Geometric RSK Correspondence and Relative Positions Within Springer Fibers

Calvin Yost-Wolff

Advisor: Oron Propp

Project Suggested by Roman Bezrukavnikov

Abstract

A classical result of Spaltenstein–Tits states that for a pair of Borel subgroups B_1, B_2 in relative position w within a Springer fiber $\mathcal{B}_e$ and a reduced expression $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_r}$, there exists a unique sequence of projective lines of type s_{α_i} lying within $\mathcal{B}_e$ connecting B_1 to B_2. In this paper, we study these sequences (which we call *RSK-paths*) by means of a geometric RSK correspondence between irreducible components of the Steinberg variety and $C_G(e)$-orbits of pairs of irreducible components of $\mathcal{B}_e$. We begin by showing that the $C_G(e)$-orbits of irreducible components of the variety of pairs of Borel subgroups in $\mathcal{B}_e$ in relative position w are in bijection with the irreducible components of a certain subvariety of the nilpotent orbit of e, which we term a *w-orbital variety*. Using this bijection, we exactly characterize the sequence of irreducible components of $\mathcal{B}_e$ through which an RSK-path passes via the Jordan normal forms of a sequence of adjoint actions on an element of the w-orbital variety associated to the reduced expression of w. We then prove a formula relating the length of this sequence of irreducible components to the dimension of the corresponding irreducible component of the w-orbital variety. Finally, we conjecture that a certain "convexity" property for irreducible components of $\mathcal{B}_e$ with respect to RSK-paths is equivalent to a condition on the Weyl group elements produced by the inverse RSK correspondence, and give an explicit description of the corresponding classes of tableaux in types A, B and C.

Contents

1 Introduction

Let G be a complex connected semisimple algebraic group, B be a Borel subgroup of G containing a maximal torus T, and $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$ denote their respective Lie algebras. The flag variety $\mathcal{B}$ of G can be identified with the quotient G/B and parameterizes the Borel subgroups of G. The Springer fiber $\mathcal{B}_e$ over a nilpotent $e \in \mathfrak{g}$ is the subvariety of $\mathcal{B}$ consisting of Borel subgroups whose Lie algebra contains e. A classical result of Spaltenstein–Tits states that any two Borel subgroups in relative position w within a Springer fiber can be connected by a sequence of projective lines lying within the Springer fiber determined by the sequence of simple reflections in a reduced expression for w [Spa82]. We call these sequences *RSK-paths*, due to their connections to a geometric version of the Robinson–Schensted–Knuth (RSK) correspondence (see [vL89]). This geometric correspondence was first observed by Steinberg in type A where he discovered a description of the irreducible components X_i of Springer fibers in terms of tableaux and noticed that the relative position of a generic pair of Borel subgroups $(B_1, B_2) \in X_1 \times X_2$ is the same as inverse RSK of the pair of tableaux [Spa82, 2.9.8]. Later, this correspondence was generalized to other types (see [vL89]) and now has a solid theoretical background in the irreducible components of the Steinberg variety (see [CG97]). While properties of irreducible components of Springer fibers have been studied extensively (see [Tym16] for a survey), the sequence of irreducible components of Springer fibers which RSK-paths passes through has not been thoroughly studied. In this paper, we study which irreducible components of Springer fibers an RSK-path passes through.

The projective lines corresponding to simple reflections in the RSK-paths are called lines of type ρ for the simple reflection ρ. Spaltenstein first observed that in type A, an irreducible component X of Springer is a unions of lines of type ρ for certain ρ based on a condition on the tableaux corresponding to X under the geometric RSK correspondence. We find this generalizing to arbitrary types based on the conditions on Weyl group elements which correspond under geometric RSK correspondence to (X, X). We also note that his tableaux criteria generalizes to other types as well.

Next, we study the structure of the RSK-paths themselves, demonstrating a natural way to give an algebraic variety structure to the set of RSK-paths and showing it is isomorphic to the subvariety of Springer fibers consisting of Borel subgroups in relative position w. This motivates our interest in the pairs of Borel subgroups within a Springer fiber in relative position w, which we denote by $Y_w \cap \mathcal{B}_e \times \mathcal{B}_e$.

Let $\mathfrak{n}$ denote the subvariety of nilpotents in $\mathfrak{b}$ and let $\mathfrak{n}_w$ denote the intersection of $\mathfrak{n}$ with the Lie algebra of wBw^{-1} where we choose a representative w for the equivalence class of the normalizer of T which w represents as an element of the Weyl group. Another classical result of Spaltenstein [Spa77] states that for a nilpotent $e \in \mathcal{N} \subseteq \mathfrak{g}$ with nilpotent orbit $\mathcal{O}$, the irreducible components of the orbital variety $\mathfrak{n} \cap \mathcal{O}$ are in bijection with $C_G(e)$-orbits of irreducible components of $\mathcal{B}_e$, where $C_G(e)$ denotes the centralizer of e in G. We extend this result to $\mathfrak{n}_w$:

Theorem 1.1. *There is a bijection between irreducible components of the w-orbital variety $\mathfrak{n}_w \cap \mathcal{O}$ and $C_G(e)$-orbits of irreducible components of $Y_w \cap \mathcal{B}_e \times \mathcal{B}_e$.*

When $w = 1$, we recover the classical result of Spaltenstein. While Springer fibers and orbital varieties are equidimensional [CG97], the subvarieties $Y_w \cap \mathcal{B}_e \times \mathcal{B}_e$ and $\mathfrak{n}_w \cap \mathcal{O}$ are not necessarily equidimensional. Nevertheless, we demonstrate a method of computing the dimensions of irreducible components of $\mathfrak{n}_w \cap \mathcal{O}$ and $Y_w \cap \mathcal{B}_e \times \mathcal{B}_e$ in terms of RSK-paths.

Theorem 1.2. *For a reduced expression $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_r}$ and an irreducible component $Y \subset Y_w \cap \mathcal{B}_e \times \mathcal{B}_e$, there exists explicit projections based on the RSK-paths, $\pi_i : Y \to \mathcal{B}_e$ and $\pi_{P,i} : Y \to (G/P_{\alpha_i})$ such that*

$$\dim Y = \sum_i \dim \pi_i(Y) - \sum_i \dim \pi_{P,i}(Y).$$

In the case that the image of the projections $\pi_{P,i}$ have the maximal possible dimension $\dim \pi_{P,i}(Y) = \dim \mathcal{B}_e - 1$,

$$\dim Y = \dim \mathcal{B}_e + \#\{i | T_i = T_{i+1}\} + 1$$

where the T_i are the tableaux corresponding under the geometric RSK correspondence to the irreducible components of $\mathcal{B}_e$ which the RSK-paths of pairs of Borel subgroups in Y pass through.

Notably, this theorem applies for any reduced expression for w. Based on computational examples, it seems that for many if not all irreducible components of $Y \subset Y_w \cap \mathcal{B}_e \times \mathcal{B}_e$, it is possible to choose a reduced expression for w, where all the projections $\pi_{P,i}$ satisfy $\dim \pi_{P,i}(Y) = \dim \mathcal{B}_e - 1$.

This gives further motivation to the question of which irreducible components of a Springer fiber an RSK-path does pass through or, at least, which tableaux correspond to those components under the geometric RSK correspondence. Using techniques in our proof of Theorem 1.1, we construct a conjugation of an RSK-path which is contained in $\mathfrak{n} \cap \mathcal{O}$ and starts in $\mathfrak{n}_w \cap \mathcal{O}$. Using the bijection between $C_G(e)$-orbits of irreducible components of $\mathcal{B}_e$ and irreducible components of $\mathfrak{n} \cap \mathcal{O}$, we find the irreducible components which an RSK-path passes through based on the irreducible components of $\mathfrak{n} \cap \mathcal{O}$ which this conjugation of the RSK path passes through:

Theorem 1.3. *Fix $g \in G, e \in \mathfrak{n}_w \cap \mathcal{O}$ such that $\mathrm{Ad}(g)e = e'$ and a reduced expression $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_r}$. Let $V_\bullet$ be standard flag which B fixes. Then the RSK-path from ${}^gB$ to ${}^{gw}B$ within the Springer fiber $\mathcal{B}_{e'}$ passes through irreducible components of $\mathcal{B}_{e'}$ which can be described by the Jordan normal forms of $\mathrm{Ad}(\sigma_i^{-1})(e)$ acting on V_i where $\sigma_i = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_i}$.*

Motivated by the fact that every RSK-path whose endpoints lie within a Springer fiber is contained in the Springer fiber, we conjecture:

Conjecture 1.4. *For an irreducible component $X \subset \mathcal{B}_e$, every RSK-path whose endpoints lie in X is contained in X if and only if under the geometric RSK correspondence, the pair (X, X) corresponds to an element $w \in W$ such that*

1. *w is an involution (this will always be the case for a pair (X, X))*

2. *For every simple reflection s which appears in a reduced expression for $w_{X,X}$,*

$$\ell(s w_{X,X}) < \ell(w_{X,X}).$$

It may appear as though Theorem 1.3 should imply Conjecture 1.4, however this is not the case since certain portions of the RSK-path may lie in the intersections of irreducible components of $\mathcal{B}_e$. We prove the condition in our conjecture is sufficient and present an approach to proving the necessary direction. When G is of type A, B, or C, the irreducible components X which satisfy the conditions in the conjecture are exactly the those corresponding to tableau T which satisfy the condition $x - 1$ is in the row one above x whenever x is not in the topmost row. These tableaux are exactly the tableaux that are constructed from left-justifying a sequence of columns as the following picture shows:

1	3	7	8
2	4		9
	5		10
	6		

$\Longrightarrow$

1	3	7	8
2	4	9	
5	10		
6			

This paper is organized as follows. In §2, I develop the algebraic and combinatorial preliminaries necessary to digest the results of this paper. An experienced reader should feel free to skip this section and refer to it as needed. In §3, we parameterize the irreducible components of the Steinberg and Springer varieties following [CG97, vL89] and analyze which components are unions of parabolic lines of type ρ. In §4, we introduce the RSK-paths and construct an algebraic variety parametrizing them. In §5, we present the correspondence between w-orbital varieties and $Y_w \cap \mathcal{B}_e \times \mathcal{B}_e$, proving our theorems 1.1, 1.2, and 1.3. In §6, we give combinatorial reformulations of the conditions in Conjecture 1.4 on Weyl group elements.

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2 Preliminaries

Here we develop much of the preliminary knowledge needed to digest the results of our paper along with some technical lemmas we will use later. We will also provide relevant sources covering additional background. In the first three subsections we discuss some of the theory associated to linear algebraic groups and the last two subsections cover combinatorial objects and bijections we will use.

2.1 Weyl Groups

The Weyl group of a connected reductive algebraic group is a finite group associated to G which has many combinatorial properties. We will only touch the surface of the theory associated to Weyl groups, providing a few technical facts we will need later. For more thorough references, see [Spr98] or [Hum75].

The *Weyl group* W of a connected reductive algebraic group G with Borel subgroup B containing a maximal torus T is the quotient of the normalizer of T by the centralizer of T (which turns out to be T):

$$W := \frac{N_G(T)}{C_G(T)} = \frac{N_G(T)}{T}.$$

Throughout this paper, when we act on a subgroup or element of G by w we view w as an element of G which is a representative of the equivalence class of $w \in N_G(T)/C_G(T)$.

The Weyl group W also corresponds to the group generated by reflections which preserves the root system of G. For any root α with corresponding unipotent subgroup U_α, there is a homomorphism $\alpha : SL(2) \to G$ which sends the upper triangular matrices $U \mapsto U_\alpha$ and diagonal matrices to T. This map has the additional property that $\alpha(I)$ is trivial and $\alpha\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = s_\alpha$ is the simple reflection of the root group through the hyperplane perpendicular to the simple root α. The action of s_α on T is an involution and the Weyl group is generated by simple reflections.

Restricting α to the compact real form of $SL(2)$ yields a map

$$\alpha : SU(2) \to G.$$

Furthermore

$$\frac{SU(2)}{U(1)} = \frac{SL(2)}{\{\text{ upper triangular matrices }\}} = \mathbb{P}^1.$$

It follows that $\alpha(SL(2))B = \alpha(SU(2))B$ is a parabolic subgroup of G with Levi quotient having two roots $\pm\alpha$.

A *finite Coxeter group* is a finite group C which is generated by involutions from a set S. For an element $g \in C$, a *reduced expression* for g is a product of ℓ (not necessarily distinct) involutions of S, $s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_1 \ell}$ which is equal to g such that for any product of (not necessarily distinct) elements of S which is equal to g, there are at least ℓ (not necessarily distinct) elements in the product. The *length* of an element g in a finite Coxeter group is the integer ℓ such that there exists a reduced expression for g of length ℓ. From now on we view ℓ as a function $\ell : C \to \mathbb{Z}_{\geq 0}$.

The length function ℓ induces a partial order on C given by

$$h \leq g \text{ if } \ell(h) + \ell(h^{-1}g) = \ell(g).$$

Theorem 2.1. *[Hum90] In a Coxeter group C with simple reflections $\{s_{\alpha_i}\}$*

1. *$\ell(s_{\alpha_i}\sigma) \in \{\ell(\sigma) \pm 1\}$.*
2. *(Exchange Condition) If $s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_\ell}$ is a reduced expression for w and $\ell(sw) < \ell(w)$, then for some r,*
$$sw = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_r} s_{\alpha_{r+2}} s_{\alpha_{r+3}} \dots s_{\alpha_\ell}.$$
3. *(Matsumoto's Theorem) Any two reduced words for g can be related by changing substring of simple reflections $xyx \cdots x$ to $yxy \cdots y$ where $xyx \cdots x = yxy \cdots y$ or commuting relations $xy = yx$*

Corollary 2.2 (Corollary of Matsumoto's Theorem)**.** *If some minimal word of w contains a simple reflection s_{α_i}, then every minimal word of w contains s_{α_i}.*

A corollary we will use later is:

Corollary 2.3. *Suppose that $\rho w < w$ and $\rho' w < w$ and $x \in W$ satisfies*

1. $x \geq \rho w$
2. $x \geq \rho' w$
3. $\ell(x) \leq \ell(w)$

Then $x = w$.

Proof. From the definition of our order and Theorem 2.1(i), there exist simple reflections σ, σ' such that

$$x = \sigma \rho w \qquad x = \sigma' \rho' w.$$

Thus either $\sigma = \rho$ and $\sigma' = \rho'$ or by Theorem 2.1(iii), since

$$\sigma \rho = \sigma' \rho',$$

$\sigma = \rho'$ and $\sigma' = \rho$. In the former case $x = w$, and in the latter case $x = \rho' \rho w$. By the exchange condition in Theorem 2.1(iii) $\ell(\rho' \rho w) = \ell(w) - 2$. However since $x \geq \rho w$ and $x \geq \rho' w$ and

$$\ell(\rho w) = \ell(\rho' w) = \ell(w) - 1,$$

$\ell(x) \geq \ell(w)$. Thus $x \neq \rho' \rho w$. □

2.2 Flag Variety and Bruhat Decomposition

As with Weyl groups, I will introduce the basics of the Bruhat decomposition and the flag variety. For a more involved exposition, see [Hum75, §8, §10].

Throughout this subsection, fix a standard Borel subgroup B and torus T with $T \subset B \subset G$.

The *flag variety* $\mathcal{B}$ of an algebraic group G is the variety of $(\dim B)$-dimensional solvable subspaces of the Lie algebra $\mathfrak{g}$. From the Plucker embedding, $\mathcal{B}$ is a projective variety. The adjoint action of G on $\mathfrak{g}$ acts transitively on $\mathcal{B}$ and the stabilizer of $\mathfrak{b} = Lie(B)$ is simply B. This parameterizes the points of $\mathcal{B}$ as Borel subgroups of G and allows us to identify $\mathcal{B} \cong G/B$.

An alternate description of the flag variety which will be important for us is the language of flags. A *flag* $V_\bullet$, is a sequence of vector subspaces

$$\{0\} \subsetneq V_1 \subsetneq V_2 \ldots \subsetneq V_n = \mathbb{C}^n$$

where $\dim V_i = i$. An *isotropic flag* for a certain binary form $\langle \cdot, \cdot \rangle$ is a flag $V_\bullet$ such that $V_i = V_{n-i}^\perp$. By Borel's fixed point theorem, for a finite dimensional representation $\phi : G \to GL(\mathbb{C}^n)$, B fixes a 1-dimensional subspace of $\mathbb{C}^n$, and thus iterating this argument, fixes some flag. For a suitable representation ϕ, each flag is fixed by at most one Borel subgroup. This allows us to identify the variety G/B with the G-orbit of a certain flag. For type A, this identifies G/B with the variety of n-dimensional flags. In other types, from embedding into a certain $SL(n)$, this identifies G/B with the variety of *isotropic flags* for a certain binary form $\langle \cdot, \cdot \rangle$.

The *Bruhat decomposition* takes the form of three equivalent bijections between the Weyl group and related objects concerning G and B:

1. There is a bijection between $B \backslash G / B$ double cosets and the Weyl group:

$$G = \sqcup_{w \in W} BwB.$$

2. There is a bijection between the B-orbits in $\mathcal{B}$ and the Weyl group.

3. The diagonal action of G on $\mathcal{B} \times \mathcal{B} \cong G/B \times G/B$ defined by

$$g \cdot (h_1 B, h_2 B) = (gh_1 B, gh_2 B)$$

has orbits in bijection with the Weyl group W.

The equivalence of (i) and (ii) is seen from the flag variety parameterizing Borel subgroups. The equivalence of (ii) and (iii) is then a general fact that that the B-orbits of G/B are in bijection with the G-diagonal orbits in $G/B \times G/B$. There are many different ways to prove either $B\backslash G/B$, $\{B\text{-orbits in } \mathcal{B}\}$, or $\{G\text{-diagonal orbits in } \mathcal{B} \times \mathcal{B}\}$ are in bijection with the Weyl group.

We will be primarily concerned with the geometric implications of (ii) and (iii) above. Algebraically, (ii) can be refined to say for each Borel subgroup B', there exists a unique $w \in W$ such that

$$B' = bwBw^{-1}b^{-1}$$

for some $b \in B$. Since throughout this paper we will conjugate Borel subgroups often, we adopt the notation

$${}^{g}B_1 := gB_1 g^{-1}.$$

In the same vein, (iii) says

Definition 2.4. *For each pair of Borel subgroup B_1, B_2, there exists a unique $w \in W$ such that*

$$B_2 = {}^{bw}B_1$$

for some $b \in B_1$. This w is called the relative position *of B_1, B_2 or the relative position of B_2 with respect to B_1.*

Geometrically, relative position can be used to find a cellular decomposition of the flag variety with cells $C(w)$ consisting of the set of Borel subgroups in relative position w to B (i.e. the B-orbits of $\mathcal{B}$). Similarly relative position gives a partition of the variety $\mathcal{B} \times \mathcal{B}$ into subvarieties Y_w which consist of all pairs of Borel subgroups (B_1, B_2) with B_2 in relative position w to B_1 (i.e. the G-diagonal orbits of $\mathcal{B} \times \mathcal{B}$).

This raises an important question about the Bruhat decomposition: Suppose that B_1, B_2 is in relative position w, what is the relative position of $B_1, b_1\rho B_2 \rho b_1^{-1}$ for some $b_1 \in B_1$? This question is answered by Tits's analyses of additional combinatorial properties of the Bruhat decomposition (see [Hum75, §29]).

Tits showed that the following two axioms hold for a simple reflection $\rho \in W$ and $\sigma \in W$:

1. $\rho B \sigma \subset B\sigma B \cup B\rho\sigma B$

2. $\rho B \rho \neq B$.

Using these, one can determine:

Lemma 2.5. *[Hum75, Lemma 10.29.3A]*

1. *If $\ell(\rho\sigma) > \ell(\sigma)$ then $\rho B \sigma \subset B\rho\sigma B$.*

2. *If $\ell(\rho\sigma) < \ell(\sigma)$ then for all $b \in B$ such that $\rho b \rho \notin B$, we have $\rho b \sigma \in B\sigma B$.*

By inducting on length, this can be used to show if B_1, B_3 are in relative position $w_1 w_2$ where $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$, then there exists a unique B_2 such that B_1, B_2 are in relative position w_1 and B_2, B_3 are in relative position $\ell(w_1 w_2)$. We record this in the following lemma along with an important converse of the above statement:

Lemma 2.6. *Suppose that $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$, then*

(i) If B_1, B_3 are in relative position $w_1 w_2$, there is exactly one Borel subgroup B_2 such that B_1, B_2 are in relative position w_1 and B_2, B_3 are in relative position w_2.

(ii) If B_1, B_2 are in relative position w_1 and B_2, B_3 are in relative position w_2, then B_1, B_3 are in relative position $w_1 w_2$.

2.3 Springer and Steinberg Varieties

The Springer and Steinberg varieties are the main varieties which we will analyze. These varieties are fundamental in the representation theory of Weyl group (see [CG97]): the irreducible representations of Weyl groups are all found in the top-dimensional cohomology and in the top-dimensional Borel-Moore homology of the Springer fibers. The proof of these facts often use the embedding of Springer fibers into the Steinberg variety.

The Steinberg variety Z is the subvariety of $\mathcal{N} \times \mathcal{B} \times \mathcal{B}$, where $\mathcal{N}$ is the subvariety of nilpotents of $\mathfrak{g}$, consisting of triples (e, B_1, B_2) such that e is in $\operatorname{Lie} B_1$ and $\operatorname{Lie} B_2$.. In the equivalent formulation of the flag variety in terms of $(\dim B)$-dimensional solvable subspaces of $\mathfrak{g}$, the condition for $(e, \mathfrak{b}_1, \mathfrak{b}_2)$ to be in Z is that $e \in \mathfrak{b}_1$ and $e \in \mathfrak{b}_2$. The Steinberg variety has two natural surjective projections $\pi : Z \to \mathcal{B} \times \mathcal{B}$ and $\nu : Z \to \mathcal{N}$.

$$\begin{array}{ccc} & Z, & \\ \swarrow^{\nu} & & \searrow^{\pi} \\ \mathcal{N} & & \mathcal{B} \times \mathcal{B} \end{array} .$$

The Springer fiber above a nilpotent $e \in \mathcal{N}$ is the subvariety of the flag variety consisting of Borel subgroups B whose Lie algebras contain e. In the formulation of the flag variety in terms of (isotropic) flags, it is the subvariety of (isotropic) flags $V_\bullet$ such that $eV_i \subseteq V_{i-1}$ for all i. In the equivalent formulation of the flag variety in terms of $(\dim B)$-dimensional solvable subspaces of $\mathfrak{g}$, the condition is that e is contained in the subspace. We denote the Springer fiber over e by $\mathcal{B}_e$.

There is a natural embedding of $\mathcal{B}_e \times \mathcal{B}_e$ into the Steinberg variety via

$$((e, B_1), (e, B_2)) \mapsto (e, B_1, B_2).$$

In §3 we will use this alongside conjugation by G to relate the irreducible components of the Steinberg variety to pairs of irreducible components in the Springer fiber; this embedding will also be important in many of our other endeavors.

2.4 Standard Young Tableaux

Standard tableaux are a combinatorial object commonly used in representation theory of the symmetric group. Unsurprisingly, they appear in the representation theory of other Weyl groups. We will use them to specify certain subvarieties of Springer fibers and later of w-orbital varieties.

Let Λ denote the positive orthant $\mathbb{N} \times \mathbb{N}$ with order $(a_1, b_1) \leq (a_2, b_2)$ if $a_1 \leq a_2$ and $b_1 \leq b_2$. It is common to represent the elements of Λ via a square tiling of the positive orthant. A *partition* λ is a sequence $(\lambda_1, \cdots, \lambda_\ell)$ of nonnegative integers with $\lambda_1 \geq \cdots \geq \lambda_\ell$. We define $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$. Associated to a partition is its *Young diagram*, which is a downward closed subset of $\mathbb{N} \times \mathbb{N}$ consisting of boxes λ_i consecutive boxes in a row starting at $(1, i)$ for $i = 1, \ldots, \ell$. Each Young diagram naturally defines a poset consisting of its boxes with partial order induced from Λ.

A *filling* of shape $\lambda \subset \Lambda$ is a function $f : \lambda \to X$ for some set X. When $X = \{1 < 2 < 3 \ldots < |\lambda|\}$, a bijective filling which satisfies $x < y$ implies $T(x) < T(y)$ is called a *standard Young tableau*. From here on out, we will refer to standard Young tableaux as simple tableaux. The *shape* of a tableau T is the domain of T. For any tableau T, we define $T|_{[1,n]}$ to be the tableau of shape $T^{-1}([1, n])$ such that $T|_{[1,n]}(x) = T(x)$ for all $x \in T^{-1}([1, n])$.

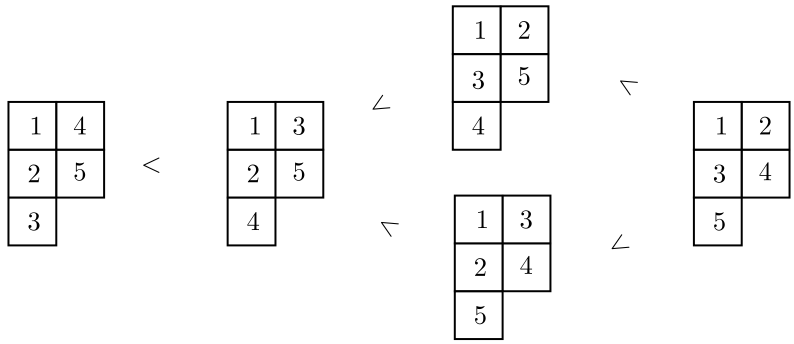


Figure 1: An example of the partial order on tableaux

Remark 2.7. *There are many alternate definitions of standard Young tableaux. The above definition defines a tableau as a bijective poset map $T : \lambda \to \{1 < 2 < 3 \ldots < |\lambda|\}$. This gives a bijection between tableaux of λ and linear extensions $p_1 \le p_2 \ldots \le p_\lambda$ of λ via $\{p_1, p_2 \ldots p_n\} = T^{-1}([1, n])$. This bijection mimics our later use of tableaux to define subvarieties of Springer fibers.*

We define a partial ordering on partitions of size n which yields a partial ordering on the tableaux of shape λ called the *dominance ordering*. We say that $\lambda \le \lambda'$ if for all i,

$$\sum_{j=1}^{i} \lambda_i \le \sum_{j=1}^{i} \lambda'_i.$$

Similarly we say that $T \le T'$ if for all i,

$$\text{shape}(T^{-1}([1, i])) \le \text{shape}(T'^{-1}([1, i])).$$

Furthermore, we define an action of a subset of S_n on tableaux: Let $\sigma * T$ denote the filling of shape(T) defined by

$$(\sigma * T)(x) := \sigma \circ T(x).$$

Let σT denote the filling equal to $\sigma * T$ with the columns rearranged in increasing order.

Example 2.8.

$$\left((12) * \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \right) = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad \left((12) \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \right) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1} \end{array}$$

$$\left((34) * \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \right) = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \left((34) \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} \right) = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1} \end{array}$$

2.5 Robinson–Schensted–Knuth Correspondence

The classical Robinson–Schensted–Knuth (RSK) correspondence gives a bijection between the symmetric group S_n and pairs of tableau of the same shape consisting of n blocks via a row-insertion and column-insertion algorithm. Later a geometric generalization of this correspondence having to do with the Steinberg variety was found and is discussed in §3.

Since this paper will not deal with combinatorial results involving the RSK insertion algorithms, we do not delve into these algorithms. Instead, we will give a few well-known properties of the RSK bijection.

Lemma 2.9. *If $RSK(w) = (T, T')$, then $RSK(w^{-1}) = (T', T)$.*

A *descent of a permutation* σ is an integer i such that $\sigma^{-1}(i) > \sigma^{-1}(i+1)$. A *descent of a tableau* T is an integer i such that $\text{row}(T^{-1}(i)) < \text{row}(T^{-1}(i+1))$. A standard fact for the RSK row insertion algorithm is:

Lemma 2.10. *An integer i is a descent of σ if and only if it is a descent of the tableau T which is the row insertion of σ.*

Example 2.11. *The permutation in two line notation*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 2 & 4 \end{pmatrix}$$

has descents 2 and 4 and under RSK row insertion maps to

1	2	4
3	5	

which has descents 2 and 4.

3 Irreducible Components of Springer and Steinberg Varieties

The irreducible components of the Steinberg variety and Springer varieties both have been thoroughly studied. In the beginning of this section, we describe the irreducible components of the Steinberg and Springer varieties. In §3.3, we provide different criteria for irreducible components of Springer fibers to be unions of certain projective lines corresponding to parabolic subgroups. In §3.4, we study certain intersections of irreducible components of Springer fibers using the lines in §3.3.

3.1 Steinberg Variety

There are two common constructions of the irreducible components of the Steinberg variety. In this subsection, we demonstrate both approaches and how comparing the two yields a geometric RSK correspondence. We largely follow the exposition in [DR00] and in [CG97].

The Steinberg variety has a surjective projection $\pi : Z \to \mathcal{B} \times \mathcal{B}$. From the Bruhat decomposition (see §2.2), the G orbits in $\mathcal{B} \times \mathcal{B}$ correspond to the different relative positions. Let $w \in W$, and let Y_w represent the G orbit $G \cdot (B, {}^wB)$ in $\mathcal{B} \times \mathcal{B}$. We define

$$Z_w := \pi^{-1}(Y_w) = G \times^{B \cap {}^wB} \{(e, B, {}^wB) | e \in \mathfrak{n} \cap \text{Ad}(w)\mathfrak{n}\}, \tag{3.0.1}$$

which forms a partition of the space Z. For later in our paper we note the preimage of the closed set $\bigcup_{w \leq \sigma} Y_w$ is closed:

Proposition 3.1. *For any $w \in W$,*

$$\bigcup_{w \leq \sigma} Z_w$$

is closed.

Looking at the second expression for Z_w in (3.0.1), we see that Z_w has a certain bundle structure over Y_w. In fact:

Lemma 3.2. *[CG97] Z_w is the co-normal tangent bundle over Y_w. Z_w is a locally closed irreducible subset of Z of dimension $2\dim(\mathcal{B})$.*

Note that the dimension can be computed as

$$\dim(Z_w) = \dim(G) - \dim(B \cap {}^wB) + \dim(\mathfrak{n} \cap \mathrm{Ad}(w)\mathfrak{n}) = 2\dim(\mathcal{B}).$$

Since the Z_w partition the space Z, we have

Corollary 3.3. *[CG97] The irreducible components of Z are the set of closures $\overline{Z_w}$ for $w \in W$.*

The Steinberg variety has another surjective projection $\nu : Z \to \mathcal{N}$. Let $\mathcal{O}$ denote a G orbit in $\mathcal{N}$ and fix a nilpotent $e \in \mathcal{O}$. We define

$$Z_{\mathcal{O}} := \nu^{-1}(\mathcal{O}) = G \times^{C_G(e)} \mathcal{B}_e \times \mathcal{B}_e$$

which again partition Z. Using the fact that $x + \mathfrak{n} = B \cdot x$ for a regular semisimple element x [CG97, Proposition 3.1.43], Chriss and Ginzburg compute the dimension of $Z_{\mathcal{O}}$ to be $2\dim(\mathcal{B})$ and find that $Z_{\mathcal{O}}$ is equidimensional. Furthermore from the second expression for $Z_{\mathcal{O}}$ in our above equation, we see that the irreducible components of $Z_{\mathcal{O}}$ correspond to $C_G(e)$-orbits of pairs of components of $\mathcal{B}_e$.

Proposition 3.4. *[CG97] The irreducible components of Z are the set of closures of $\overline{G \times^{C_G(e)} X_1 \times X_2}$ for pairs of irreducible components X_1, X_2 of $\mathcal{B}_e$.*

Comparing our two indexings of irreducible components of Z, we recover a geometric version of the RSK correspondence:

Corollary 3.5 (RSK Correspondence)**.** *There is a bijection between elements of the Weyl group and the data of a nilpotent orbit $\mathcal{O}$ and the $C_G(e)$-orbit of a pair of irreducible components of Springer fibers for a chosen $e \in \mathcal{O}$ i.e.*

$$W \Longleftrightarrow [\mathcal{O}, (X_1, X_2)/C_G(e)]$$

via the equality of irreducible components of the Steinberg variety

$$\overline{T_{Y_\sigma}} = \overline{G \times^{C_G(e)} X_1 \times X_2}.$$

For the rest of the paper, for two irreducible components X_1, X_2 of a Springer fiber $\mathcal{B}_e$, let w_{X_1,X_2} be the element of the Weyl group which corresponds to X_1, X_2 under the RSK correspondence. In the proceeding section, we will see that $C_G(e)$-orbits of irreducible components in Springer fibers are given by certain simple tableaux. In type A (when $G = SL(n)$ for example), this description yields the RSK bijection between the symmetric group and pairs of tableaux of the same shape consisting of n blocks.

3.2 Springer Variety

The irreducible components of the Springer variety are more difficult to describe than those of the Steinberg variety. In [Spa82, §2] and later in [vL89] and [Pie02], the authors show the irreducible components of Springer fibers in type A correspond to tableaux and in other types correspond to certain equivalence classes of a combinatorial objects called signed domino tableaux. Since our results depend only on $C_G(e)$-invariant properties of irreducible components, we can use a cruder description of the irreducible components of the Springer fibers based on the RSK correspondence.

Theorem 3.6. *[vL89] In type A, irreducible components of Springer fibers are indexed by tableaux T of shape(e). In other types, the $C_G(e)$-orbits of irreducible components of Springer fibers are indexed by tableaux T of shape(e) such that RSK correspondence (T, T) corresponds to $w \in W(G) \subset W(SL(n))$*

under the embedding of G into $SL(n)$ that identifies G/B with the set of isotropic flags. In both instances, the component corresponding to T is the closure of the set of flags $V_\bullet$ such that

$$shape(e|_{V_i}) = T|_{[1,i]}.$$

The irreducible components within a $C_G(e)$-orbit of irreducible components in $\mathcal{B}_e$ are the connected components of the orbit.

Example 3.7. *For $G = Sp(6)$ with $\mathbb{C}^6$ having the ordered basis $(e_1, e_2, e_3, f_1, f_2, f_3)$ with the standard sympletic form $\langle e_i, f_j\rangle = \delta_{i,j}$, $\langle e_i, e_j\rangle = 0$, $\langle f_i, f_j\rangle = 0$. For the nilpotent*

$$n = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

the Springer fiber $\mathcal{B}_n$ consists of three irreducible components given by the closures of the sets of isotropic flags for the symplectic form with corresponding tableaux pictured below:

$$\left\{ \begin{matrix} \text{isotropic} \\ V_\bullet \end{matrix} \;\middle|\; \begin{matrix} V_1 = \langle ae_1 + bf_3\rangle \\ V_2 = \langle e_1, f_3\rangle \\ V_3 = \langle e_1, f_3, ce_2 + df_2\rangle \end{matrix} \right\}$$

1	3	5
2	4	6

$$\left\{ \begin{matrix} \text{isotropic} \\ V_\bullet \end{matrix} \;\middle|\; \begin{matrix} V_1 = \langle ae_1 + bf_3\rangle \\ V_2 = \langle ae_1 + bf_3, ae_2 - bf_2 \;\; + c(be_1 + af_3)\rangle \\ V_3 = \langle e_1, f_3, ae_2 - bf_2\rangle \end{matrix} \right\}$$

1	2	4
3	5	6

$$\left\{ \begin{matrix} \text{isotropic} \\ V_\bullet \end{matrix} \;\middle|\; \begin{matrix} V_1 = \langle ae_1 + bf_3\rangle \\ V_2 = \langle ae_1 + bf_3, ae_2 - bf_2\rangle \\ V_3 = \langle ae_1 + bf_3, ae_2 - bf_2, ae_3 + bf_1 + c(be_1 + af_3)\rangle \end{matrix} \right\}$$

1	2	3
4	5	6

Since we will use the RSK correspondence throughout our paper, it is important we understand the map sending a pair of irreducible components X_1, X_2 of $\mathcal{B}_e$ to the w_{X_1,X_2} i.e. the $w \in W$ such that in the Steinberg variety

$$\overline{G \times^{C_G(e)} X_1 \times X_2} = \overline{Z_w}. \tag{3.7.1}$$

First notice from this description that we can find the dimensions of the irreducible components of Springer fibers based on the dimension of the corresponding irreducible component in the Steinberg variety:

Corollary 3.8. *[CG97] The dimension of an irreducible component $X \subset \mathcal{B}_e$ is $\dim G/B - 1/2 \dim \mathcal{O}$. Where $\mathcal{O} \subset \mathcal{N}$ is the G-orbit of e.*

Proof. Put $X_1 = X_2 = X$ into (3.7.1). □

Now we turn our attention to the particular details relating pairs of components in Springer fibers to relative positions:

Proposition 3.9. *Let X_1, X_2 be two components in $\mathcal{B}_e$. Then*

(i) Generically, two Borel subsets $B_1 \in X_1, B_2 \in X_2$ are in relative position w_{X_1,X_2}.

(ii) For two Borel subsets $B_1 \in X_1, B_2 \in X_2$, B_1, B_2 are in a relative position $\leq w_{X_1,X_2}$.

Proof. (i) From the RSK correspondence, the intersection $Z_{w_{X_1,X_2}} \cap G \times^{C_G(e)} X_1 \times X_2$ is dense in $G \times_{C_G(e)} X_1 \times X_2$. Since the relative position of two elements is independent of conjugation, the statement follows.
(ii) From Proposition 3.1,

$$\overline{Z_w} \subseteq \bigcup_{\sigma \leq w} Z_\sigma.$$

Since $X_1 \times X_2 \subseteq \overline{Z_w}$, the statement follows. □

3.3 Parabolic Projective Lines in the Springer Variety

In some sense the most combinatorial way to connect relative positions in the Steinberg variety is via projective lines corresponding to P/B for a parabolic subgroups P with two roots $\pm\alpha$ in its Levi quotient. In the Springer fibers, Spaltenstein–Tits used these lines to prove that Springer fibers are connected [Spa82]. In this subsection, we analyze which irreducible components of the Springer fibers are unions of these projective lines.

Definition 3.10. *For a simple root ρ, a* line of type ρ *in the flag variety is the image of a map $P_\rho/B \to G/B$ sending ${}^gB \mapsto {}^{hg}B$ for some fixed $h \in G$. A line of type ρ in a Springer fiber $\mathcal{B}_e$ is a line of type ρ in the flag variety which is contained in $\mathcal{B}_e$.*
A subvariety $X \subset \mathcal{B}_e$ is called a ρ-variety if it is a union of lines of type ρ.

An important fact, we will implicitly use throughout this paper is that any Borel subgroup is contained in a unique line of type ρ [Spr98, Cor. 6.4.11]. Spaltenstein showed the Springer fibers are connected based on the following proposition:

Proposition 3.11. *[Ste74, §3.9 Prop. 1] If $B_1, B_2 \in \mathcal{B}_e$ are in relative position w and $\rho w < w$, then the line of type ρ containing B_1 is contained in $\mathcal{B}_e$.*

Proposition 3.12. *An irreducible component $X \subset \mathcal{B}_e$ is a ρ-variety if and only if*

$$\ell(\rho w_{X,X}) < \ell(w_{X,X}).$$

Furthermore every irreducible ρ-subvariety $X \subset \mathcal{B}_e$ is contained in an irreducible component of $\mathcal{B}_e$ which is a ρ-variety.

Before we jump into the proof, we define the product of lines of types in products of the flag variety $\mathcal{B}_e \times \mathcal{B}_e$.

Definition 3.13. *A* line of type $\rho \times 1$ (resp. $\rho \times \rho'$) *in $\mathcal{B}_e \times \mathcal{B}_e$ is a line of type ρ times a point (resp. a line of type ρ'). In the Steinberg variety, a line of type of $\rho \times 1$ (resp. $\rho \times \rho'$) is a line of type $\rho \times 1$ (resp. $\rho \times \rho'$) over a fixed nilpotent e. We define a $\rho \times 1$ (resp. $\rho \times \rho'$) variety as a variety which is a union of lines of type $\rho \times 1$ (resp. $\rho \times \rho'$).*

Notice that our proposition is independent of the $C_G(e)$-action on irreducible components of $\mathcal{B}_e$. This motivates us to first prove the analogous statement for the Steinberg variety:

Lemma 3.14. *The irreducible component $\overline{Z_w}$ is a $\rho \times 1$(resp. $1 \times \rho$)-variety if and only if $\ell(\rho w) < \ell(w)$ (resp. $\ell(w\rho) < \ell(w)$). Every irreducible $\rho \times 1$ (resp. $1 \times \rho$)-subvariety of Z is contained in an irreducible component of Z which is a $\rho \times 1$ (resp. $1 \times \rho$) variety.*

Proof. We prove the $\rho \times 1$ statement as the proof of the $1 \times \rho$ statement is analogous.

($\Leftarrow$) By Lemma 2.6(i), a dense subset of the pairs of Borel subgroups $({}^{b\rho}B, {}^{w}B)$, for $b \in B$ are in relative position w. It follows that $Z_w \subset Z$ and $Y_w \subset \mathcal{B} \times \mathcal{B}$ are $\rho \times 1$-varieties. Since $G/B \times G/B$ is a $\mathbb{P}^1$-bundle over $G/P_\rho \times G/B$ and Y_w is a union of fibers of this fiber bundle, its closure is also a union of fibers, hence $\overline{Y_w}$ is a $\rho \times 1$-variety. Thus, the preimage $\pi^{-1}(\overline{Y_w}) \subseteq Z$ of this subvariety is a $\rho \times 1$-variety and hence is a $\mathbb{P}^1$-bundle over its image in the generalized Steinberg variety

$$\{(e, {}^{g}_{1}P, {}^{g}_{2}B) \subset \mathcal{N} \times G/P_\rho \times G/B | e \in \mathrm{Ad}(g_1)\mathfrak{p}, e \in \mathrm{Ad}(g_2)\mathfrak{b}\}.$$

Since Z_w is a union of fibers in this fiber bundle, $\overline{Z_w} \cap \pi^{-1}(\overline{Y_w})$ is a union of fibers in this bundle. Since $\pi^{-1}(\overline{Y_w})$ is a closed set containing Z_w, $\overline{Z_w} \cap \pi^{-1}(\overline{Y_w}) = \overline{Z_w}$ is a union of fibers in this bundle and hence a $\rho \times 1$-variety.

($\Rightarrow$) Suppose that $\ell(\rho w) > \ell(w)$. By the forwards direction, $\overline{Z_{\rho w}}$ is a $\rho \times 1$-variety and every line of type $\rho \times 1$ in $\overline{Z_{\rho w}}$ intersects Z_w at a point. Now $\overline{Z_w}$ cannot by a $\rho \times 1$-variety or else it would contain $Z_{\rho w}$.

(irreducible $\rho \times 1$-subvarieties) If X is an irreducible $\rho \times 1$-subvariety of Z, then X contains Borel subgroups in relative position $\rho\sigma$ for some $\sigma \in W$ with $\ell(\rho\sigma) > \ell(\sigma)$. Thus by Proposition 3.9, X is contained in an irreducible component $\overline{Z_w}$ with $\rho\sigma \leq w$ and thus by Theorem 2.1, $\ell(\rho w) < \ell(w)$. □

Proof of Proposition 3.12. We prove that X is a ρ-variety if and only if $\overline{Z_{w_{X,X}}}$ is a $\rho \times 1$ variety. By Lemma 3.14 this implies the "if and only if" statement in our proposition.

($\Leftarrow$) By Corollary 3.5 and theorem 3.6, $X \times X$ is isomorphic to a connected component of $\overline{Z_{w_{X,X}}} \cap \nu^{-1}(e)$. Since $\overline{Z_{w_{X,X}}}$ is a $\rho \times 1$ variety, so is $\overline{Z_{w_{X,X}}} \cap \nu^{-1}(e)$. Since the lines of type ρ are all connected, this implies $X \times X$ is a $\rho \times 1$-variety and hence X is a ρ-variety.

($\Rightarrow$) If X is a ρ-variety then $G \times^{C_G(e)} X \times X$ is a $\rho \times 1$ subvariety of $Z_{\mathcal{O}}$. Arguing as in the reverse direction of Lemma 3.14, $\overline{G \times^{C_G(e)} X \times X}$ is a $\rho \times 1$ variety. Since $\overline{G \times^{C_G(e)} X \times X} = \overline{Z_{w_{X,X}}}$, the latter is a $\rho \times 1$ variety.

The last statement is proved in the same way as the last statement in Lemma 3.14. □

Remark 3.15. *In fact, from Lemma 3.14, the above proof extends to show X is a ρ-variety if and only if for every irreducible component Y, $\ell(\rho w_{X,Y}) < \ell(w_{X,Y})$.*

The above gives us a characterization of which irreducible components are ρ-varieties in terms of the Weyl group. The proof of Lemma 3.14 shows that the lines of type ρ in some sense connect the different irreducible components in the Steinberg variety. Now we present a characterization of which irreducible components X are ρ-varieties in terms of a condition on tableaux. Our discussion generalizes Spaltenstein's arguments in type A [Spa82, §2]. Let T_X be the tableau associated to the irreducible component X under Theorem 3.6 and let $\mathrm{comp}(T)$ be the union of the $C_G(e)$-orbit irreducible components of $\mathcal{B}_e$ associated to the tableau T.

Lemma 3.16. *For a simple reflection $\rho \in W$, X is a ρ-variety if and only if $\rho T_X \leq T_X$ in the dominance ordering on tableaux.*

Before we dive into the proof, let's remark a bit about how G acts on flags. Let ρ be the simple reflection corresponding to a simple root α and $\alpha : SL(2) \to G$ the map from §2.1. Then in type A, when $\alpha = (i, i+1)$,

$$\alpha(SL(2))V_\bullet = \{V'_\bullet | V'_j = V_j \text{ for all } j \neq i\}.$$

In other types, under the embedding of G into $SL(n)$, a similar statement is true:

$$\begin{aligned}\alpha(SL(2))V_\bullet &= \{\text{ isotropic } V'_\bullet | V'_j = V_j \text{ for all } j \notin \{i, n-i\}\} \\ &= \{V'_\bullet | V'_j = V_j \text{ for all } j \notin \{i, n-i\} \text{ and } V_i^\perp = V_{n-i}\}.\end{aligned}$$

Lemma 3.17. *Let ρ be the simple reflection corresponding to a simple root α and T be some tableau such that $\rho T \leq T$ in the dominance ordering. For a flag $V_\bullet$ in the open subset defined by the condition*

$$(\text{shape}(e|_{V_i}) = T|_{[1,i]})$$

for all i, a dense subset of $\alpha(SL(2))V_\bullet$ is contained in the open subset defined by

$$(\text{shape}(e|_{V_i}) = T|_{[1,i]}).$$

Proof. We first prove this in type A. Suppose that α corresponds to the simple transposition $(i, i+1)$. For all integers s, $column(T^{-1}(s))$ is the smallest integer k such that $\dim\ker(e^k|_{V_s}) > \dim\ker(e^k|_{V_{s-1}})$. If $(i, i+1)T \leq T$, then $column(T^{-1}(i)) \geq column(T^{-1}(i+1))$. Choose $v_i \in V_i \setminus V_{i-1}$ such that the smallest integer k such that $e^k v = 0$ is $k = column(T^{-1}(i))$. Similarly choose $v_{i+1} \in V_{i+1} \setminus V_i$. Then

$$\{V'_\bullet \in \alpha(SL(2))V_\bullet | V_i/V_{i-1} \neq \langle v_{i+1}\rangle\}$$

is contained (and dense) in the open subset

$$(\text{shape}(e|_{V_j}) = T|_{[1,j]}).$$

In other types the same proof works by focusing on the first half of the vector subspaces $V_1, V_2 \ldots V_{[n/2]}$ in the isotropic flags, as the complete isotropic flag is determined from the first half of it from the condition $V_i^\perp = V^{2n-i}$. □

Proof of lemma 3.16. (⇐) From our previous lemma and the fact that the closure of the open subset defined by for all i

$$(\text{shape}(e|_{V_i}) = T_X|_{[1,i]})$$

is equal to X, a dense subset of X is a ρ-variety. Therefore this dense subset is a union of fibers in the fiber bundle $G/B \to G/P_\rho$. Thus its closure (in the flag variety) is also a union of fibers. Since $\mathcal{B}_e$ is closed, X is equal to this union of fibers.

(⇒) In this step we will use that X is a ρ-variety if and only if comp(T) is a ρ-variety, which follows from 3.12. Suppose for the sake of contradiction that $\rho T_X > T_X$. Then by the ⇐ direction, comp(T') is a ρ-variety. Also from our argument in (ii), every line of type ρ in comp(T') intersects comp(T) and thus if comp(T') was a ρ-variety it would be contained comp(T) which is a contradiction. □

Example 3.18. *Following example 3.7 with $G = Sp(6)$ and for the $(3,3)$-nilpotent. With W embedded into S_6 generated by $(12)(56), (23)(45), (34)$, we can see that*

1	3	5
2	4	6

is a $(12)(56)$-variety and a (34)-variety;

1	2	4
3	5	6

is a $(23)(45)$-variety;

1	2	3
4	5	6

is a (34)-variety.

This can be seen on the level flags as well; for instance, the flags in the irreducible component corresponding to

1	3	5
2	4	6

have $V_1 \subseteq ker(e)$, $V_2 = \ker(e)$, and for any flag $V_\bullet$ in this subspace, acting on it by an element in the image of $(12)(56) : SL(2) \to Sp(6)$ will change the V_1-subspace while fixing V_2 and V_3, which is in the same irreducible component.

Now we present a uniqueness result on the lines of the type above relating them further to the Weyl group:

Lemma 3.19. *Given irreducible components X and $Y \neq Y'$ of $\mathcal{B}_e$, if $\ell(w_{Y',X}) \leq \ell(w_{Y,X})$, then both*

(i) Y' is a ρ-variety and

(ii) Every line of type ρ in Y' intersects Y

hold if and only if both $w_{Y,X} = \rho w_{Y',X}$ and Y' intersects Y. Additionally, there exists at most one ρ which satisfies (i) and (ii) for fixed X, Y, Y', and for fixed X, Y', the irreducible component $Y \neq Y'$ which satisfies (ii) is unique.

The last statement in this lemma is important when the centralizer of e acts non-trivially on the irreducible components of $\mathcal{B}_e$.

Proof. ($\Rightarrow$) By Lemma 3.9(ii), $w_{Y,X} \geq \rho w_{Y',X}$ as for generic points $B_1 \in Z, B_2 \in X$, and for a point B_1' in the line of type ρ through B_1, we have

$$(B_1', B_2) \in \{w_{Y',X}, \rho w_{Y',X}\}.$$

Since $\ell(w_{Y,X}) \leq \ell(w_{Y',X})$, we must have $\ell(\rho w_{Z,X}) < \ell(w_{Z,X})$. Applying Corollary 2.3 yields $w_{Y,X} = \rho w_{Y',X}$.

($\Leftarrow$) Since $\ell(w_{Y,X}) \leq \ell(w_{Y',X})$, we must have $\ell(\rho w_{Y',X}) \leq \ell(w_{Y',X})$. By Lemma 3.16, Y' is a ρ-variety. Since every line of type ρ in $Z_{w_{Y,X}}$ intersects $Z_{w_{Y',X}}$, every line of type ρ in a dense subset of $Y' \times X$ intersects $C_G(e)(Y \times X)$. Now we want to show that there is a single irreducible component Y which every line of type ρ in Y' intersects. Since Y' is connected the projection $Y' \to ZY'P_\rho$ has connected image where P_ρ is the parabolic group whose Levi quotient has roots $\pm\rho$. Using Lemma 2.6(i) as in the proof of Lemma 3.14, we can determine that generically a single point in a line of type ρ in $Z_{w_{Y',X}}$ intersects $Z_{w_{Y,X}}$. It follows that a dense subset of Y'/P_ρ is isomorphic to a dense subset of the intersection $Y' \cap C_G(e)Y$. Thus the intersection $Y' \cap C_G(e)Y$ is connected and therefore by Theorem 3.6, Y is unique.

Lastly given $w_{Y,X} = \rho w_{Y',X}$, by Corollary 2.3 there is no $\rho' \neq \rho$ such that $\rho' w_{Y',X} \leq w_{Y,X}$ and therefore by Proposition 3.9(ii), there exists at most one ρ which satisfies (i) and (ii) for fixed X, Y, Y'. □

Remark 3.20. *A similar result to Lemma 3.19 for Type A can be recovered from [MP06, Section 3.1].*

4 Geometric Robinson–Schensted–Knuth Path

The line of type ρ in §3.3 closely relate to the relative position of Borel subgroups. In this section, we construct a sequence of lines of various types that Spaltenstein used to proved the connectivity of Springer fibers. We call these "paths" RSK-paths based on their relation to the relative positions of pairs of irreducible components of $\mathcal{B}_e$ which correspond to Weyl group elements via the RSK correspondence. We construct the variety consisting of the RSK-paths in this section and later introduce Conjecture 1.4.

4.1 RSK-path Within $\mathcal{B} \times \mathcal{B}$

In this subsection, we introduce the RSK-path as a path consisting of lines of type ρ for different simple reflections $\rho \in W$, which starts at a point in Y_w and ends in Y_e.

Let $(B_1, B_2) \subset \mathcal{B} \times \mathcal{B}$ be in relative position w and let $w = s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_r}$ be a reduced expression for w. By Lemma 2.6 (i), there is exactly one point of the $s_{\alpha_1} \times 1$ line containing (B_1, B_2) which is contained in $Y_{s_{\alpha_1} w}$, and the rest of the line is contained in Y_w. Let $(B_{1,s_{\alpha_1}}, B_2)$ be the point in the $s_{\alpha_1} \times 1$ line containing (B_1, B_2) where $B_{1,s_{\alpha_1}}, B_2$ are in relative position $s_{\alpha_1} w = s_{\alpha_2} s_{\alpha_3} \ldots s_{\alpha_r}$. Then again by Lemma 2.6 (i), there is exactly one point of the $s_{\alpha_2} \times 1$ line containing $(B_{1,s_{\alpha_1}}, B_2)$ which is contained in $Y_{s_{\alpha_2} s_{\alpha_1} w}$, and the rest of the line is contained in $Y_{s_{\alpha_1} w}$. Continuing in this fashion, we get a sequence of projective lines of type s_{α_i} which together form a path from $(B_1, B_2) \in Y_w \subset \mathcal{B} \times \mathcal{B}$ to $(B_2, B_2) \in Y_1 \subset \mathcal{B} \times \mathcal{B}$.

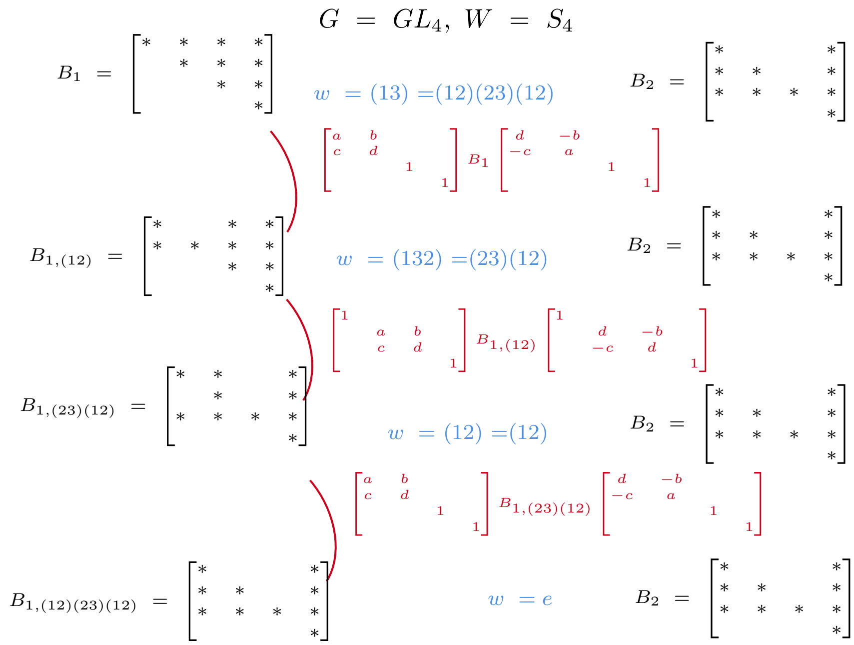


Figure 2: Here the path connecting (B_1, B_2) to (B_2, B_2) is displayed. The red lines indicate the lines of type $s_i \times 1$ connecting $(B_{1,\sigma}, B_2)$ to $(B_{1,\sigma s_i}, B_2)$. The blue Weyl group elements are the relative positions of B_2 with respect to $B_{1,\sigma}$

4.2 RSK-Path Variety

Intuitively, since for each pair of Borel subgroups and a reduced expression for their relative position there is a unique RSK-path between them, we should expect the data of the sequence of parabolic subgroups in the RSK-path between a pair of Borel subgroups to be the same as the data of the pair. In this subsection, we make this precise by realizing it as an isomorphism of algebraic varieties.

First consider the simplest version of an RSK-path: a line of type ρ. Such an RSK-path describes a subset of $G/B \times G/B$ where the points are connected by a line corresponding to P_ρ/B. Concretely, we can identify the space of RSK-paths for $w = \rho$ with the open subset

$$\{(g_1B, g_2B) \in G/B \times_{G/P_\rho} G/B | g_1B \neq g_2B\}$$

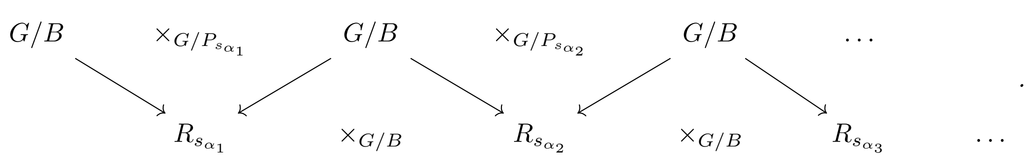
of the pullback variety $G/B \times_{G/P_\rho} G/B$. We shall call this space R_ρ. It is a quasi-projective variety isomorphic to $\mathcal{B} \times A^1$. Similarly we define the quasi-projective variety $R_{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_r}}$ as the open subset

$$\{(g_iB) \in G/B \times_{G/P_{s_{\alpha_1}}} G/B \times_{G/P_{s_{\alpha_2}}} \cdots \times_{G/P_{s_{\alpha_r}}} G/B \mid g_iB \neq g_{i+1}B\}.$$

Remark 4.1. *An alternate way to think about $R_{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_r}}$ is as the concatenation of $R_{s_{\alpha_1}}, R_{s_{\alpha_2}}, \ldots R_{s_{\alpha_r}}$. Specifically, we have*

$$R_{s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_r}} \cong R_{s_{\alpha_1}} \times_{G/B} R_{s_{\alpha_2}} \times_{G/B} R_{s_{\alpha_3}} \times_{G/B} \cdots \times_{G/B} R_{s_{\alpha_r}}.$$

via the diagram



There are many projections from $R_{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_r}}$ onto $\mathcal{B}\times\mathcal{B}$. We shall let proj denote the projection from the first and last factors:

$$\begin{array}{ccc} G/B & \times_{G/P_{s_{\alpha_1}}} G/B \times_{G/P_{s_{\alpha_2}}} \ldots \times_{G/P_{s_r}} & G/B \\ & \searrow \qquad \swarrow & \\ & \mathcal{B}\times\mathcal{B} & \end{array}$$

We have already claimed that $R_{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_r}}$ is the space of RSK-paths. Now we make this precise. Define the map

$$\text{RSK-path}_{w=s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_r}} : Y_w \longrightarrow R_{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_r}}$$

by sending

$$({}^{g}B, {}^{gw}B) \mapsto ({}^{g}B, {}^{gs_{\alpha_1}}B, {}^{gs_{\alpha_r}}B, {}^{gs_{r-1}s_{\alpha_r}}B, \ldots, {}^{gw}B).$$

Then composing the map $\text{RSK-path}_{w=s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_r}}$ with the map

$$\begin{array}{ccccccccccc} G/B & & \times_{G/P_{s_{\alpha_1}}} & & G/B & & \times_{G/P_{s_{\alpha_2}}} & & G/B & & \ldots \\ & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & \\ & G/P_{s_{\alpha_1}} & & \times_{G/P_{s_{\alpha_1,\alpha_2}}} & & G/P_{s_{\alpha_2}} & & \times_{G/P_{s_{\alpha_2,\alpha_3}}} & & G/P_{s_{\alpha_3}} & \ldots \end{array}$$

gives the chain of lines of type s_{α_i} in $\mathcal{B}\times\mathcal{B}$ which the RSK-path applied to $({}^{g}B, {}^{gw}B)$ follows. Here $P_{s_{\alpha_i,\alpha_j}}$ is the parabolic subgroup of G with Levi quotient generated by simple reflections for the roots $\{\pm\alpha_i, \pm\alpha_j\}$.

Lemma 4.2. *For any reduced expression $w = s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_r}$, the map*

$$\text{RSK-path}_{w=s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_r}} : Y_w \to R_{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_r}}$$

is an algebraic map which determines an isomorphism with inverse proj $: R_{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_r}} \to Y_w$.

This is a well-known lemma. To prove that it is an isomorophism, from the fact that

$$\text{proj} \circ \text{RSK-path}_{w=s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_r}} = 1_{Y_w},$$

it is enough to show that proj is injective with image in Y_w. These follow from Lemma 2.6 (i) and (ii) respectively. Algebraicity follows from proj being algebraic.

Remark 4.3. *There is also a direct way to show the map* $\text{RSK-path}_{w=s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_r}}$ *is algebraic. Consider the commutative diagram*

$$\begin{array}{ccc} G/B\times G/B & \xrightarrow{\;q\;} & G/P_{s_{\alpha_1}} \times G/B \\ \uparrow & & \uparrow \\ Y_\sigma & \xrightarrow{\;\phi_{\sigma\to\rho\sigma}\;} & Y_{\rho\sigma} \end{array}.$$

where q is the algebraic quotient and

$$\phi_{w\to\rho w} : ({}^{g}B, {}^{gw}B) \to ({}^{g\rho}B, {}^{gw}B).$$

Since $Y_{\rho\sigma} \cong q(Y_\sigma)$, the map $\phi_{\sigma\to\rho\sigma}$ is algebraic. By inductively applying this procedure, the map $Y_\sigma \to Y_{\tau^{-1}\sigma}$ is algebraic for all $\tau \le \sigma$. These maps can be combined to the map $Y_\sigma \to$ $\text{RSK-path}_{w=s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_r}}$.

4.3 RSK-path Within Springer Fibers and the Steinberg Variety

Proposition 3.11 tells us for any point (e, B, B') in the Steinberg variety Z, the RSK-path in $\mathcal{B}_e \times \mathcal{B}_e$ connecting $\pi(e, B, B')$ to $\pi(e, B', B')$ lifts to a path in $\nu^{-1}(e) \subset Z$ consisting of projective lines of type $s_{\alpha_i} \times 1$ for the s_{α_i} in a minimal decomposition of w. This path connects $(e, B, B') \in Z_w \subset Z$ to $(e, B', B') \in Z_1 \subset Z$. Projecting onto the first Borel subgroup we get the following result of Spaltenstein:

Corollary 4.4. *[Ste74, §3.9 Prop. 1] The Springer fibers $\mathcal{B}_e$ are connected.*

Similarly, Proposition 3.11 tells us that the restriction of RSK-path$_{w=s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_r}}$ to $\mathcal{B}_e \times \mathcal{B}_e$ has image contained in the quasi-projective variety

$$(R_{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_r}})_e := \{(g_iB) \in (G/B)_e \times_{(G/P_{s_{\alpha_1}})_e} (G/B)_e \times_{(G/P_{s_{\alpha_2}})_e} (G/B)_e \ldots \times_{(G/P_{s_{\alpha_r}})_e} (G/B)_e | g_iB \neq g_{i+1}B\}.$$

Restricting the result of Lemma 4.2 to this space we have:

Proposition 4.5. *For any reduced expression $w = s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_r}$, the map*

$$\text{RSK-path}_{w=s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_r}} : (Y_w \cap \mathcal{B}_e \times \mathcal{B}_e) \to (R_{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_r}})_e$$

is an algebraic map which determines an isomorphism with inverse the restriction of proj : $R_{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_r}} \to Y_w$ to $(R_{s_{\alpha_1},s_{\alpha_2},\ldots,s_{\alpha_r}})_e$.

In other words, the algebraic isomorphism of §4.2 between $Y_w \subset \mathcal{B} \times \mathcal{B}$ and $R_{s_{\alpha_1},s_{\alpha_2},\ldots s_{\alpha_r}}$ restricts to an algebraic isomorphism between the subvariety of elements in relative position w within each Springer fiber and the RSK-paths corresponding to the reduced expression $w = s_{\alpha_1}s_{\alpha_2}\ldots s_{\alpha_r}$ contained within the Springer fiber.

Recall our conjecture from the introduction:

Conjecture 1.4. *For an irreducible component $X \subseteq \mathcal{B}_e$, every RSK-path with start and endpoints in X stays in X if and only if under the geometric RSK correspondence, the pair (X, X) corresponds to an element $w \in W$ such that*

1. *w is an involution (this will always be the case for a pair (X, X))*
2. *For every simple reflection s which appears in a reduced expression for $w_{X,X}$,*

$$\ell(sw_{X,X}) < \ell(w_{X,X}).$$

By Corollary 2.2 of Matsumoto's theorem, the second condition is independent of the reduced expression for $w_{X,X}$. We now have the tools to prove that our condition on w is sufficient:

Proof. If for every simple reflection s_{α_i} in a reduced word for $s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}\ldots s_{\alpha_r} = w_{X,X}$,

$$\ell(s_{\alpha_i}w_{X,X}) < \ell(w_{X,X}),$$

then X is a s_{α_i}-variety for all i. Since every RSK-path with start and endpoints in X is a sequence of connected lines of type s_{α_i}, it stays in X. □

Remark 4.4. *A similar question one might hope to answer is the following: Which irreducible components $X \subseteq \mathcal{B}_e$ satisfy the property that for any B_1, B_2, there* ***exists*** *an RSK-path between B_1 and B_2 which stays within X? This is not quite equivalent to the conjecture above: Consider the irreducible component X corresponding to the tableau*

1	3
2	
4	

of the $2+1+1$ nilpotent for $SL_4(\mathbb{C})$. It can be determined that a flag $V_\bullet \in X$ if and only if $im(e) \subset V_2$. In this instance, for any two flags in X, the RSK-path associated to $(12)(34)(23)(34)(12)$ stays within X. However the RSK-path associated to $(12)(23)(34)(23)(12)$ does not stay within X.

5 Relative Positions and w-Orbital Varieties

In the previous section, we show that the data of the RSK-paths within a Springer fiber is isomorphic to the data of the variety $Y_w \cap \mathcal{B}_e \times \mathcal{B}_e$. To study RSK-paths further, we investigate the variety $Y_w \cap (\mathcal{B}_e \times \mathcal{B}_e)$.

In this section, we show that there is a natural correspondence between $Y_w \cap (\mathcal{B}_e \times \mathcal{B}_e)$ and the w-orbital variety $\mathfrak{n}_w \cap \mathcal{O} = (\mathfrak{n} \cap \mathrm{Ad}(w)\mathfrak{n}) \cap \mathcal{O}$, where $\mathcal{O}$ denotes the nilpotent orbit of e. We extend this correspondence to show that the w-orbital varieties concretely describe the sequence of $C_G(e)$-orbits of components that RSK-paths take. In this way, we show that the w-orbital variety encodes the information of the corresponding RSK-paths in $\mathcal{B}_e$.

5.1 Correspondence between relative positions within Springer fibers and w-orbital varieties

There is a natural duality

$$\mathcal{B}_e = \{B | e \in \mathfrak{b}, e \text{ is nilpotent }\} \Longleftrightarrow \mathfrak{n} = \{e | e \in \mathfrak{b}, e \text{ is nilpotent }\}.$$

or more generally a duality

$$Y_\sigma \cap (\mathcal{B}_e \times \mathcal{B}_e) \Longleftrightarrow \mathfrak{n}_\sigma := \mathfrak{n} \cap \mathrm{Ad}(w)\mathfrak{n}.$$

In this subsection we make this duality precise by analyzing the orbits of these spaces inside the Steinberg variety. We then prove Theorem 1.1, which when $w = 1$, recovers a classic result by Spaltenstein [Spa77].

Theorem 1.2. *There is a bijection between irreducible components of the w-orbital variety $\mathfrak{n}_w \cap \mathcal{O}$ and $C_G(e)$-orbits of irreducible components of $Y_w \cap \mathcal{B}_e \times \mathcal{B}_e$ given by taking corresponding irreducible components of $Z_w \cap \mathcal{O}$ via (5.0.1).*

Proof. First embed $Y_w \cap \mathcal{B}_e \times \mathcal{B}_e$ in the Steinberg variety as the subvariety

$$\begin{aligned} Y_w \cap \nu^{-1}(e) &= \{(e, B_1, B_2) | e \in \mathfrak{b}_1, e \in \mathfrak{b}_2, B_1 \text{ in relative position } w \text{ to } B_2\} \\ &\cong Y_w \cap \mathcal{B}_e \times \mathcal{B}_e. \end{aligned}$$

Similarly, for a fixed Borel subgroup B, embed the w-Orbital variety $\mathfrak{n}_w \cap \mathcal{O}$ as

$$\begin{aligned} Y_w \cap \pi^{-1}(B, {}^wB) &= \{(e, B, {}^wB) | e \in \mathfrak{n}_w \cap \mathcal{O}\} \\ &\cong \mathfrak{n}_w \cap \mathcal{O}. \end{aligned}$$

Now the conjugation action of G on both these spaces generates the subspace $Z_w \cap Z_{\mathcal{O}}$, i.e.,

$$G \times^{C_G(e)} (Y_w \cap \mathcal{B}_e \times \mathcal{B}_e) = Z_w \cap Z_{\mathcal{O}} = G \times^{B \cap {}^wB} (\mathfrak{n}_w \cap \mathcal{O}). \tag{5.0.1}$$

This yields a bijection

$$\mathrm{Irr}(Y_w \cap \mathcal{B}_e \times \mathcal{B}_e)/C_G(e) \Longleftrightarrow \mathrm{Irr}(Z_w \cap Z_{\mathcal{O}}) \Longleftrightarrow \mathrm{Irr}(\mathfrak{n}_w \cap \mathcal{O})/(B \cap {}^wB).$$

Now notice

$$B \cap {}^wB = T \cdot \prod_{\substack{\alpha > 0, \\ w(\alpha) > 0}} U_\alpha$$

where $U_\alpha \cong \mathbb{G}_a$ is the unipotent subgroup corresponding to the root α. It follows that $B \cap {}^wB$ is a connected subgroup and therefore $B \cap {}^wB$-orbits in $\mathfrak{n}_w \cap \mathcal{O}$ are irreducible. Therefore $B \cap {}^wB$ acts trivially on the set of irreducible components of $\mathfrak{n}_w \cap \mathcal{O}$. $\square$

$Z_\sigma \cap Z_{\mathcal{O}}$ is a subvariety of the Steinberg variety

$G\times^{B\cap{}^\sigma B}(e,B,{}^\sigma B)$ for $e \in Lie(B)\cap Lie({}^\sigma B)$ with e nilpotent $=\!=$ $Z_\sigma \cap Z_{\mathcal{O}}$ $=\!=$ $G\times^{C_G(e)}(e,B_1,B_2)$

$\|$ $\qquad\qquad$ $\|$

$G\times^{B\cap{}^\sigma B}(n_\sigma \cap \mathcal{O})$ where $n_\sigma = Lie(B)\cap Lie({}^\sigma B)\cap\mathcal{N}$ $\qquad$ $G\times^{C_G(e)}(Y_\sigma \cap (\mathcal{B}_e\times\mathcal{B}_e))$

$n_\sigma \cap \mathcal{O}$ is a σ-orbital variety

$Y_\sigma \cap (\mathcal{B}_e\times\mathcal{B}_e)$ is a subvariety of the product of Springer fibers

Irreducible components are $G\times^{B_\sigma} X$ for X irreducible component of $(n_\sigma \cap \mathcal{O})$

Irreducible components are $G\times^{C_G(e)} X$ for X irreducible component of $(Y_\sigma \cap (\mathcal{B}_e\times\mathcal{B}_e))$

When σ corresponds to orbit $\mathcal{O}$, $n_\sigma \cap \mathcal{O}$ is dense in n_σ

When σ corresponds to orbit $\mathcal{O}$, $(Y_\sigma \cap (\mathcal{B}_e\times\mathcal{B}_e))$ is a dense subset of $C_G(e)(X_1\times X_2)$ for irreducible components $X_1, X_2 \subseteq \mathcal{B}_e$

Figure 3: The correspondence between σ-orbital varieties and $Y_\sigma \cap \mathcal{B}_e \times \mathcal{B}_e$

Remark 5.1. *When w corresponds to a $C_G(e)$ orbit of pairs of irreducible components of $\mathcal{B}_e$, then for any pair of components in this orbit X, Y, the subset of pairs of Borel subgroups in $X\times Y$ in relative position w is dense in $X\times Y$. This is reflected on the w-orbital variety side by $\mathfrak{n}_w \cap Z_{\mathcal{O}}$ being dense in $\mathfrak{n}_w$ where $\mathcal{O}$ is the G orbit of e.*

In this case, $\overline{Z_w} = \overline{X}$ for an irreducible component X of $Z_{\mathcal{O}}$. Then

$$Z_{\mathcal{O}} \cap Z_w \subseteq Z_{\mathcal{O}} \cap \overline{X} = X,$$

and

$$X = Z_{\mathcal{O}} \cap X \subseteq Z_{\mathcal{O}} \cap \overline{Z_w}.$$

Therefore $Z_w \cap Z_{\mathcal{O}}$ has one irreducible component X' which is dense in X. Since X is dense in $\overline{Z_w}$, X' is dense in Z_w. That is, $Z_w \cap Z_{\mathcal{O}} = G\times^{B\cap{}^wBw}\mathfrak{n}_w \cap \mathcal{O}$ is dense in $Z_w = G\times^{B\cap{}^wB}\mathfrak{n}_w$ and so $\mathfrak{n}_w \cap \mathcal{O}$ is dense in $\mathfrak{n}_w$.

When $w = 1$, we recover a classical theorem of Spaltenstein:

Corollary 5.2. *[Spa77] There is a bijection between irreducible components of the $\mathfrak{n}\cap\mathcal{O}$ and $C_G(e)$-orbits of irreducible components of $\mathcal{B}_e$.*

We now explicitely write out about the bijection in Theorem 1.1 based on 5.0.1. For a $C_G(e)$-orbit of an irreducible component X of $Y_w \cap \mathcal{B}_e \times \mathcal{B}_e$, the corresponding irreducible component X' of $\mathfrak{n}_w \cap \mathcal{O}$

satisfies

$$G \times^{C_G(e)} C_G(e) \cdot X = G \times^{B \cap {}^w B} X',$$

which implies

$$X' = \{\mathrm{Ad}(g^{-1})e \in \mathfrak{n}_w | ({}^g B, {}^{gw} B) \in C_G(e) \cdot X\}$$

In the case $w = 1$, from the tableaux description of $C_G(e)$-orbits of irreducible components of $\mathcal{B}_e$ in Theorem 3.6, we find:

Theorem 5.3. *The irreducible components of $\mathfrak{n} \cap \mathcal{O}$ are in bijection with tableaux T of* shape($\mathcal{O}$) *such that under RSK, the pair (T, T) corresponds to $w \in W(G) \subset W(SL(n))$ under the embedding of G into $SL(n)$ that identifies G/B with the set of isotropic flags.*
The component corresponding to T is the closure of the set of nilpotents e such that for the standard flag $V_\bullet$ corresponding to the standard Borel subgroup B,

$$shape(e|_{V_i}) = T|_{[1,i]}.$$

Example 5.4 (Continuation of Example 3.7)**.** *For $G = Sp(6)$ with W embedded as the subset of S_6 generated by $(12)(56), (23)(45), (34)$, is generated by the root spaces e_w for $w \in W$ pictured below*

$$\mathfrak{n} = \begin{pmatrix} 0 & e_{(12)(56)} & e_{(13)(46)} & e_{(14)(36)} & e_{(15)(26)} & e_{(16)} \\ 0 & 0 & e_{(23)(45)} & e_{(24)(35)} & e_{(25)} & e_{(15)(26)} \\ 0 & 0 & 0 & e_{(34)} & e_{(24)(35)} & e_{(14)(36)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_{(12)(56)} & 0 & 0 \\ 0 & 0 & 0 & -e_{(13)(46)} & -e_{(23)(45)} & 0 \end{pmatrix}.$$

and for the orbit of the nilpotent with $(3,3)$-Jordan normal form, $\mathfrak{n} \cap \mathcal{O}$ has 3 irreducible components of dimension 7 which are dense in the subsets of $\mathfrak{n}$,

$$\begin{pmatrix} 0 & 0 & e_{(13)(46)} & 0 & e_{(15)(26)} & e_{(16)} \\ 0 & 0 & e_{(23)(45)} & e_{(24)(35)} & e_{(25)} & e_{(15)(26)} \\ 0 & 0 & 0 & e_{(34)} & e_{(24)(35)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_{(13)(46)} & -e_{(23)(45)} & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & e_{(12)(56)} & e_{(13)(46)} & e_{(14)(36)} & e_{(15)(26)} & e_{(16)} \\ 0 & 0 & e_{(23)(45)} & \lambda e_{(12)(56)} & -\lambda e_{(13)(46)} & e_{(15)(26)} \\ 0 & 0 & 0 & e_{(34)} & \lambda e_{(12)(56)} & e_{(14)(36)} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_{(12)(56)} & 0 & 0 \\ 0 & 0 & 0 & -e_{(13)(46)} & -e_{(23)(45)} & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & e_{(12)(56)} & e_{(13)(46)} & 0 & e_{(15)(26)} & e_{(16)} \\ 0 & 0 & e_{(23)(45)} & 0 & e_{(25)} & e_{(15)(26)} \\ 0 & 0 & 0 & e_{(34)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e_{(12)(56)} & 0 & 0 \\ 0 & 0 & 0 & -e_{(13)(46)} & -e_{(23)(45)} & 0 \end{pmatrix}.$$

The standard flag $V_\bullet$ in this case is given by the permuation of the standard basis $v_1, v_2, v_3, v_6, v_5, v_4$ in that V_i is the span of the first i vectors in that list. By computing the Jordan normal form of a generic element in each subvariety of $\mathfrak{n} \cap \mathcal{O}$'s action of the standard flag $V_\bullet$ we get the corresponding tableaux

1	3	5
2	4	6

,

1	2	4
3	5	6

,

1	2	3
4	5	6

respectively which are the same tableaux as in Example 3.7.

There is an analog in $\mathfrak{n}\cap\mathcal{O}$ of the projective lines of type ρ in Springer fibers. Recall from Section 2.1 that for any root α with corresponding unipotent subgroup U_α, there is a map $\alpha : SL(2) \to G$ which restricts to a map $\alpha : SU(2) \to G$ generating the parabolic group containing B with Levi quotient having roots $\pm\alpha$.

Definition 5.5. *A line of type s_α in $\mathfrak{n}\cap\mathcal{O}$ is a projective line of the form*

$$\mathrm{Ad}(\alpha(SU(2))e.$$

Example 5.6. *Consider the nilpotent for $Sl(3)$*

$$e = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0. \end{pmatrix}$$

and the root $\alpha = (12)$. Then

$$\alpha\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathrm{Ad}\left(\alpha\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\right)e = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & c \\ 0 & 0 & 0. \end{pmatrix}$$

Thus $\mathrm{Ad}(\alpha(SU(2))e \cong P^1$.

Remark 5.7. *The reason we restrict α to the subgroup $SU(2)$ is so that our lines of type s_α are projective lines as in the flag variety case. Alternatively, one could define a very similar subvariety to our lines of type s_α which involve the adjoint action of the whole of $\alpha(SL(2))$ and achieve roughly the same results. This could be advantageous in some situations where the complex geometry is very important and also generalizes more easily to other fields.*

As with subvarieties of the flag variety, we call a subvariety of $\mathfrak{n}\cap\mathcal{O}$ an s_α-variety if it is a union of projective lines of type s_α.

From (5.0.1), we see the following.

Lemma 5.8. *A Borel subgroup ${}^gB \in \mathcal{B}_e$ is in a $C_G(e)$-orbit of an irreducible component $X \subset \mathcal{B}_e$ if and only if $\mathrm{Ad}(g^{-1})e$ is in the component $Y \subset \mathfrak{n}\cap\mathcal{O}$ which corresponds to X via the bijection in Theorem 1.1.*

Lemma 5.9. *An irreducible component X of $\mathfrak{n}\cap\mathcal{O}$ corresponding to a tableau T is an s_α-variety if and only if the $C_G(e)$-orbit of irreducible components $\mathcal{B}_e$ corresponding under RSK to (T, T) is an s_α-variety.*

Proof. For notational simplicity, let Y be the $C_G(e)$-orbit of irreducible components $\mathcal{B}_e$ corresponding under RSK to (T, T). In the proof of Theorem 5.3 we see X corresponds to T if under the bijection of 1.1, X corresponds to Y.

($\Rightarrow$) If X is an s_α-variety, then for any ${}^gB \in Y$, $\mathrm{Ad}(g^{-1})e \in X$. Since X is an s_α-variety, $\mathrm{Ad}(\alpha(x)^{-1}g^{-1})e \in X$ for any $x \in SU(2)$ and thus Y contains the line of type s_α containing ${}^gB$.

($\Leftarrow$) If Y is an s_α-variety for any $x \in SU(2)$ we have ${}^{\alpha(x)}B \in Y$ and acting on e by $\mathrm{Ad}(\alpha(x)^{-1})$ implies X is an s_α-variety. ∎

5.2 Dual Robinson–Schensted–Knuth Path in $\mathfrak{n}\cap\mathcal{O}$

Similar to the RSK-paths within Springer fibers, there is a notion of an RSK-path within $\mathfrak{n}\cap\mathcal{O}$. In this subsection, we describe this RSK-path and demonstrate how to find the sequence of irreducible components of $\mathfrak{n} \cap \mathcal{O}$ which it passes through. We then show how this can be used to give the irreducible components of $\mathcal{B}_e$ which an RSK-path passes through.

We use conjugation to construct RSK-paths within $\mathfrak{n} \cap \mathcal{O}$ similar to the RSK-paths within Springer fibers or more generally the Steinberg variety. Recall the RSK-path in section 4 for a reduced expression $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_r}$ and point in the Steinberg variety (e, B_1, B_2) with B_1 in relative position w to B_2 is the unique sequence of projective lines of types s_{α_i} which connect (e, B_1, B_2) to (e, B_2, B_2). The corresponding RSK-path in $\mathfrak{n} \cap \mathcal{O}$ we find through conjugation by $\alpha(SU(2))$:

Given a point $e \in \mathfrak{n}_w \cap \mathcal{O}$ and a reduced expression $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_r}$. We define the RSK-path e takes for the expression as

$$\mathrm{Ad}(\alpha_{i+1}(x)^{-1} \sigma^{-1}) e$$

for $\sigma = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_i}$ and $x \in SU(2)$. This defines a path of projective lines from $e \in \mathfrak{n}_w \cap \mathcal{O} \subseteq \mathfrak{n} \cap \mathcal{O}$ to $\mathrm{Ad}(w^{-1}) e \in \mathfrak{n} \cap \mathcal{O}$.

As in the RSK-path in the Springer fiber where there is a single point between two adjacent projective lines, the RSK-path in the orbital variety has the unique point $\mathrm{Ad}(\sigma^{-1}) e$ connecting the adjacent projective lines.

Proposition 5.10. *A path of components the dual RSK-path passes through in $\mathfrak{n} \cap \mathcal{O}$ when applied to a reduced expression $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_r}$ is the sequence of irreducible components corresponding to tableaux T where*

$$\mathrm{Ad}(\sigma^{-1})(e)|_{V_i} = T|_{[1,i]}$$

or equivalently,

$$e|_{\sigma(V_i)\sigma^{-1}} = T|_{[1,i]}$$

where $\sigma = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_i}$.

Proof. Let X_T refer to the component of $\mathfrak{n} \cap \mathcal{O}$ which corresponds to T under RSK. Let T_σ be the tableaux such that

$$\mathrm{Ad}(\sigma^{-1})(e)|_{V_i} = T|_{[1,i]}.$$

Clearly $\mathrm{Ad}(\sigma^{-1})(e) \in X_{T_\sigma}$. Also note that $T_{\rho\sigma} = \rho T_\sigma$. Now by lemma 5.9 and lemma 3.16, at least one of $T_{\rho\sigma}$ or ρT_σ is a ρ-variety. Therefore the projective line connecting $\mathrm{Ad}(\sigma^{-1})(e)$ and $\mathrm{Ad}(\sigma^{-1}\rho^{-1})(e)$ in the dual RSK-path is contained in one of $T_{\rho\sigma}$ or ρT_σ. □

From Lemma 5.8 and arguing as in Lemma 5.9, our dual RSK-path is dual to the normal RSK-path:

Theorem 1.3. *Fix $g \in G$ such that $\mathrm{Ad}(g)e = e'$. Then if P is the dual RSK-path from e to $\mathrm{Ad}(w^{-1})e$ for a reduced expression $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_r}$, then the image of the map defined by*

$$\mathrm{Ad}(\alpha_{i+1}(x)^{-1} \sigma^{-1}) e \mapsto {}^{g\sigma\alpha_{i+1}(x)}B$$

is the RSK-path from ${}^gB$ to ${}^{gw}B$. Furthermore, a sequence of components the RSK-path passes through corresponds to the sequence of tableaux in corresponding to the sequence T in Proposition 5.10.

This gives us a way of computing a sequence of paths which the RSK-path passes through in type A. In other types where the $C_G(e)$ action on components may be non-trivial, the uniqueness of Y in the proof of lemma 3.19 allows one to trace out the path of irreducible components which RSK passes through with this result.

5.3 Irreducible Components of w-Orbital Varieties

In this subsection, we study the irreducible components of $\mathfrak{n}_w \cap \mathcal{O}$ and $Y_w \cap \mathcal{B}_e \times \mathcal{B}_e$ through use of the RSK-path variety. In particular we will give formulae to compute their dimensions.

By Theorem 1.1 and Proposition 4.5 we have that for any reduced expression $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_r}$,

$$Irr((R_{s_{\alpha_1}, s_{\alpha_2}, \dots s_{\alpha_r}})_e) \cong Irr(Y_w \cap \mathcal{B}_e \times \mathcal{B}_e) \to_{/C_G(e)} Irr(\mathfrak{n} \cap \mathcal{O}).$$

We wish to find dimensions for each irreducible component in the above spaces. Now for an irreducible component $X \subseteq \mathfrak{n}_w \cap \mathcal{O}$ which corresponds to the $C_G(e)$-orbit of an irreducible component $Y \subset Y_w \cap \mathcal{B}_e \times \mathcal{B}_e$, 5.0.1 tells us

$$\dim G - \dim(B \cap {}^wB) + \dim X = \dim G - \dim C_G(e) + \dim Y. \tag{5.10.1}$$

Using the following two equations (the first is 3.3.24 in [CG97]),

$$1/2 \dim \mathcal{O} + \dim \mathcal{B}_e = \dim G/B \tag{5.10.2}$$
$$\dim(B \cap {}^wB) = \dim B - \ell(w), \tag{5.10.3}$$

we get

$$\begin{aligned}\dim X - 1/2\dim\mathcal{O} &= \dim X - \dim\mathcal{O} + \dim(G/B) - \dim\mathcal{B}_e \\ &= \dim X - \dim\mathcal{O} + \dim G - \dim(B \cap {}^wB) - \ell(w) - \dim\mathcal{B}_e \\ &= \dim Y - \dim\mathcal{O} + \dim G - \dim C_G(e) - \ell(w) - \dim\mathcal{B}_e \\ &= \dim Y - \ell(w) - \dim\mathcal{B}_e\end{aligned}$$

where the blue indicates our use of 5.10.1. When w is trivial, we find that $\mathfrak{n} \cap \mathcal{O}$ is an equidimensional variety of dimension $1/2 \dim \mathcal{O}$.

Remark 5.11. *While $\mathfrak{n} \cap \mathcal{O}$ is equidimensional, $\mathfrak{n}_w \cap \mathcal{O}$ need not be. For example when $G = SL(4)$, $\mathcal{O}$ is the nilpotent orbit associated with the $2,1,1$-Jordan normal form, and $w = (132)$, then*

$$\mathfrak{n}_w = \begin{pmatrix} 0 & e_{(12)} & 0 & e_{(14)} \\ 0 & 0 & 0 & e_{(24)} \\ 0 & 0 & 0 & e_{(34)} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\mathfrak{n}_w \cap \mathcal{O}$ has two irreducible components which are dense subsets of

$$\begin{pmatrix} 0 & e_{(12)} & 0 & e_{(14)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & e_{(14)} \\ 0 & 0 & 0 & e_{(24)} \\ 0 & 0 & 0 & e_{(34)} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which have different dimensions.

Now we try to compute the dimension of irreducible components of $Y_w \cap \mathcal{B}_e \times \mathcal{B}_e$ via the isomorphism

$$Y_w \cap \mathcal{B}_e \times \mathcal{B}_e \cong_{RSK} (R_{s_{\alpha_1}, s_{\alpha_2}, \ldots s_{\alpha_r}})_e.$$

Let π_i be the projection of

$$R_{s_{\alpha_1}, s_{\alpha_2}, \ldots s_{\alpha_r}} = G/B \times_{G/P_{s_{\alpha_1}}} \times G/B \times_{G/P_{s_{\alpha_2}}} G/B \ldots \times_{G/P_{s_{\alpha_r}}} G/B$$

onto the i-th copy of G/B and let $\pi_{P,i}$ denote the projection onto $G/P_{s_{\alpha_i}}$. Since the projection onto $\pi_{P,i}(Y)$ from $\pi_i(Y)$ and $\pi_{i+1}(Y)$ is surjective, the dimension of an irreducible subset $Y \subset (R_{s_{\alpha_1}, s_{\alpha_2}, \ldots s_{\alpha_r}})_e$ is

$$\dim Y = \sum \dim \pi_i(Y) - \sum \dim \pi_{P,i}(Y). \tag{5.11.1}$$

This equation is the first part of Theorem 1.2.
For the second part of Theorem 1.2, notice for $Y \subset (R_{s_{\alpha_1}, s_{\alpha_2}, \ldots s_{\alpha_r}})_e$, the maximal dimension of $\dim \pi_{P,i}(Y)$ can be $\dim \mathcal{B}_e - 1$. Since map $\mathcal{G}/B \to G/P_{\alpha_i}$ is a P^1-bundle, when Y is irreducible and $\dim \pi_{P,i}(Y) = \dim \mathcal{B}_e - 1$, the preimage of $\pi_{P,i}(Y)$ is a dimension $\dim \mathcal{B}_e$ irreducible subvariety of $\mathcal{B}_e$. It follows that there is a unique irreducible component of $\mathcal{B}_e$ which the preimage of $\pi_{P,i}(Y)$ is contained in. We are now ready to prove theorem 1.2:

Theorem 1.2. *For an irreducible component $Y \subseteq (R_{s_{\alpha_1}, s_{\alpha_2}, \ldots s_{\alpha_r}})_e$,*

$$\dim Y = \sum_i \dim \pi_i(Y) - \sum_i \dim \pi_{P,i}(Y).$$

In the case that the image of the projections $\pi_{P,i}$ have the maximal possible dimension $\dim \pi_{P,i}(Y) = \dim \mathcal{B}_e - 1$,

$$\dim Y = \dim \mathcal{B}_e + \#\{i | T_i = T_{i+1}\} + 1$$

Proof. The first part of the theorem is . For the second part, throughout our proof we will let Y_n and Y_n^* be the image of Y under the maps

$$(R_{s_{\alpha_1}, s_{\alpha_2}, \ldots s_{\alpha_r}})_e \to (R_{s_{\alpha_{n+1}}, s_{\alpha_{n+2}}, \ldots s_{\alpha_r}})_e$$
$$(R_{s_{\alpha_1}, s_{\alpha_2}, \ldots s_{\alpha_r}})_e \to (R_{s_{\alpha_1}, s_{\alpha_2}, \ldots s_{\alpha_{n-2}}})_e$$

respectively. Notably, Y_n, Y_n^* are irreducible. Also let X_i be the unique irreducible component containing the preimage of $\pi_{P,i}(Y)$ in $(G/B)_e$.

First, we claim that $\dim \pi_1(Y) = \dim \pi_r(Y) = \dim \mathcal{B}_e$. We will prove $\dim \pi_1(Y) = \dim \mathcal{B}_e$ as the proof of $\dim \pi_r(Y) = \dim \mathcal{B}_e$ is analogous. From Lemma 2.6(ii),

$$X_1 \times_{G/P_{\alpha_1}} Y_1 \cap (R_{s_{\alpha_1}, s_{\alpha_2}, \ldots s_{\alpha_r}})_e$$

is an $\mathbb{A}^1$-bundle over Y_1 and hence irreducible. Since Y is contained in $X_1 \times_{G/P_{\alpha_i}} Y_1$ and Y is maximal among irreducible sets in $R_{s_{\alpha_1}, s_{\alpha_2}, \ldots s_{\alpha_r}}$, $\pi_1(Y)$ is dense in X_1 and has dimension $\mathcal{B}_e$.

Second, we claim that for $i \neq 1, r$,

$$\dim \pi_i(Y) = \begin{cases} \dim \mathcal{B}_e & \text{if } T_{i-1} = T_i, \\ \dim \mathcal{B}_e - 1 & \text{if } T_{i-1} \neq T_i. \end{cases}$$

For the first case, for any point $p \in Y$, $\pi_{i-1}(p)$ is contained in a line of type α_{i-1} which is contained in $T_{i-1} = T_i$ and similarly $\pi_{i+1}(p)$ is contained in a line of type α_i which is contained in $T_i = T_{i-1}$. Since $\alpha_{i-1} \neq \alpha_i$ ($s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_r}$ is a reduced expression), for every $x \in \pi_{i-1}(Y)$ and $y \in \pi_i(Y)$, there exists a sequence of lines of type α_{i-1}, α_i which connected x to y. Therefore

$$Y_i^* \times_{G/P_{\alpha_{i-1}}} X_i \times_{G/P_{\alpha_i}} Y_i \cong Y_i^* \times Y_i.$$

Now from lemma 2.6,

$$Y_i^* \times_{G/P_{\alpha_{i-1}}} X_i \times_{G/P_{\alpha_i}} Y_i \cap (R_{s_{\alpha_1}, s_{\alpha_2}, \ldots s_{\alpha_r}})_e$$

is a dense open subset of

$$Y_i^* \times_{G/P_{\alpha_{i-1}}} X_i \times_{G/P_{\alpha_i}} Y_i$$

and therefore irreducible. Since Y is contained in $Y_i^* \times_{G/P_{\alpha_{i-1}}} X_i \times_{G/P_{\alpha_i}} Y_i$ and Y is maximal among irreducible sets in $(R_{s_{\alpha_1}, s_{\alpha_2}, \ldots s_{\alpha_r}})_e$, $\pi_i(Y)$ is dense in X_i and has dimension $\mathcal{B}_e$.

For the second case, since the intersection of any two distinct irreducible components of $\mathcal{B}_e$ have strictly smaller dimension than that of $\mathcal{B}_e$, $\dim \pi_i(Y) \leq \dim \mathcal{B}_e - 1$. Furthermore since $\pi_i(Y)$ surjects onto $\pi_{P,i-1}(Y)$ and $\pi_{P,i}(Y)$ which both have dimension $\mathcal{B}_e - 1$, we must have $\dim \pi_i(Y) = \dim \mathcal{B}_e - 1$.

Plugging both our claims into equation 5.3, we find

$$\dim Y = r \dim \mathcal{B}_e - r + 2 + \#\{i | T_i = T_{i+1}\} - (r-1)(\dim \mathcal{B}_e - 1) = \dim \mathcal{B}_e + \#\{i | T_i = T_{i+1}\} + 1.$$

□

One of the interesting things about computing the dimension of an irreducible component Y of $Y_w \cap \mathcal{B}_e \times \mathcal{B}_e$ is that you can choose which reduced expression $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_r}$ to use. It would be very nice if for every irreducible component Y of $Y_w \cap \mathcal{B}_e \times \mathcal{B}_e$, one could find a reduced expression $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_r}$ such that the irreducible component of $(R_{s_{\alpha_1}, s_{\alpha_2}, \dots s_{\alpha_r}})_e$ corresponding to Y satisfies the conditions in the above proposition. I have not found found a counterexample to this; however, I have found counterexamples to slight strengthenings of this statement. I am very interested in whether this happens to be true or not.

If for every irreducible component X of $\mathcal{B}_e$, the unique irreducible component of $Y_{w,X,X} \cap \mathcal{B}_e \times \mathcal{B}_e$ satisfies the hypothesis of the proposition above for some reduced expression $w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_r}$, then we can prove conjecture 1.4:

Proof of conjecture 1.4 assuming hypothesis discussed above. The if direction is covered at the end of §4. For the only if direction:
Suppose that $w_{X,X}$ (which is an involution) does not meet condition (2) in our conjecture i.e. there exists some s_{α_i} in our reduced expression for w such that $\ell(s_{\alpha_i} w) > \ell(w)$. Then by 3.14, X is not a s_{α_i} variety. From the RSK correspondence, there is an irreducible component Y of $Y_{w_{X,X}} \cap \mathcal{B}_e \times \mathcal{B}_e$ which is a dense subset of $X \times X$. Then as in the proof of proposition **??**, there exists a unique irreducible component Z of $\mathcal{B}_e$ which contains the lines of type s_{α_i} in the RSK paths associated to Y. This component is an s_{α_i} component and thus is not X. □

6 Investigating the Weyl Group Condition in Conjecture 1.4

Here we combinatorially investigate the conditions in conjecture 1.4. To cut down on page turns, I have included the statement here:

Conjecture 6.1. *For an irreducible component $X \subset \mathcal{B}_e$, every RSK-path with start and endpoints in X stays in X if and only if under the geometric RSK correspondence, the pair (X, X) corresponds to an element $w \in W$ such that*

1. *w is an involution*

2. *For every simple reflection s which appears in a reduced expression for $w_{X,X}$,*

$$\ell(s w_{X,X}) < \ell(w_{X,X}).$$

6.1 Type A

In this case, we identify the Weyl group with S_n and its simple reflections with the simple transpositions $(i, i+1)$.

Lemma 6.2. *The following are equivalence for $\sigma \in S_n$:*

1. *σ avoids the pattern 231 and the pattern 312*

2. *σ written in two-line notation is*

$$\begin{pmatrix} 1 & 2 & \dots & n_1 & n_1+1 & n_1+2 & \dots & n_2 & n_2+1 & \dots \\ n_1 & n_1-1 & \dots & 1 & n_2 & n_2-1 & \dots & n_1+1 & n_3 & \dots \end{pmatrix}$$

3. *σ corresponds under RSK to (T, T) where T is a tableaux such that for any x such that $T^{-1}(x)$ is not in the first row,*

$$row(T^{-1}(x-1)) = row(T^{-1}(x)) - 1.$$

Proof. $(i \Rightarrow ii)$ Suppose that $\sigma(1) = n_1$, then I claim that for $1 \leq i \leq n_1$, $\sigma(i) = n_1 + 1 - i$. The proof is by induction on i with the base case of $i = 1$ already given. For the inductive step assume that $\sigma(i) = n_1 + 1 - i$ for all $j \leq i$. If $\sigma(i+1) < n_1 - i$ then σ does not avoid 312 as $\sigma^{-1}(n_1 - i) > i + 1$. If $\sigma(i+1) > n_1 - i$, then then σ does not avoid 231 as $\sigma^{-1}(n_1 - i) > i + 1$. This induction argument can then be used on the intervals $(n_1 + 1, \sigma(n_1 + 1)), (\sigma(n_1 + 1) + 1, \sigma(\sigma(n_1 + 1) + 1)) \ldots$ and so on. This proves $(i \Rightarrow ii)$.
$(ii \Rightarrow iii)$ Clear from the row insertion algorithm and the fact that w is an involution.
$(iii \Rightarrow i)$ From lemma 2.10, we see that if the $row(T^{-1}(x)) > 1$, then $\sigma^{-1}(x) < \sigma^{-1}(x-1)$. On the other hand if $row(T^{-1}(x)) = 1$, then by lemma 2.10 $\sigma^{-1}(x-1) > \sigma^{-1}(x)$. Combining these two imply that σ^{-1} avoids 231. Since σ corresponds to (T, T) it is an involution. Thus σ also avoids 312 since $(123)^{-1} = (132)$. ☐

Remark 6.3. *As stated in the introduction, this shows the alternate statement of our conjecture in type A which says that any irreducible component X which satisfies the condition that every RSK-path with starting point and ending point in X stays in X under RSK corresponds to a tableau T which can be constructed by left-justifying a sequence of columns together as the following picture shows*

$$\begin{array}{|c|c|c|c|} \hline 1 & 3 & 7 & 8 \\ \hline 2 & 4 & & 9 \\ \hline & 5 & & 10 \\ \cline{2-2}\cline{4-4} & 6 & & \\ \cline{2-2} \end{array} \Longrightarrow \begin{array}{|c|c|c|c|} \hline 1 & 3 & 7 & 8 \\ \hline 2 & 4 & 9 \\ \cline{1-3} 5 & 10 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}.$$

Lemma 6.4. *$(i, i+1)$ is in every reduced expression for σ if and only if either*

1. *There exists $a > i$ such that $\sigma(a) < \sigma(i)$ or*

2. *There exists $a < i$ such that $\sigma(a) > \sigma(i+1)$*

Proof. The condition $\sigma(x) \geq i$ is invariant under all simple transpositions except $(i, i+1)$. The condition $\sigma(x) \leq i+1$ is invariant under all simple transpositions except $(i, i+1)$. ☐

Theorem 6.5. *An element $\sigma \in S_n$ satisfies for every simple transposition s_i in a reduced expression for σ,*

$$\ell(s_i\sigma) < \ell(\sigma)$$

if and only if σ is 231-avoiding and 312-avoiding.

Proof. The σ avoids 231 and 312 case follows from condition (ii) in lemma 6.2 and lemma 6.4.
If σ does not avoid 231, then for some $a < b < c$, $\sigma(c) < \sigma(a) < \sigma(b)$. Since $\sigma(c) < \sigma(a)$, for all i in the range $a \leq i < c$, each simple transposition $(i, i+1)$ is in every reduced expression for σ. Since $\sigma(a) < \sigma(b)$ there exists some i in the range $a \leq i < b < c$, such that $\sigma(i) < \sigma(i+1)$. For such an i, $\ell((i, i+1)\sigma) > \ell(\sigma)$.
If σ does not avoid 312, then for some $a < b < c$, $\sigma(b) < \sigma(c) < \sigma(a)$. Since $\sigma(c) < \sigma(a)$, for all i in the range $a \leq i < c$, each simple transposition $(i, i+1)$ is in every reduced expression for σ. Since $\sigma(b) < \sigma(c)$ there exists some i in the range $a < b \leq i < c$, such that $\sigma(i) < \sigma(i+1)$. For such an i, $\ell((i, i+1)\sigma) > \ell(\sigma)$. ☐

6.2 Types B and C

In this case, we identify the Weyl group with the hyperoctahedral group: i.e. the subgroup of permuations σ of the $\{-n, -n+1 \ldots -1, 1, 2 \ldots n\}$ such that $\sigma(x) = -\sigma(-x)$. We identify the simple reflections as the set

$$\{(-i-1, -i)(i, i+1) | i > 0\} \cup \{(1, -1)\}.$$

We say a simple reflection s *increases* i if $s(i) > i$. We let s_i denote the unique simple transposition that increases i for $-n \leq i < n$.

Lemma 6.6. *s_i is in every reduced expression for σ if and only if either*

1. *There exists $a > i$ such that $\sigma(a) < \sigma(i)$ or*
2. *There exists $a < i$ such that $\sigma(a) > \sigma(i+1)$*

The proof of this lemma is analogous to that of lemma 6.4. As a corollary of the description of length in the hyperoctahedral group as

$$\ell(\sigma) = \frac{\#\{\text{inversions of } \sigma\} + \#\{i|\sigma(i) > 0, i < 0\}}{2}$$

(see [Inc03]), we find

Lemma 6.7. *$\ell(s_i\sigma) < \ell(\sigma)$ if and only if $\sigma(-1) > \sigma(1)$ if $i = -1$ or $\sigma(i) > \sigma(i+1)$ if $i \neq -1$.*

Theorem 6.8. *An element $\sigma \in W$ satisfies for every simple reflection s in a reduced expression for σ,*

$$\ell(s\sigma) < \ell(\sigma)$$

if and only if σ is 231-avoiding and 312-avoiding.

The proof is almost identical to that in type A with lemma 6.6 replacing lemma 6.4 and again using condition (ii) in lemma 6.2 for the if case.

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