots, \mu_t)$. Define i to be the minimal index such that $\lambda_i \neq \mu_i$. Since $\mu > \lambda$, we must have that $\mu_i > \lambda_i$.

Consider filling a Young diagram of shape λ with elements given by μ. Suppose for the sake of contradiction that no two elements in a column are the same. This means that the μ_1 elements y_1 must all be in distinct columns, so without loss of generality, put them in the first row. Then the μ_2 elements y_2 must all be in distinct columns, so we may place them in the second row. We can keep doing this until the μ_i elements y_i, in which case they must be in distinct rows, but $\lambda_i < \mu_i$, so we must have two elements in the same column, contradiction.

Second, we show that if Π_{λ^*} vanishes on H_μ, then $\mu > \lambda$. We claim that if $\alpha < \lambda$, then Π_{λ^*} vanishing on H_μ implies that Π_{α^*} vanishes on H_μ. This is clear because if any filling of a Young diagram of shape λ by numbers given by μ must have two of the same element in some column, then since $\alpha < \lambda$, any filling of a Young diagram of shape α has the same property. Now, assume for the sake of contradiction that $\mu \leq \lambda$ and Π_{λ^*} vanishes on H_μ. Since $\mu \leq \lambda$, then by the previous claim, Π_{μ^*} vanishes on H_μ. However, this is a contradiction because a possible filling of the Young diagram is to have all of the same elements in the same row, and this clearly does not have two elements in some column being the same.

Therefore, Π_{λ^*} vanishes exactly on the points in the set $H_{>\lambda}$. $\square$

By Hilbert's Nullstellensatz, for every $I_{\leq\lambda}$, there is a corresponding subvariety of $\mathbb{C}^n$. We know that $I_{\leq\lambda}$ is generated by $S_n \cdot \Pi_{\mu^*}$ for all $\mu \leq \lambda$. This means that the corresponding subvariety is $\bigcap_{\mu \leq \lambda} H_{>\mu} = H_{>\lambda}$. Similarly, the corresponding subvariety to $I_{<\lambda}$ is $\bigcap_{\mu \leq \lambda} H_{>\mu} = H_{\geq\lambda}$.

Definition 5.2.3. Let $\alpha \vdash n$ with $\alpha = (\alpha_1, \ldots, \alpha_l)$. Call a hypersurface $H \subset \mathbb{C}^n$ to be *of type* α if we can partition the coordinates $\{1, 2, \ldots, n\}$ into subsets $P_1, \ldots, P_l$ such that $|P_i| = \alpha_i$ for all i and if $z \in H$ with $z = (z_1, \ldots, z_n)$, then $z_i = z_j$ if $i, j \in P_k$ for all i, j, k. Note that this is not H_α, but rather a subset, because this definition fixes the partition of the coordinates; further note that the union of all hypersurfaces of type α is exactly H_α.

Lemma 5.2.4. *If $\lambda \vdash n$ and $\lambda \neq (1, \ldots, 1)$, then the intersection of two distinct hypersurfaces of type λ is a subset of $H_{>\lambda}$.*

Proof. Let the distinct hypersurfaces be H_1 and H_2, both of type λ. Their intersection must be another hypersurface H' of type α for some $\alpha \vdash n$. Furthermore, since $H' \subset H_1$, then we must have that $\alpha \geq \lambda$; if equality holds, then either $H' = H_1$, which would imply $H_1 = H_2$, or $\lambda = (1, \ldots, 1)$, both of which contradict our assumptions. Thus, we must have $\alpha > \lambda$ strictly. Therefore, we have that

$$H' \subset H_\alpha \subset H_{>\lambda},$$

as desired. $\square$

For the rest of this section, suppose $\lambda = (\lambda_1, \ldots, \lambda_m)$. This naturally defines a partition of $\{1, \ldots, n\}$ into m sets $J_1, \ldots, J_m$ such that $J_k = \{\lambda_1 + \cdots + \lambda_{k-1} + 1, \ldots, \lambda_1 + \cdots + \lambda_k\}$. Define J_λ as the ideal generated by

$$\{x_i - x_j : i, j \in J_k \text{ for all } k = 1, \ldots, m\}.$$

For example, if $\lambda = (2, 2)$, then $J_\lambda = (x_1 - x_2, x_3 - x_4)$. Denote

$$\mathbb{C}[x_{\lambda_1}, \ldots, x_{\lambda_m}] := \mathbb{C}[x_1, \ldots, x_n]/J_\lambda.$$

This is equivalent to identifying $\{x_i : i \in J_j\}$ with each other, for all $j = 1, \ldots, m$. For example, if $\lambda = (2, 2)$, then this is equivalent to identifying x_1 with x_2 and x_3 with x_4 in $\mathbb{C}[x_1, x_2, x_3, x_4]$.

Next, we know that S_n has a natural action on $\{x_1, \ldots, x_n\}$ by permutation. Denote S_λ as the subgroup of S_n which has a natural action on $\{x_{\lambda_1}, \ldots, x_{\lambda_m}\}$ by permuting the x_{λ_i} with equal λ_i. For example, if $\lambda = (3, 3, 2, 2, 2)$, then $S_\lambda \cong S_2 \times S_3$.

Define $\varphi_1 : \mathbb{C}[x_1, \ldots, x_n]^{S_n} \to \mathbb{C}[x_1, \ldots, x_n]$ as the identity map, and define $\varphi_2 : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_{\lambda_1}, \ldots, x_{\lambda_m}]^{S_\lambda}$. Define $\varphi = \varphi_2 \circ \varphi_1$.

Define D_λ as the hypersurface of type λ in the form

$$D_\lambda := \{(\underbrace{z_1, \ldots, z_1}_{\lambda_1}, \underbrace{z_2, \ldots, z_2}_{\lambda_2}, \ldots, \underbrace{z_m, \ldots, z_m}_{\lambda_m}) \in \mathbb{C}^n\}.$$

We know that $D_\lambda \subset H_\lambda$. Note that variables of $\mathbb{C}[x_{\lambda_1}, \ldots, x_{\lambda_m}]$ and the points of D_λ correspond naturally.

Lemma 5.2.5. *We have that* $\ker(\varphi) = I_{<\lambda}^{S_n}$.

Proof. First, note that the map $\varphi_2 : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_{\lambda_1}, \ldots, x_{\lambda_m}]^{S_\lambda}$ factors through $\mathbb{C}[x_1, \ldots, x_n]/I_{<\lambda}$ because $D_\lambda \subset H_{\geq\lambda}$ and $H_{\geq\lambda}$ is the subvariety corresponding to $I_{<\lambda}$. Thus, $I_{<\lambda}^{S_n} \subset \ker(\varphi)$.

On the other hand, an element of $\mathbb{C}[x_1, \ldots, x_n]$ is in $I_{\leq\lambda}$ if and only if it vanishes on $H_{>\lambda}$. If an element of $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$ is in $\ker(\varphi)$, then it vanishes on D_λ, and by symmetry, it must vanish on all of H_λ. Thus, an element in $I_{\leq\lambda}^{S_n}$ that is in $\ker(\varphi)$ must vanish on both $H_{>\lambda}$ and H_λ, so it vanishes on $H_{\geq\lambda}$. Thus it must be in $I_{>\lambda}^{S_n}$, so $\ker(\varphi) \subset I_{<\lambda}^{S_n}$.

Therefore, we have $\ker(\varphi) = I_{<\lambda}^{S_n}$, as desired. $\square$

As a result of this lemma, we know that $I_{\leq\lambda}^{S_n}/I_{<\lambda}^{S_n} \cong \varphi(I_{\leq\lambda}^{S_n})$. Next, we show the following lemma:

Lemma 5.2.6. *For $\sigma \in S_n$, we have that $\varphi_2(\sigma \cdot \Pi_{\mu^*})$ is either $\pm v_\lambda$ or 0 for all $\mu \leq \lambda$, where*

$$v_\lambda := \prod_{1 \leq i < j \leq m} (x_{\lambda_i} - x_{\lambda_j})^{\lambda_j}.$$

Proof. Suppose $\mu < \lambda$. From Proposition 5.2.2, we know that $\sigma \cdot \Pi_{\mu^*}$ vanishes on $H_{>\mu}$, so it vanishes on H_λ. In particular, it vanishes on D_λ, so $\varphi_2(\sigma \cdot \Pi_{\mu^*}) = 0$.

The only case left is when $\mu = \lambda$. Note that one way to compute a $\sigma \cdot \Pi_{\lambda^*}$ is by filling in a Young diagram of shape λ (where rows correspond to the parts of λ) with the variables $x_1, \ldots, x_n$, then taking the product of $(x_i - x_j)$ for x_i and x_j in the same column. Then, the map φ_2 will identify the variables in each of the following sets with the other variables in the same set:

$$\begin{aligned} J_1 &:= \{x_1, \dots, x_{\lambda_1}\}, \\ J_2 &:= \{x_{\lambda_1+1}, \dots, x_{\lambda_1+\lambda_2}\}, \\ &\vdots \\ J_m &:= \{x_{\lambda_1+\cdots+\lambda_{m-1}+1}, \dots, x_{\lambda_1+\cdots+\lambda_m}\} \subset \{x_1, \dots, x_n\}, \end{aligned}$$

where all of these sets are seen as subsets of $\{x_1, \dots, x_n\}$.

In order for $\sigma \cdot \Pi_{\lambda^*}$ not to vanish under φ_2, then the elements of J_1 must be in different columns. Since λ_1 is the greatest part in the partition, and since we do not care about which rows the variables end up in when computing $\sigma \cdot \Pi_{\lambda^*}$ (up to sign ± 1, since we only care about what elements are in which columns), then without loss of generality, they are all in the first row. Then the second set J_2 must all be in different columns. By similar logic, they are in the second row. We continue in this fashion.

Next, under the map φ_2, the entire first row will now be identified together, denoted as x_{λ_1}; the entire second row will be identified together, denoted as x_{λ_2}; and so on. Therefore, we have

$$\sigma \cdot \Pi_{\lambda^*} = \prod_{k=1}^{m} \prod_{x_i, x_j \in J_k, i \neq j} (x_i - x_j) = (\pm 1) \cdot \prod_{1 \leq k_1 < k_2 \leq m} (x_{\lambda_{k_1}} - x_{\lambda_{k_2}})^{\lambda_{k_2}} = \pm v_\lambda,$$

where the second equality is because the factor $(x_{\lambda_{k_1}} - x_{\lambda_{k_2}})$ will occur as many times as rows k_1 and k_2 of the partition share columns, which is exactly $\min\{\lambda_{k_1}, \lambda_{k_2}\} = \lambda_{k_2}$. Hence, if some $\sigma \cdot \Pi_{\lambda^*}$ does not vanish under φ_2, then its image must be exactly $\pm v_\lambda$. ☐

Lemma 5.2.7. *We have that* $\varphi(I_{\leq\lambda}^{S_n}) \subset v_\lambda^2 \mathbb{C}[x_{\lambda_1}, \dots, x_{\lambda_m}]^{S_\lambda}$.

Proof. Suppose that $x \in I_{\leq\lambda}^{S_n}$. Then we know we can write x in terms of the generators, so

$$x = \sum_{\sigma \in S_n} (\sigma \cdot \Pi_{\lambda^*}) h_{\sigma,\lambda} + \sum_{\sigma \in S_n} \sum_{\mu < \lambda} (\sigma \cdot \Pi_{\mu^*}) h_{\sigma,\mu}$$

for some elements $h_{\sigma,\mu}$ for $\mu \leq \lambda$. Since this element is symmetric, then we must have that it is equal, up to scaling, to its symmetrizer, i.e. averaging all permutations. Denote the symmetrizing function as Sym, which we know is linear. This means that

$$x = \mathrm{Sym}(x) = \sum_{\sigma \in S_n} \mathrm{Sym}((\sigma \cdot \Pi_{\lambda^*}) h_{\sigma,\lambda}) + \sum_{\sigma \in S_n} \sum_{\mu < \lambda} \mathrm{Sym}((\sigma \cdot \Pi_{\mu^*}) h_{\sigma,\mu}).$$

We claim that $\mathrm{Sym}(\Pi_{\lambda^*} \cdot h) \in (\{S_n \cdot \Pi_{\lambda^*}^2\})$ for any h. We know

$$\mathrm{Sym}(\Pi_{\lambda^*} \cdot h) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \cdot (\Pi_{\lambda^*} \cdot h).$$

Recall that Π_{λ^*} is created by partitioning $\{1, \dots, n\}$ to sets of size $\lambda_1^*, \lambda_2^*, \dots,$ in order, then multiplying $(x_i - x_j)$ for i, j in the same set. Similarly, $\sigma \cdot \Pi_{\lambda^*}$ partitions $\{1, \dots, n\}$, but they are not necessarily in order. This means that we can partition the set $\{\sigma : \sigma \in S_n\}$, depending on the partition of $\{1, \dots, n\}$ that $\sigma \cdot \Pi_{\lambda^*}$ is associated with. It will be split into $\binom{n}{\lambda_1^*, \lambda_2^*, \dots}$ groups of size $(\lambda_1^*)!(\lambda_2^*)! \cdots$.

Let Λ be such a group and let $\tau \in \Lambda$ be some fixed element. We know that all $\sigma \cdot \Pi_{\lambda^*} = \pm \tau \cdot \Pi_{\lambda^*}$ for $\sigma \in \Lambda$ by definition of Λ. Thus, $\sum_{\sigma \in \Lambda} \sigma \cdot (\Pi_{\lambda^*} \cdot h) = (\tau \cdot \Pi_{\lambda^*}) \sum_{\sigma \in \Lambda} (\pm 1_\sigma) \sigma \cdot h$. Suppose that $(x_i - x_j)$ is one of the factors of $\tau \cdot \Pi_{\lambda^*}$. Then the transposition $(i\ j)$ will permute the elements in Λ with each other. This means that

$$\sum_{\sigma \in \Lambda} (\pm 1_\sigma)(\sigma \cdot h) = \sum_{\sigma \in \Lambda} (\pm 1_{(i\ j)\sigma})((i\ j)\sigma \cdot h).$$

But note that the sign corresponding with σ is opposite the sign corresponding with $(i\ j)\sigma$, because if we switch i and j, then the sign of $(x_i - x_j)$ will flip. Therefore, we have

$$\sum_{\sigma \in \Lambda} (\pm 1_\sigma)(\sigma \cdot h) = \sum_{\sigma \in \Lambda} (\pm 1_{(i\ j)\sigma})((i\ j)\sigma \cdot h) = -\sum_{\sigma \in \Lambda} (\pm 1_\sigma)((i\ j)\sigma \cdot h).$$

Therefore, switching i and j in the sum $\sum_{\sigma\in\Lambda}(\pm 1_\sigma)(\sigma\cdot h)$ will negate it. If we let $h(x_i, x_j, \dots) = \sum_{\sigma\in\Lambda}(\pm 1_\sigma)(\sigma\cdot h)$, then $h(x_j, x_i, \dots) = -h(x_i, x_j, \dots)$. Identifying x_i with x_j gives $h(x_i, x_i, \dots) = -h(x_i, x_i, \dots)$, so we must have $h(x_i, x_i, \dots) = 0$. Therefore, identifying x_i with x_j makes the element 0, so the element must be divisible by $(x_i - x_j)$. Therefore, we know that $\sum_{\sigma\in\Lambda}(\pm 1_\sigma)(\sigma\cdot h)$ is divisible by $(x_i - x_j)$. Since $(x_i - x_j)$ was an arbitrary factor of $\tau\cdot\Pi_{\lambda^*}$, then we know that $\tau\cdot\Pi_{\lambda^*}$ must divide $\sum_{\sigma\in\Lambda}(\pm 1_\sigma)(\sigma\cdot h)$. Thus, $\sum_{\sigma\in\Lambda}\sigma\cdot(\Pi_{\lambda^*}\cdot h)$ is divisible by $(\tau\cdot\Pi_{\lambda^*})^2$.

Since we chose an arbitrary Λ, then we know that $\sum_{\sigma\in S_n}\sigma\cdot(\Pi_{\lambda^*}\cdot h)$ can be written as a linear combination of the elements $(\tau\cdot\Pi_{\lambda^*})^2$ for all $\tau\in S_n$. Therefore, $\mathrm{Sym}(\Pi_{\lambda^*}\cdot h)$ is in the ideal generating by $\{S_n\cdot\Pi^2_{\lambda^*}\}$, as claimed. A similar argument tells us that $\mathrm{Sym}((\sigma\cdot\Pi_{\lambda^*})h_{\sigma,\lambda})\in(\{S_n\cdot\Pi^2_{\lambda^*}\})$.

Therefore, we have

$$\begin{aligned}
\varphi(x) &= \varphi\left(\sum_{\sigma\in S_n}\mathrm{Sym}((\sigma\cdot\Pi_{\lambda^*})h_{\sigma,\lambda}) + \sum_{\sigma\in S_n}\sum_{\mu<\lambda}\mathrm{Sym}((\sigma\cdot\Pi_{\mu^*})h_{\sigma,\mu})\right)\\
&= \varphi\left(\sum_{\sigma\in S_n}\mathrm{Sym}((\sigma\cdot\Pi_{\lambda^*})h_{\sigma,\lambda})\right) + 0\\
&\subset (\{\varphi(\tau\cdot\Pi^2_{\lambda^*}) : \tau\in S_n\})\\
&= (v_\lambda^2),
\end{aligned}$$

where the second equality is because Π_{μ^*} vanishes under φ for $\mu<\lambda$, the subset is from the claim above, and the last equality is from Lemma 5.2.6.

Since x was symmetric, then $\varphi(x)$ must be invariant from S_λ action, which means that $\varphi(x)\in v_\lambda^2\mathbb{C}[x_{\lambda_1},\dots,x_{\lambda_m}]^{S_\lambda}$, as desired. □

This leads us to the main result of this section:

Theorem 5.2.8. *We have that $I^{S_n}_{\le\lambda}/I^{S_n}_{<\lambda}\cong v_\lambda^2\mathbb{C}[x_{\lambda_1},\dots,x_{\lambda_m}]^{S_\lambda}$.*

Proof. Let HS denote the Hilbert Series. Then Lemma 5.2.7 tells us that

$$\mathrm{HS}(\varphi(I^{S_n}_{\le\lambda}))\le \mathrm{HS}(v_\lambda^2\mathbb{C}[x_{\lambda_1},\dots,x_{\lambda_m}]^{S_\lambda}),$$

where the $\le$ symbol denotes that the coefficient of t_i of the left-hand side is less than or equal to the coefficient of t_i of the right-hand side.

We compute the Hilbert Series of the right-hand side. The degree of v_λ^2 is $2(\binom{\lambda_1^*}{2}+\binom{\lambda_2^*}{2}+\cdots)$. Denote $m_i(\lambda)$ as the multiplicity of part i in the partition λ. Then S_λ permutes the parts of size i in the partition λ, so any element of $\mathbb{C}[x_{\lambda_1},\dots,x_{\lambda_m}]^{S_\lambda}$ is symmetric with respect to the $m_i(\lambda)$ elements corresponding to parts in the partition of size i, i.e. symmetric with respect to permuting $\{x_{\lambda_j} : \lambda_j = i\}$. However, symmetric polynomials can be written in terms of elementary symmetric polynomials, which have degrees $1, 2, \dots, m_i(\lambda)$. The corresponding Hilbert Series will be $\frac{1}{(1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)})}$. Doing this for all i, and combining with the degree of v_λ, we get that the Hilbert Series on the right-hand side is

$$\frac{t^{2\left(\binom{\lambda_1^*}{2}+\binom{\lambda_2^*}{2}+\cdots\right)}}{\prod_{i=1}^n\left((1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)})\right)}.$$

Thus, we have

$$\mathrm{HS}(\varphi(I^{S_n}_{\le\lambda}))\le\frac{t^{2\left(\binom{\lambda_1^*}{2}+\binom{\lambda_2^*}{2}+\cdots\right)}}{\prod_{i=1}^n\left((1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)})\right)}.$$

Summing over λ, the left-hand side will become the Hilbert Series for $\mathbb{C}[x_1,\dots,x_n]^{S_n}$. The symmetric polynomials has a basis of the elementary symmetric polynomials, which have degrees $1, 2, \dots, n$, so the Hilbert Series for this is just

$$\frac{1}{(1-t)(1-t^2)\cdots(1-t^n)}.$$

Therefore, we have

$$\frac{1}{(1-t)(1-t^2)\cdots(1-t^n)} \leq \sum_{\lambda \vdash n} \frac{t^{2\left(\binom{\lambda_1^*}{2}+\binom{\lambda_2^*}{2}+\cdots\right)}}{\prod_{i=1}^{n}\left((1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)})\right)}.$$

In Lemma 5.2.10, which we prove below, we show that these two generating functions are equal to each other. This implies that we have equality everywhere, so we must have that $\varphi(I^{S_n}_{\leq \lambda}) \cong v_\lambda^2 \mathbb{C}[x_{\lambda_1}, \ldots, x_{\lambda_m}]^{S_\lambda}$. Combining this with our results from Lemma 5.2.5, we obtain the desired result that

$$I^{S_n}_{\leq \lambda}/I^{S_n}_{<\lambda} \cong v_\lambda^2 \mathbb{C}[x_{\lambda_1}, \ldots, x_{\lambda_m}]^{S_\lambda}.$$

□

To finish the last part of the proof, we need to show the equality of the two generating functions. We will prove this completely combinatorially. First, we prove a similar identity of generating functions, which straightforwardly leads to the desired lemma.

Lemma 5.2.9. *For any integer $n \geq 1$, we have*

$$\frac{t^n}{(1-t)(1-t^2)\cdots(1-t^n)} = \sum_{\lambda \vdash n} \frac{t^{||\lambda'||_2}}{\prod_{i=1}^{n}\left((1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)})\right)},$$

where $m_i(\lambda)$ is the multiplicity of i in λ and $||\lambda'||_2$ is the sum of the squares of the parts in the conjugate partition of λ.

Proof. We can reindex the summation by taking the conjugate partitions, so statement is equivalent to showing

$$\frac{t^n}{(1-t)(1-t^2)\cdots(1-t^n)} = \sum_{\lambda \vdash n} \frac{t^{||\lambda||_2}}{\prod_{i=1}^{n}\left((1-t)(1-t^2)\cdots(1-t^{m_i(\lambda')})\right)}.$$

First, the coefficient of t^m in the generating function $\frac{t^n}{(1-t)(1-t^2)\cdots(1-t^n)}$ is the number of partitions of m with exactly n parts. For any such partition μ of m with exactly n parts, we define its *inner partition* λ as follows. First, let the Ferrers diagram be justified to the top and to the left, such that the parts of the partition correspond to the rows of the diagram. Next, let λ_1 be the side-length of the largest square that fits in the top left of the diagram (this is also known as the Durfee square). Then let λ_2 be the side-length of the largest square that fits below the previous square, top and left justified. Keep repeating this until no more squares can fit, which is equivalent to repeating until we arrive at the bottom of the Ferrers diagram. Note that this inner partition λ is a partition of n, because $\lambda_1 + \lambda_2 + \cdots$ equals the number of parts of μ, which is n in this case. Also, note that the number of squares included in these squares is exactly $\lambda_1^2 + \lambda_2^2 + \cdots = ||\lambda||_2$.

We claim that the coefficient of t^m in the generating function $\frac{t^{||\lambda||_2}}{\prod_{i=1}^{n}\left((1-t)(1-t^2)\cdots(1-t^{m_i(\lambda')})\right)}$ is the number of partitions μ of m with exactly n parts that have inner partition λ.

Any such μ is built from the $||\lambda||_2$ squares above, so this contributes a factor of $t^{||\lambda||_2}$. Next, we must count the number of ways to add cells to these squares such that the diagram is still a valid partition and such that the inner partition is still λ.

In the first λ_1 rows, we can add any partition with at most λ_1 parts. The generating function for this is $\frac{1}{\prod_{i=1}^{\lambda_1}(1-t^i)}$. For the next λ_2 rows, we can add any partition with at most λ_2 parts. However, we also must keep the inner partition still as λ, which means that we cannot have this partition have any part greater than $\lambda_1 - \lambda_2$. This means that the possible partition to add to these λ_2 rows must be contained in a $(\lambda_1 - \lambda_2) \times \lambda_2$ rectangle. The generating function for this is well-known to be $\frac{\prod_{i=1}^{(\lambda_1-\lambda_2)+\lambda_2}(1-t^i)}{\prod_{i=1}^{\lambda_1-\lambda_2}(1-t^i)\prod_{i=1}^{\lambda_2}(1-t^i)} = \frac{\prod_{i=1}^{\lambda_1}}{\prod_{i=1}^{\lambda_1-\lambda_2}(1-t^i)\prod_{i=1}^{\lambda_2}(1-t^i)}$.

For the next λ_3 rows, we have the same analysis, so we can add any partition that fits in the $(\lambda_2 - \lambda_3) \times \lambda_3$ rectangle. The generating function for this, as before, is $\frac{\prod_{i=1}^{\lambda_2}(1-t^i)}{\prod_{i=1}^{\lambda_2-\lambda_3}(1-t^i)\prod_{i=1}^{\lambda_3}(1-t^i)}$.

We continue in this manner, until we are at the end of λ. Then, we find that the generating function counting this is

$$t^{||\lambda||_2} \cdot \frac{1}{\prod_{i=1}^{\lambda_1}(1-t^i)} \cdot \frac{\prod_{i=1}^{\lambda_1}(1-t^i)}{\prod_{i=1}^{\lambda_1-\lambda_2}(1-t^i)\prod_{i=1}^{\lambda_2}(1-t^i)} \cdot \frac{\prod_{i=1}^{\lambda_2}(1-t^i)}{\prod_{i=1}^{\lambda_2-\lambda_3}(1-t^i)\prod_{i=1}^{\lambda_3}(1-t^i)} \cdots \frac{\prod_{i=1}^{\lambda_{m-1}}(1-t^i)}{\prod_{i=1}^{\lambda_{m-1}-\lambda_m}(1-t^i)\prod_{i=1}^{\lambda_m}(1-t^i)}$$
$$= t^{||\lambda||_2} \cdot \frac{1}{\prod_{i=1}^{\lambda_1-\lambda_2}(1-t^i)\prod_{i=1}^{\lambda_2-\lambda_3}(1-t^i)\cdots\prod_{i=1}^{\lambda_{m-1}-\lambda_m}(1-t^i)\prod_{i=1}^{\lambda_m}(1-t^i)}$$
$$= \frac{t^{||\lambda||_2}}{\prod_{i=1}^{n}\left((1-t)(1-t^2)\cdots(1-t^{m_i(\lambda')})\right)},$$

as desired. □

Lemma 5.2.10. *For any integer $n \geq 1$, we have*

$$\frac{1}{(1-t)(1-t^2)\cdots(1-t^n)} = \sum_{\lambda \vdash n} \frac{t^{2\cdot\left(\binom{\lambda_1'}{2}+\binom{\lambda_2'}{2}+\cdots\right)}}{\prod_{i=1}^{n}\left((1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)})\right)},$$

where $\lambda' = (\lambda_1', \lambda_2', \dots)$ is the conjugate partition of λ and $m_i(\lambda)$ is the multiplicity of i in partition λ.

Proof. By Lemma 5.2.9, we know that

$$\frac{t^n}{(1-t)(1-t^2)\cdots(1-t^n)} = \sum_{\lambda \vdash n} \frac{t^{||\lambda'||_2}}{\prod_{i=1}^{n}\left((1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)})\right)}.$$

Dividing by t^n, we get

$$\begin{aligned}
\frac{1}{(1-t)(1-t^2)\cdots(1-t^n)} &= \sum_{\lambda \vdash n} \frac{t^{||\lambda'||_2} - n}{\prod_{i=1}^{n}\left((1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)})\right)} \\
&= \sum_{\lambda \vdash n} \frac{t^{\sum_i (\lambda_i')^2 - \sum_i \lambda_i'}}{\prod_{i=1}^{n}\left((1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)})\right)} \\
&= \sum_{\lambda \vdash n} \frac{t^{\sum_i \lambda_i'(\lambda_i'-1)}}{\prod_{i=1}^{n}\left((1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)})\right)} \\
&= \sum_{\lambda \vdash n} \frac{t^{2\sum_i \binom{\lambda_i'}{2}}}{\prod_{i=1}^{n}\left((1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)})\right)},
\end{aligned}$$

as desired. □

Again, this implies that $I_{\leq\lambda}^{S_n}/I_{<\lambda}^{S_n} \cong v_\lambda^2 \mathbb{C}[x_{\lambda_1}, \dots, x_{\lambda_m}]^{S_\lambda}$.

Conjecture 5.2.11. *We expect $I_\lambda \subset \mathbb{C}[x_{\lambda_1}^{\pm 1}, \dots, x_{\lambda_m}^{\pm 1}]$ to be a free module with respect to $\mathbb{C}[x_{\lambda_1}^{\pm 1}, \dots, x_{\lambda_m}^{\pm 1}]$. (Note that here we set I_λ to be an ideal in the ring of Laurent polynomials.)*

Showing that ideal $I_\lambda \subset \mathbb{C}[x_{\lambda_1}^{\pm 1}, \dots, x_{\lambda_m}^{\pm 1}]$ is free as a module over $\mathbb{C}[x_{\lambda_1}^{\pm 1}, \dots, x_{\lambda_m}^{\pm 1}]$ is an integral step towards rigorously describing the relationship between our distinguished ideals and the distinguished weights, which we explain in Section 5.3. Therefore, it is an essential question as to whether these ideals are free. We have shown that this is true for the S_n-invariant elements for $I_{\leq\lambda}^{S_n}/I_{<\lambda}^{S_n}$, even at the level of polynomial rings (which immediately implies that it also holds at the level of S_n-invariant Laurent polynomials and the corresponding ideals), which provides evidence suggesting that the same may hold for the full rings (without taking invariants).

5.3. **Construction of ideals.** It is generally expected that the distinguished ideals we described are related to the distinguished weights in some way. Here, we introduce an algorithm to construct some particular ideals of $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. We expect that these constructed ideals have immediate parallels with LV_p and distinguished weights, and give examples to suggest this relationship. This algorithm and relationship heavily relies on Conjecture 5.2.11 being true.

The construction is as follows. First, pick a finite sequence $(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(k)})$ where $\lambda^{(1)}$ is a partition of n, and $\lambda^{(i+1)}$ is a partition refinement of $\lambda^{(i)}$ for $1 \leq i \leq k-1$. Start with $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and consider the quotient ideal $I_{\lambda^{(1)}}$. By Conjecture 5.2.11, this should decompose freely. Then we could take each free summand, which is a polynomial ring, and take the quotient ideal corresponding to $I_{\lambda^{(2)}}$, which should again be a free module and decompose into a direct sum of smaller polynomial rings. We iterate this procedure for s steps.

Then, to construct the ideal corresponding to this sequence, we iterate the following steps. As a base case, we start at the end of the previous steps, and we choose the ideals of those summands to just be the entire ring. Next, suppose we had an ideal for each of the free summands at some step. We lift each quotient ideal back to the original ring by adding powers of p to the variables, and we intersect these ideals to get an ideal of the original (non-quotient) ring. Then this ring is part of a summand in a free module decomposition of a larger ring, and we can iterate the next step.

We give an example to show this construction more concretely.

Example 5.3.1. Suppose that $n = 4$. In our construction, suppose we started with the partition $\lambda = (2, 2)$. Then we have

$$I_\lambda = (x_{12} - x_{34})^2\mathbb{C}[x_{12}^{\pm1}, x_{34}^{\pm1}] \oplus (x_{13} - x_{24})^2\mathbb{C}[x_{13}^{\pm1}, x_{24}^{\pm1}] \oplus (x_{14} - x_{23})^2\mathbb{C}[x_{14}^{\pm1}, x_{23}^{\pm1}].$$

Then suppose we took the refinement $((1, 1), (1, 1))$ of $\lambda = (2, 2)$. Then we have the ideals

$$\begin{aligned}
(x_{12} - x_{34})\mathbb{C}[x_{12}^{\pm1}, x_{34}^{\pm1}] &\subset \mathbb{C}[x_{12}^{\pm1}, x_{34}^{\pm1}]\\
(x_{13} - x_{24})\mathbb{C}[x_{13}^{\pm1}, x_{24}^{\pm1}] &\subset \mathbb{C}[x_{13}^{\pm1}, x_{24}^{\pm1}]\\
(x_{14} - x_{23})\mathbb{C}[x_{14}^{\pm1}, x_{23}^{\pm1}] &\subset \mathbb{C}[x_{14}^{\pm1}, x_{23}^{\pm1}].
\end{aligned}$$

We lift these ideals up once, adding powers of p, to get the ideals

$$\begin{aligned}
(x_{12} - x_{34})^2(x_{12}^p - x_{34}^p)\mathbb{C}[x_{12}^{\pm1}, x_{34}^{\pm1}] &\subset (x_{12} - x_{34})^2\mathbb{C}[x_{12}^{\pm1}, x_{34}^{\pm1}]\\
(x_{13} - x_{24})^2(x_{13}^p - x_{24}^p)\mathbb{C}[x_{13}^{\pm1}, x_{24}^{\pm1}] &\subset (x_{13} - x_{24})^2\mathbb{C}[x_{13}^{\pm1}, x_{24}^{\pm1}]\\
(x_{14} - x_{23})^2(x_{14}^p - x_{23}^p)\mathbb{C}[x_{14}^{\pm1}, x_{23}^{\pm1}] &\subset (x_{14} - x_{23})^2\mathbb{C}[x_{14}^{\pm1}, x_{23}^{\pm1}].
\end{aligned}$$

Finally, we lift these ideals back to $\mathbb{C}[x_1^{\pm1}, x_2^{\pm1}, x_3^{\pm1}, x_4^{\pm1}]$ and add the generators from the quotient to get the following ideals:

$$\begin{aligned}
I_1 :=& \langle (x_1 - x_3)^2(x_1^p - x_3^p), x_1 - x_2, x_3 - x_4\rangle\\
I_2 :=& \langle (x_1 - x_2)^2(x_1^p - x_2^p), x_1 - x_3, x_2 - x_4\rangle\\
I_3 :=& \langle (x_1 - x_2)^2(x_1^p - x_2^p), x_1 - x_4, x_2 - x_3\rangle.
\end{aligned}$$

Using the computer algebra system Magma, we find that $I_1 \cap I_2 \cap I_3$ is generated by

$$\{(x_2 - x_4)^2(x_2^p - x_4^p), (x_3 - x_4)(x_2 - x_4)(x_3^p - x_4^p), (x_3 - x_4)^2(x_3^p - x_4^p), (x_3 - x_4)(x_2 - x_4)(x_2 - x_3),$$
$$(x_1 - x_3)(x_1 - x_2 + x_3 - x_4), (x_2 - x_4)(x_1 - x_2 + x_3 - x_4), (x_3 - x_4)(x_1 + x_2 - x_3 - x_4)\}.$$

This is the ideal that arises under this construction given the above choice of partitions and successive refinements.

This constructs ideals which are indexed by the same elements as distinguished weights. In particular, both are indexed by a sequence $(\lambda^{(1)}, \lambda^{(2)}, \dots)$ in which $\lambda^{(1)}$ is a partition of n and each successive element is a refinement of the element before it. This means that every ideal constructed has a corresponding distinguished weight. For instance, consider the above example again. In the final lift to $\mathbb{C}[x_1^{\pm1}, x_2^{\pm1}, x_3^{\pm1}, x_4^{\pm1}]$, we could have chosen this equivalent lift of generating I_1 by

$$\{(x_1 - x_3)(x_2 - x_4)(x_1^{\frac{p+1}{2}} x_2^{\frac{p-1}{2}} - x_3^{\frac{p-1}{2}} x_4^{\frac{p+1}{2}}), x_1 - x_2, x_3 - x_4\}$$

and similarly for I_2 and I_3. The first element in this new generating set has a leading term with the powers on the x_1 and x_2 being $\frac{p+1}{2}+1$ and $\frac{p+1}{2}$, which exactly is from the distinguished weight $\left(\frac{p+1}{2} + 1, \frac{p+1}{2}, -\frac{p+1}{2}, -\frac{p+1}{2} - 1\right)$ that corresponds with this sequence of partition refinements. The exact lift to take could be explained by using the map κ^{-1} that was part of the algorithm for the Lusztig–Vogan bijection.

Example 5.3.2. Suppose that we start with the partition $\lambda = (1, \dots, 1)$ and choosing $(1, \dots, 1)$ as a refinement. Then I_λ is a free module of rank 1, given by

$$I_\lambda = \Pi_{\lambda^*} \cdot \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

where $\Pi_{\lambda^*} = \prod_{1 \le i < j \le n}(x_i - x_j)$. Define $v(a) = \prod_{1 \le i < j \le n}(x_i^a - x_j^a)$. Then the ideal corresponding to being k steps into the construction, when lifted to the original ring, is

$$v(1)v(p)v(p^2) \cdots v(p^{k-1})\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

The generator is $v(1)v(p)v(p^2) \cdots v(p^{k-1})$. The leading degree monomial is

$$x_1^{(n-1)(1+p+\cdots+p^{k-1})} x_2^{(n-3)(1+p+\cdots+p^{k-1})} \cdots x_{\lfloor \frac{n}{2} \rfloor}^{(n-2\lfloor \frac{n}{2} \rfloor)(1+p+\cdots+p^{k-1})},$$

whose powers clearly correspond to the distinguished weight in Corollary 4.1.2 of

$$\rho_{(1,\dots,1)} \cdot \frac{p^k - 1}{p - 1} = (n-1, n-3, \dots, 1-n) \cdot \frac{p^k - 1}{p - 1}.$$

This relationship is more explicit if we use the inverse variables x_i^{-1}. Then we could define

$$\tilde{v}(a) = \prod_{1 \le i < j \le n} (x_i^a x_j^{-a} - 1),$$

which is a multiple of $v(a)$. Then a leading degree monomial is

$$x_1^{(n-1)(1+p+\cdots+p^{k-1})} x_2^{(n-3)(1+p+\cdots+p^{k-1})} \cdots x_n^{(1-n)(1+p+\cdots+p^{k-1})},$$

whose powers are exactly $\rho_{(1,\dots,1)} \cdot \frac{p^k-1}{p-1}$.

The appearance of these distinguished weights give evidence for a conjectural natural relationship between the ideals constructed in this manner and the algorithm for LV_p and is a place for further study.

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