Abstract. Given a collection of $\alpha$-dimensional $\delta$-tubes and $\beta$-dimensional $\delta$-balls in the plane, we consider the problem of finding the maximum number of incidences, or pairs $(t,b)$ of tubes and balls such that $b$ intersects $t$. We find and prove an essentially sharp estimate for this maximum number in terms of $\alpha$ and $\beta$. For the upper bound, we use a combinatorial argument for small $\alpha, \beta$ and a Fourier analytic argument for large $\alpha, \beta$.

1. Introduction

Let $\delta \ll 1$ be a small parameter. We will work with $\delta$-tubes and $\delta$-balls on the plane. A $\delta$-ball is a ball of radius $\delta$. A $\delta$-tube is a $\delta \times 1$ rectangle. The direction of a rectangle is the vector pointing in the direction of its longest side. (This vector is only determined up to $\pm 1$.) Throughout this paper, a ball will refer to a $\delta$-ball and a tube will refer to a $\delta$-tube unless otherwise stated.

Two balls $p$ and $q$ are essentially distinct if $|p \cap q| \leq \frac{1}{2}|p|$. Likewise, two tubes $s$ and $t$ are essentially distinct if $|s \cap t| \leq \frac{1}{2}|s|$. From now on, we assume all of our sets of balls and tubes are essentially distinct.

Definition 1. Let $P$ be a set of $\delta$-balls and $T$ be a set of $\delta$-tubes. The number of incidences $I(P,T)$ is the number of pairs $(p,t)$ of balls $p \in P$ and tubes $t \in T$ such that $|p \cap t| \neq \emptyset$.

The basic problem we will consider is the following:

Problem 1. Given a set of balls $P$ and a set of tubes $T$ in the square $[0,2]^2$, what is the maximum number of incidences $I(P,T)$?

We will impose a spacing condition on the set of balls and the set of tubes. The dimension condition on the balls appeared in Katz and Tao [2].

Definition 2. A set of essentially distinct balls $P$ in $[0,2]^2$ is at most $\beta$-dimensional if for every $w \in [\delta,1]$ and every ball $B_w$ of radius $w$,

$$\#\{p \in P \mid p \subset B_w\} \leq 100 \left(\frac{w}{\delta}\right)^\beta.$$

We say $P$ is $\beta$-dimensional if it is at most $\beta$-dimensional and $|P| \geq 0.1 \left(\frac{w}{\delta}\right)^\beta$.

We will impose an analogous condition on the set of tubes.
Definition 3. A set of essentially distinct tubes $T$ in $[0, 2]^2$ is at most $\alpha$-dimensional if for every $w \in [\delta, 1]$ and every $w \times 2$ tube $T_w$,

$$\#\{t \in T \mid t \subset T_w\} \leq 100 \left(\frac{w}{\delta}\right)^{\alpha}.$$  

We say $T$ is $\alpha$-dimensional if it is at most $\alpha$-dimensional and $|T| \geq 0.1 \left(\frac{w}{\delta}\right)^{\alpha}$.

The exact values of the constants are not important, and when describing the constructed examples in Section 2, we will replace 100 and 0.1 by appropriate constants. We also remark that the dimension of an essentially distinct set of balls is a real number between 0 and 2, and likewise for a set of tubes.

We can rephrase the problem as follows: given a set of $\beta$-dimensional balls $P$ and a set of $\alpha$-dimensional tubes $T$, what is the maximum number of incidences $I(P, T)$?

Guth et. al. in [1] consider a few related problems with well-spaced tubes. They fix a parameter $1 \leq W \leq \delta^{-1}$ and choose $T$ to be a collection of $W^2$ tubes, such that every $W^{-1} \times 1$ rectangle contains at most one tube in $T$. Well-spaced tubes thus loosely correspond to the zero-dimensional tubes setting. Guth et. al. were able to prove sharp incidence estimates for their well-spaced tubes.

To resolve our question, we will prove the following main theorem:

**Theorem 1.** Suppose $\alpha, \beta$ satisfy $0 \leq \alpha, \beta \leq 2$. For every $\varepsilon > 0$, there exists $C_\varepsilon = C(\varepsilon, \alpha, \beta)$ with the following property: for every $\beta$-dimensional set of balls $P$ and $\alpha$-dimensional set of tubes $T$, the following bound holds:

$$I(P, T) \leq C_\varepsilon \delta^{-f(\alpha, \beta)-\varepsilon},$$

where $f(\alpha, \beta)$ is defined as in Figure 1. These bounds are sharp up to $\delta^\varepsilon$.

![Figure 1. The answer](image-url)
In Section 2, we will show the estimates in Theorem 1 are sharp up to \( \delta \), by constructing suitable examples.

We will then prove the upper bound of Theorem 1 by analyzing different cases for \( \alpha, \beta \). In Section 3, we will use a combinatorial argument to resolve the case where \( \alpha \leq 1 \) or \( \beta \leq 1 \). In Section 4, we will use a Fourier analytic argument based on induction to resolve the case where \( \alpha, \beta \geq 1 \). In Section 5, we will complete the proof of Theorem 1.

Notation. We will use \( A \gtrsim B \) to represent \( A \geq CB \) for a constant \( C \), and \( A \lesssim B \) to represent \( A \leq CB \). The constant \( C \) might depend on the dimensions \( \alpha \) or \( \beta \). We will use \( A \sim B \) to represent \( A \gtrsim B \) and \( A \lesssim B \).

The angle between two tubes \( s \) and \( t \), or \( \angle(s,t) \), is the acute angle between their directions.

2. Constructions

We divide the constructions into four cases. Case 1 is the main construction that works for most \( \alpha \) and \( \beta \). Cases 2, 3, 4 can be considered as auxiliary constructions which take care of exceptional values of \( \alpha, \beta \) not covered in Case 1. The constructions will all take place inside a \( 1 \times 1 \) square.

The correctness of the constructions depends on the following angle separation lemma.

**Lemma 1.** Let \( c \geq 1 \) be a constant. If \( R \) is a \( w \times c \) rectangle and \( t_1, t_2 \) are tubes inside \( R \), then \( \angle(t_1, t_2) \lesssim w \). This bound is sharp.

**Proof.** Let the direction of \( w \) be horizontal. The maximum possible angle of a tube from horizontal is \( \sin^{-1} w \lesssim w \), so the maximum possible angle between two tubes is \( \lesssim w \). The bound is sharp as witnessed by Figure 2.

\[ \Box \]

2.1. **Case 1.** In this case, we assume
\[ (1) \quad \alpha < \beta + 1, \beta < \alpha + 1, \text{ and } \alpha + \beta < 3. \]

Let \( D = \delta^{-1} \), and refer to Figure 3. The left picture depicts a single bundle with \( \sim D^{(1-\gamma)a} \) tubes and \( \sim D^{\gamma b} \) balls. The tubes are rotates of a single central tube \( t \), and the angle spacing between tubes is \( \sim \delta^{\gamma + (1-\gamma)a} \). The balls are contained in a \( \delta \times \delta^{1-\gamma} \) rectangle with the same center and direction as the central tube \( t \), and they are spaced distance \( \sim \delta^{1-\gamma + \gamma b} \) apart. Here, \( a = \min(\alpha, 1), b = \min(\beta, 1), \gamma = a - \alpha + b \). We will later prove that \( 0 \leq \gamma \leq 1 \).

Some observations are in order.

- The maximum angle separation between two tubes is \( \sim \delta^{\gamma} \). Thus, the entire bundle fits inside a \( \delta^{\gamma} \times 1 \) rectangle by Lemma 1.
• Let $O$ be the center of the central tube $t$. The maximum distance of a ball $p$ from $O$ is $\sim \delta^{1-\gamma}$. For any tube $s$ in the bundle, we have $\angle(s, t) \lesssim \delta^\gamma$ (where $t$ is the central tube). Thus, a trigonometric argument shows that $s$ intersects $p$. In other words, every ball lies in every tube of the bundle.

In the right picture, there are $\sim D^\kappa$ bundles. The bundles are arranged in a $D^{(1-\lambda)\kappa} \times D^\lambda\kappa$ grid, with the horizontal spacing $\delta^{(1-\lambda)\kappa}$ and the vertical spacing $\delta^{\lambda\kappa}$. The bundles in the same row are translates of each other; two adjacent bundles in the same column are $\delta^{\lambda\kappa}$ rotates of each other. Here, $0 \leq \lambda \leq 1$ is a parameter and $\kappa = \alpha - (1-\gamma)a = \frac{a\beta + b\alpha - ab}{a+b}$. We will later prove $0 \leq \kappa < 1$.

The right picture thus has $\sim D^{(1-\gamma)a+\kappa} = D^a$ tubes and $\sim D^{\gamma b+\kappa} = D^\beta$ balls.

$\lambda$ is a parameter satisfying the defining condition

- $(1-\gamma)a(a+1-a) + \max(\lambda, 1-\lambda)\kappa \leq a$;
- $(1-\gamma)b(b+1-\beta) + \max(\lambda, 1-\lambda)\kappa \leq b$;
- $\gamma + (1-\gamma)\min(a, b) \geq \max(\lambda, 1-\lambda)$;
- $0 \leq \lambda \leq \min(\gamma, 1-\gamma) \leq 1$.

We will show later that there exists a valid choice for $\lambda$.

The prototypical example is $\alpha = 1, \beta = 1$, in which $\gamma = \kappa = 0.5$. If we then choose $\lambda = 0$, then we get a series of $D^{0.5}$ horizontally spaced, parallel bundles, as in Figure 4.

We will show five facts:

1. $0 \leq \gamma \leq 1$.
2. $0 \leq \kappa \leq \min(a, b)$ and $\kappa < 1$.
3. The tubes are $\alpha$-dimensional.
Figure 4. The special case $\alpha = \beta = 1$.

(4) The tubes are $\beta$-dimensional.

(5) $I(P, T) = D^{\alpha + \beta + \alpha \beta} / \gamma \kappa$.

These facts allow us to verify $\lambda = \min(\gamma, 1 - \gamma)$ satisfies the defining condition. We perform the following computation using Facts 1-2 and $(a - 1)(a - \alpha) = 0$:

\[
(1 - \gamma)a(a + 1 - \alpha) + \gamma \kappa \leq a(1 - \gamma) + \gamma a = a,
\]

\[
(1 - \gamma)(a(a + 1 - \alpha) + (\alpha - (1 - \gamma)a)) = (1 - \gamma)(a(a - \alpha) + \alpha + \gamma a)
\]

\[
= (1 - \gamma)(a - \alpha + \alpha + \gamma a) = (1 - \gamma)(1 + \gamma)a \leq a.
\]

\[
\gamma + (1 - \gamma) \min(a, b) \geq \gamma \geq \gamma \kappa,
\]

\[
\gamma + (1 - \gamma) \min(a, b) \geq (1 - \gamma) \min(a, b) \geq (1 - \gamma) \kappa,
\]

To show Facts 1-2, we look at the following table.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\leq 1$</td>
<td>$\leq 1$</td>
<td>$\beta$</td>
<td>$\alpha \beta$</td>
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<tr>
<td>$\leq 1$</td>
<td>$\geq 1$</td>
<td>$\alpha + \beta$</td>
<td>$\alpha + \beta$</td>
</tr>
<tr>
<td>$\geq 1$</td>
<td>$\leq 1$</td>
<td>$1 - \alpha + \beta$</td>
<td>$1 + \beta$</td>
</tr>
<tr>
<td>$\geq 1$</td>
<td>$\geq 1$</td>
<td>$1 - \alpha + \beta$</td>
<td>$\alpha + \beta - 1$</td>
</tr>
</tbody>
</table>

From these and the conditions in equation (1), we can easily show Facts 1 and 2. Now, we will verify the tubes are $\alpha$-dimensional. Fix $w$ and a $w \times 2$ rectangle $\mathcal{R}$; we will count how many tubes are in $\mathcal{R}$. We find three main contributions to our count:

- $\mathcal{R}$ can intersect bundles of $\lesssim \left\lceil \frac{w}{\gamma \kappa} \right\rceil$ different directions (Lemma 1 and $\gamma > \lambda \kappa$).
- For each direction, $\mathcal{R}$ can intersect $\lesssim \left\lceil \frac{w}{\delta^{1/2} \lambda \kappa} \right\rceil$ bundles in that direction.
For each bundle, $\mathcal{R}$ can contain $\lesssim \min\left(\frac{w}{\delta^{\gamma + (1-\gamma)a}}, D^{(1-\gamma)a}\right)$ tubes. (Lemma 1)

Thus, $\mathcal{R}$ contains at most $n$ tubes, for
\[ n \lesssim \min\left(\frac{w}{\delta^{\gamma + (1-\gamma)a}}, D^{(1-\gamma)a}\right) \cdot \left(\frac{w}{\delta^{1-\lambda}\kappa} + 1\right) \left(\frac{w}{\delta^{3\lambda}\kappa} + 1\right). \]

Suppose $w < \delta^{\gamma + (1-\gamma)a}$. From property of $\lambda$, we get $w < \delta^{\lambda\kappa}$ and $w < \delta^{(1-\lambda)\kappa}$. Hence,
\[ n \lesssim 2 \cdot 2 = 8 < 8 \left(\frac{w}{\delta}\right)^a. \]

Thus we may assume $w > \delta^{\gamma + (1-\gamma)a}$. In this case, we can use $|x| \leq x + 1 \leq 2x$ to write
\[ n \lesssim \min\left(\frac{w}{\delta^{\gamma + (1-\gamma)a}}, D^{(1-\gamma)a}\right) \cdot \left(\frac{w}{\delta^{1-\lambda}\kappa} + 1\right) \left(\frac{w}{\delta^{3\lambda}\kappa} + 1\right). \]

We expand the product and bound each term separately. Let $m = \min\left(\frac{w}{\delta^{\gamma + (1-\gamma)a}}, D^{(1-\gamma)a}\right)$. We will use the fact $\min(x, y) \leq xy^{-\gamma}$ for any $0 \leq c \leq 1$.

\begin{align*}
(2) & \quad m \leq \left(\frac{w}{\delta^{\gamma + (1-\gamma)a}}\right)^a \left(D^{(1-\gamma)a}\right)^{1-a} = \left(\frac{w}{\delta}\right)^a \leq \left(\frac{w}{\delta}\right)^a. \\
(3) & \quad m \cdot \frac{w}{\delta^{(1-\lambda)\kappa}} \leq \left(\frac{w}{\delta}\right)^a \left(D^{(1-\gamma)a}\right)^{1-a} \cdot \frac{w}{\delta^{(1-\lambda)\kappa}} \leq \frac{w^{a-1}}{\delta^a} \leq \left(\frac{w}{\delta}\right)^a, \\
(4) & \quad m \cdot \frac{w}{\delta^{3\lambda\kappa}} \leq \left(\frac{w}{\delta}\right)^a \left(D^{(1-\gamma)a}\right)^{1-a} \cdot \frac{w}{\delta^{3\lambda\kappa}} \leq \frac{w^{a-1}}{\delta^a} \leq \left(\frac{w}{\delta}\right)^a, \\
(5) & \quad m \cdot \frac{w}{\delta^{(1-\lambda)\kappa}} \cdot \frac{w}{\delta^{3\lambda\kappa}} \leq D^{(1-\gamma)a} \cdot \frac{w}{\delta^{3\lambda\kappa}} \cdot \frac{w}{\delta^{(1-\lambda)\kappa}} = \frac{w^2}{\delta^a} \leq \left(\frac{w}{\delta}\right)^a. 
\end{align*}

Equation (2) follows from $a \leq \alpha$, $\frac{w}{\delta} \geq 1$, and
\[ a(\gamma + (1-\gamma)a) + (1-\gamma)a(1-a) = 1. \]

Equation (5) follows from the definition $\kappa = \alpha - (1-\gamma)a$. To justify (3) and (4), we use the defining relation of $\lambda$. Hence, we get $n \lesssim \left(\frac{w}{\delta}\right)^a$, as desired. On the other hand, we saw earlier that there are $\sim D^a$ tubes in $T$. Finally, to verify the tubes are essentially distinct, we observe that if $s$ and $t$ are not parallel, then they are $\delta$-separated (since $1 \geq \gamma > \lambda\kappa$); otherwise, they are separated by a translation of at least $\delta^{(1-\lambda)\kappa} \geq \delta$. Thus, the tubes are $\alpha$-dimensional.

Now, we verify the balls are $\beta$-dimensional. Fix $w$ and a ball $\mathcal{B}$ of radius $w$; we will count how many balls are in $\mathcal{B}$. There are three main contributions.

- $\mathcal{B}$ can intersect bundles of $\lesssim \left[\frac{w}{\delta^{\gamma + (1-\gamma)a}}\right]$ different directions.
- For each direction, $\mathcal{B}$ can intersect $\lesssim \left[\frac{w}{\delta^{(1-\lambda)\kappa}}\right]$ bundles in that direction.
- For each bundle, $\mathcal{B}$ can contain $\lesssim \min\left(\left[\frac{w}{\delta^{(1-\lambda)\kappa}}\right], D^{\beta}\right)$ balls.

Thus, $\mathcal{B}$ contains at most $n$ balls, for
\[ n \lesssim \min\left(\left[\frac{w}{\delta^{1-\gamma+\gamma b}}\right], D^{\beta}\right) \cdot \left(\frac{w}{\delta^{(1-\lambda)\kappa}} + 1\right) \left(\frac{w}{\delta^{3\lambda\kappa}} + 1\right). \]

Using a similar method to the tubes case, we get $n \lesssim \left(\frac{w}{\delta}\right)^\beta$, as desired. On the other hand, we saw earlier that there are $\sim D^\beta$ tubes in $P$. Finally, to verify the balls are essentially distinct, we observe that if $p$ and $q$ are not in the same bundle, then they are separated by a translation of at least $\delta^{\max(\lambda, (1-\lambda)\kappa)} \geq \delta$; and two balls in the same bundle are $\delta$-separated. Thus, the balls are $\beta$-dimensional.
Finally, since there are $\sim D^\alpha$ tubes, and each tube contains $D^\gamma$ balls, we get
\[ I(P, T) \sim D^\alpha D^\gamma = D^{a\alpha + b\beta + ab}. \]

2.2. Case 2. For this case, we will assume $\alpha \geq \beta + 1$.

Refer to Figure 5. In each bundle, there are $\sim D$ tubes, each separated by angle $\delta$. By Lemma 1, we can fit the bundle inside a $\frac{1}{4} \times 1$ rectangle. We arrange $\sim D^\alpha$ bundles as in the right figure, separated by distance $\sim \delta^{\alpha - 1}$, such that the centers of the bundles lie within a segment of length $\frac{1}{2}$ centered at the unit square’s center. Then, we place $D^\beta$ balls at some of the centers of the bundles, such that the balls are $\delta^\beta$-separated. Thus, there are $D^\alpha$ tubes and $D^\beta$ balls.

We will show the tubes are $\alpha$-dimensional and the balls are $\beta$-dimensional.

Fix $w$ and a $w \times 2$ rectangle $R$; we will count how many tubes are in $R$. There are two main contributions.

- $R$ can intersect $\lesssim \left\lceil \frac{w}{\delta^\alpha} \right\rceil$ bundles.
- For each bundle, $R$ can contain $\lesssim \frac{w}{\delta}$ tubes.

Thus, $R$ contains at most $n$ tubes, where
\[ n \lesssim \left( \frac{w}{\delta^\alpha - 1} + 1 \right) \cdot \frac{w}{\delta} \leq \left( \frac{w}{\delta} \right)^\alpha + \left( \frac{w}{\delta} \right) = 2 \cdot \left( \frac{w}{\delta} \right)^\alpha. \]

The tubes are essentially distinct because non-parallel tubes are $\delta$-angle separated and parallel-tubes are $\delta$-distance separated. Thus, the tubes are $\alpha$-dimensional.

Now, we verify the balls are $\beta$-dimensional. Fix $w$ and a ball $B$ of radius $w$; we will count how many balls are in $B$. Note that $B$ can intersect at most $n$ balls, where
\[ n \lesssim \left\lceil \frac{w}{\delta^\beta} \right\rceil \leq 2 \left( \frac{w}{\delta} \right)^\beta. \]
The balls are essentially distinct, so they are $\beta$-dimensional. Finally, each ball lies in $\sim D$ tubes, so $I(P,T) \sim D^\beta \cdot D = D^\beta + 1$.

2.3. **Case 3.** For this case, we will assume $\beta \geq \alpha + 1$.

2.4. **Case 4.** For this case, we will assume $\alpha + \beta \geq 3$. 

Refer to Figure 6. There are $D^\beta - 1$ columns of $D$ balls each. On $D^\alpha$ of the columns, there is a tube. The tube-containing columns are separated by distance $\delta^\alpha$. Thus, there are $D^\beta$ balls and $D^\alpha$ tubes.

The tubes are $\alpha$-dimensional and the balls are $\beta$-dimensional by a similar argument to Case 2. Finally, each tube contains $D$ balls, so $I(P,T) = D^\alpha \cdot D = D^{\alpha+1}$.

Refer to Figure 7. Construction for Case 4.
Refer to Figure [7]. The bundles of tubes are the same as Case 2. We then arrange \( \sim D^{\beta} \) balls in a \( D^{\beta/2} \times D^{\beta/2} \) grid, such that adjacent balls are separated by distance \( \sim \delta^{\beta/2} \). We confine the balls to a \( \frac{1}{2} \times \frac{1}{2} \) square concentric with the large \( 1 \times 1 \) square.

From Case 2, the tubes are \( \alpha \)-dimensional. We now show the balls are \( \beta \)-dimensional.

Fix \( w \) and a ball \( \mathcal{B} \) of radius \( w \); we will count how many balls are in \( \mathcal{B} \). Note that \( \mathcal{B} \) can intersect at most \( n \) balls, where

\[
n \lesssim \left( \frac{w}{\delta^{\beta/2}} \right)^2 \leq \left( \frac{w}{\delta^{\beta/2}} + 1 \right)^2 \leq 2 \left( \frac{w^2}{\delta^\beta} + 1 \right) \leq 4 \left( \frac{w}{\delta} \right)^\beta.
\]

(Here, we used \([x] \leq x + 1\) and Cauchy-Schwarz.) The balls are essentially distinct, so they are \( \beta \)-dimensional.

Finally, we will count the number of incidences. For a bundle centered at \( O \), the tubes in the bundle cover a double cone with apex \( O \) and angle \( \frac{\pi}{2} \). This double cone contains a positive fraction of all balls. Hence, the number of incidences per bundle is \( \gtrsim D^\beta \). There are \( D^{\alpha-1} \) bundles, so \( I(P, \mathcal{T}) \gtrsim D^{\beta}D^{\alpha-1} = D^{\alpha+\beta-1} \).

3. Combinatorial upper bound

We will first prove the upper bound for \( \alpha \leq 1 \) or \( \beta \leq 1 \). We will further casework on whether \( \alpha \leq \beta \) or \( \alpha \geq \beta \), which we call the balls-dominated and tubes-dominated case, respectively.

Notation. For a tube \( t \) and ball \( p \), we use \( p \in t \) to denote \( p \cap t \neq \emptyset \). It does not mean \( p \subset t \), which means \( p \) is contained in \( t \).

3.1. Intersection Lemma. We will need the following intersection lemma, illustrated in Figure [8]

**Lemma 2.** Let \( \delta \leq w \leq \frac{\alpha}{w} \).

(a) The intersection of two tubes separated by angle \( w \) is contained in a rhombus with side length \( \sim \frac{\delta}{w} \) and angle \( w \).

(b) Let \( t \) be a tube. The set of all tubes \( s \) such that \( s \cap t \neq \emptyset \) and \( \angle(s, t) \leq w \) is contained in a tube with dimensions \( \sim w \times 3 \).

(c) Let \( \mathcal{T} \) be an \( \alpha \)-dimensional set of tubes, and let \( t \in \mathcal{T} \). The number of tubes \( s \in \mathcal{T} \) such that \( s \cap t \neq \emptyset \) and \( \angle(s, t) \leq w \) is \( \lesssim \left( \frac{w}{\delta} \right)^\alpha \).

(d) Let \( p \) and \( q \) be two balls whose centers are distance \( w \) apart, and let \( \mathcal{T} \) be an \( \alpha \)-dimensional set of tubes. If \( s \) and \( t \) are any two tubes that intersect both \( p \) and \( q \), then \( \angle(s, t) \leq \frac{\delta}{w} \). The number of tubes in \( \mathcal{T} \) that intersect both \( p \) and \( q \) is \( \lesssim \left( \frac{w}{\delta} \right)^{\min(\alpha, 1)} \).

The proof of (a) relies on basic trigonometry and the approximation \( \sin \alpha \sim \alpha \), which holds if \( 0 < \alpha \leq \frac{\pi}{4} \). Part (b) follows from trigonometry.

For part (c), we invoke part (b) and cover the \( \sim w \times 3 \) tube with two \( \sim w \times 2 \) tubes, spaced 1 apart. Each \( \delta \)-tube must be contained in one of the \( \sim w \times 1 \) tubes. Now use dimension.

The first claim of part (d) follows from part (a). The second claim follows from part (c) if \( \alpha \leq 1 \).

Suppose \( \alpha > 1 \), and let \( A \) be the set of tubes in \( \mathcal{T} \) that intersect both \( p \) and \( q \). Then the set of directions of the tubes in \( A \) is contained inside an arc of length \( \sim \frac{\delta}{w} \). We can split this arc into \( \sim \frac{1}{w} \) sub-arcs of length \( \delta \). Within each \( \delta \)-arc, we
can fit $\lesssim 1$ essentially distinct tubes that intersect $p$. Thus, $A$ can contain $\lesssim \frac{1}{w}$ essentially distinct tubes.

3.2. Balls dominated.

**Theorem 2.** Fix $\varepsilon > 0$. Let $P$ be a set of $\beta$-dimensional $\delta$-balls and $T$ be a set of $\alpha$-dimensional $\delta$-tubes. Let $D = \delta^{-1}$. Let $b = \min(\beta, 1)$, and assume $b \geq \alpha$. Then for any $\varepsilon > 0$, there exists $C_\varepsilon$ such that

$$I(P, T)^{\alpha + b} \leq C_\varepsilon D^{\alpha(1 + \varepsilon)} |P|^b |T|^{\alpha}.$$ 

If $\alpha = 0$ then the result is trivial, so assume $\alpha > 0$. Recall that $I(P, T)$ is the number of pairs $(t, p) \in T \times P$ such that $p \in t$ (i.e. $p \cap t \neq \emptyset$). Let $d(p)$ be the number of tubes containing $p$. Define

$$J(P, T) = \sum_{p \in P} d(p)^{(b+\alpha)/\alpha}.$$ 

We first relate $J$ to $I$. By the generalized AM-GM inequality, we have

$$J^\alpha \geq \frac{1}{|P|^b} \left( \sum_{p \in P} d(p) \right)^{b+\alpha} = \frac{I(P, T)^{b+\alpha}}{|P|^b}.$$ 

Next, we estimate $J(P, T)$. For a given tube $t \in T$, let

$$j(t) = \sum_{p \in t} d(p)^{b/\alpha}.$$ 

Clearly, we have

$$J(P, T) = \sum_{t \in T} j(t).$$ 

Now we will show $j(t) \leq D^{b(1+\varepsilon)}$ for any $t \in T$. Let $T(\delta) = \{s \in T \mid s \cap t \neq \emptyset, \angle(s, t) \leq 2\delta\}$ and for $w \geq 2\delta$,

$$T(w) = \{s \in T \mid s \cap t \neq \emptyset, w \leq \angle(s, t) \leq 2w\}.$$ 

From the Intersection Lemma, $|T(\gamma)| \lesssim \left( \frac{w}{\delta} \right)^\alpha$. For a ball $p$, let $N(p)$ be the tubes containing $p$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8.png}
\caption{The Intersection Lemma.}
\end{figure}
Lemma 3 (Handshake Lemma),
\[ \sum_{p \in t} |N(p) \cap T(w)| \lesssim |T(w)| \cdot \frac{1}{w^b}. \]

Proof. The left-hand side counts the number of pairs \((p, s)\) with \(p \in t, s \in T(w)\) such that \(p \in s\). For each \(s \in T(w)\), \(s \cap t\) is contained in a \(\delta \times \frac{2}{w}\) rectangle \(R\). If \(\alpha \geq 1\), \(R\) intersects \(\lesssim 1\) essentially distinct balls. If \(\alpha < 1\), then \(R\) is contained in a ball of radius \(\frac{\delta}{w}\), so a dimension argument shows \(s \cap t\) intersects \(\lesssim \left(\frac{1}{w}\right)^b\) many \(\delta\)-balls.

Now let \(K = \{\delta, 2\delta, 4\delta, \ldots, 1\}\), \(C_1 = |K|^{b/\alpha} - 1\), and \(C = C_1 |K|\). Since \(|K| = \log D\), we have \(C \lesssim D^{\varepsilon/\alpha}\). Recall that \(b \geq \alpha\). By Holder’s inequality, we have:

\[ j(t) = \sum_{p \in t} \left( \sum_{w \in K} |N(p) \cap T(w)| \right)^{b/\alpha} \leq \sum_{p \in t} C_1 \sum_{w \in K} |N(p) \cap T(w)|^{b/\alpha} \leq C_1 \sum_{w \in K} \sum_{p \in t} |N(p) \cap T(w)| \cdot |T(w)|^{b/\alpha - 1} \lesssim C_1 \sum_{w \in K} |T(w)|^{b/\alpha} \cdot \frac{1}{w^b} \lesssim C \cdot D^b \lesssim D^{b+\varepsilon/\alpha}. \]

As a result,

\[ I(P, T)^{b+\alpha} \leq |P|^b J^\alpha \lesssim |P|^b \left( \sum_{t \in T} D^{b+\varepsilon/\alpha} \right)^\alpha = |P|^b |T|^\alpha D^{b\alpha+\varepsilon}. \]

This is the desired result.

3.3. Tubes-dominated case.

Theorem 3. Fix \(\varepsilon > 0\). Let \(P\) be a set of \(\beta\)-dimensional \(\delta\)-balls and \(T\) be a set of \(\alpha\)-dimensional \(\delta\)-tubes. Let \(D = \delta^{-1}\). Let \(a = \min(\alpha, 1)\) and assume \(a \geq \beta\). Then for any \(\varepsilon > 0\), there exists \(C_\varepsilon\) such that

\[ I(P, T)^{a+\beta} \leq C_\varepsilon D^{a\beta(1+\varepsilon)} |P|^\beta |T|^a. \]

If \(\beta = 0\) then the result is trivial, so assume \(\beta > 0\). Recall that \(I(P, T)\) is the number of pairs \((t, p) \in T \times P\) such that \(p \in t\) (i.e. \(p \cap t \neq \emptyset\)). Let \(d(t)\) be the number of balls inside \(t\). Define

\[ J(P, T) = \sum_{t \in T} d(t)^{(a+\beta)/\beta}. \]

We first relate \(J\) to \(I\). By the generalized AM-GM inequality, we have

\[ J^\beta \geq \frac{1}{|T|^a} \left( \sum_{t \in T} d(t) \right)^{(a+\beta)/\beta} = \frac{I(P, T)^{a+\beta}}{|T|^a}. \]
Next, we estimate $J(P, T)$. For a given ball $p \in P$, let

$$j(p) = \sum_{t \ni p} d(t)^{\alpha/\beta}.$$ 

Clearly, we have

$$J(P, T) = \sum_{p \in P} j(p).$$

Now we will show $j(p) \leq D^a(1+\varepsilon)$ for any $t \ni p$. If $d(p, q)$ is the distance between the centers of $p$ and $q$, let

$$P(w) = \{ q \in P \mid w \leq d(p, q) \leq 2w \}.$$ 

By dimension, we have $|P(\gamma)| \lesssim (\frac{w}{\delta})^{\beta}$. For a tube $t$, let $N(t)$ be the balls contained in $t$.

**Lemma 4** (Handshake Lemma),

$$\sum_{t \ni p} |N(t) \cap P(w)| \lesssim |P(w)| \cdot \frac{1}{w^a}.$$ 

**Proof.** The left-hand side counts the number of pairs $(t, q)$ with $t \ni p$, $q \in P(w)$ such that $q \in t$. For each $q \in P(w)$, $p$ and $q$ are contained in at most $\frac{1}{w^a}$ many $\delta$-tubes (Intersection Lemma 2, part (d)). \qed

Now let $\mathcal{K} = \{ \delta, 2\delta, 4\delta, \ldots, 1 \}$, $C_1 = |\mathcal{K}|^{a/\beta - 1}$, and $C = C_1 |\mathcal{K}|$. Since $|\mathcal{K}| = \log D$, we have $C \lesssim D^{\varepsilon/\beta}$. Recall that $a \geq \beta$. By Holder’s inequality, we have:

$$j(t) = \sum_{t \ni p} \left( \sum_{w \in \mathcal{K}} |N(t) \cap P(w)| \right)^{a/\beta}$$

$$\leq C_1 \sum_{w \in \mathcal{K}} |N(t) \cap P(w)|^{a/\beta}$$

$$\leq C_1 \sum_{w \in \mathcal{K}} \sum_{t \ni p} |N(t) \cap P(w)| \cdot |P(w)|^{a/\beta - 1}$$

$$\lesssim C_1 \sum_{w \in \mathcal{K}} |P(w)|^{a/\beta} \cdot \frac{1}{w^a}$$

$$\lesssim C \cdot D^a \lesssim D^{a+\varepsilon/\beta}.$$ 

As a result,

$$I(P, T)^{a+\beta} \leq |T|^a |P|^\beta \left( \sum_{p \in P} D^{a+\varepsilon/\beta} \right)^\beta = |T|^a |P|^\beta D^{a+\beta+\varepsilon}.$$ 

This is the desired result.

## 4. Fourier analytic upper bound

We will now prove the upper bound for $\alpha \geq 1$ and $\beta \geq 1$.

### 4.1. A reduction

Define $I'(P, T)$ to be number of pairs $(p, t)$ with $p \in P, t \in T$, and $|p \cap t| \geq \frac{|t|}{2}$. Note that $I'(P, T) \leq I(P, T)$, but $I'$ can be much less than $I$, e.g. $I' = 0$ while $I$ is large. However, in Section 4.5, we will prove that an upper bound for $I'$ translates into an upper bound for $I$. 
4.2. A Fourier Analytic result. We will slightly modify Proposition 2.1 from [1], which was inspired by ideas of Orponen [3].

**Proposition 1.** Fix \( \varepsilon > 0 \). Suppose that \( P \) is a set of unit balls in \([0, 2D]^2\) and \( T \) is a set of essentially distinct \( 1 \times D \) rectangles in \([0, 2D]^2\). Let \( S = D^{2/20} \), and for \( q \in P \), define \( B_S(q) \) to be the ball of radius \( S \) concentric with \( q \), and \( W_S(q) = \# \{ T \in T \mid T \cap B_S(q) \neq \emptyset \} \). Assume \( S > 2 \) and \( D^3 > 2 \). Then

\[
I'(P, T) \lesssim S \cdot D^{1/2} P |P|^{1/2} |\mathbb{T}|^{1/2} + S^{-1} D^3 \sum_{q \in P} W_S(q).
\]

**Remarks.** (1) By rescaling, the bound for \( I' \) still holds if we replace \( P \) by a set of \( \delta \)-balls and \( T \) by a set of \( \delta \)-tubes in \([0, 2D]^2\). We also change \( B_S(q) \) to be a \( \delta \)-ball concentric with \( q \).

(2) In Proposition 2.1 from [1], the set of balls and tubes are contained inside \([0, D]^2\), rather than \([0, 2D]^2\). Changing the \( D \) to \( 2D \) does not affect the structure of the proof.

**Proof.** We will only outline our modifications to the Fourier analytic proof from [1]. The first modification is that we skip the \( \text{“basic reduction”} \) of removing balls from \( P \) to form \( P' \).

Now, we will proceed as in [1]. We first copy the definitions. For each unit ball \( q \) of \( P \), define \( \psi_q \) to be a smooth bump function approximating \( \chi_q \), such that \( \text{supp} \psi_q \subseteq 2q \) and \( \psi_q = 1 \) on \( q \). Let \( f = \sum_{q \in P} \psi_q \). Likewise, for each tube \( T \in \mathbb{T} \), let \( \psi_T \) be a smooth bump approximating \( \psi_T \), and let \( g = \sum_{T \in \mathbb{T}} \psi_T \). Let \( \rho = S^{-1} D^3 \), \( \eta_0 \) be a smooth bump approximating the unit ball, and \( \eta(\omega) = \eta_0(\rho^{-1} \omega) \). Then

\[
I'(P, T) \lesssim \int \eta \tilde{f} \tilde{g} + \int (1 - \eta) \tilde{f} \tilde{g}.
\]

As in [1], we bound the first term by

\[
\int (1 - \eta) \tilde{f} \tilde{g} \lesssim \rho^{-1} D^{3+2\varepsilon} |P|^{1/2} |\mathbb{T}|^{1/2} = S \cdot D^{1/2} |P|^{1/2} |\mathbb{T}|^{1/2}.
\]

For the second term, we require a slight modification. We copy the proof of [1]:

\[
\int \eta \tilde{f} \tilde{g} = \int f (g \ast \eta^\wedge) = \sum_{q \in P} \sum_{T \in \mathbb{T}} \int \psi_q (\psi_T \ast \eta^\wedge).
\]

We know \( \psi_T \ast \eta^\wedge \) is rapidly decaying outside the \( \rho^{-1} \times D \) around \( T \), and \( |\psi_T \ast \eta^\wedge| \lesssim \rho \). Since \( S = D^{20} \rho^{-1} \), we have that \( \psi_T \ast \eta^\wedge \) is in fact negligible outside the \( \frac{S}{2} \times D \) tube around \( T \), as long as we assume \( D^3 \geq 2 \). (For comparison, [1] used the weaker statement that \( \psi_T \ast \eta^\wedge \) is negligible outside the \( S \times D \) tube around \( T \).) As a result, \( \int \psi_q (\psi_T \ast \eta^\wedge) \) is non-negligible only when \( q \) intersects the \( \frac{S}{2} \times D \) tube around \( T \), or equivalently when the \( S/2 \)-2-neighborhood \( N_S/2(q) \) intersects \( T \). Note that \( N_S/2(q) \) is a ball of radius \( \frac{S}{2} + 1 \) concentric with \( q \), so we have \( N_S/2(q) \subset B_S(q) \) as long as \( S \geq 2 \). Putting all the pieces together, we have \( \int \psi_q (\psi_T \ast \eta^\wedge) \) is non-negligible for only \( W_S(q) \) many tubes \( T \). (Recall \( W_S(q) = \# \{ T \in \mathbb{T} \mid T \cap B_S(q) \neq \emptyset \} \).

Thus, as long as \( D^3 \geq 2 \) and \( S \geq 2 \), we have for any \( q \in P \),

\[
\sum_{T \in \mathbb{T}} \int \psi_q (\psi_T \ast \eta^\wedge) \lesssim \rho \cdot W_S(q) = S^{-1} D^3 W_S(q),
\]
and so plugging into (9) gives

$$\int \eta \hat{f} \hat{g} \lesssim S^{-1} D^{\beta} \sum_{q \in P} W_S(q).$$

Combining our estimates in (8) and (11) completes the proof. □

4.3. The Partitioning Argument. Proposition 1 hints at an inductive approach to upper bound $I'$. If the first term in (7) dominates, we get our desired upper bound. If the second term dominates, we get a sum of $W_S(q)$, which encourages us to thicken the $\delta$-balls in $P$ to $S\delta$-balls. (Here, $S = D^{\epsilon/20}$.) We would then thicken the $\delta$-tubes in $\mathbb{T}$ to $S\delta$-tubes. We thus change the scale from $\delta$ to $S\delta$, and we can apply induction. The problem with this approach is that by thickening the balls and tubes, we may violate the dimension property, which in the scale $\delta'$ reads for balls,

$$\# \{ q \in P \mid q \subset B_w \} < 100 \cdot \left( \frac{w}{S\delta} \right)^{\beta}.$$

If we just naively thicken all the $\delta$-balls, then we can still have up to $100 \cdot \left( \frac{w}{\delta} \right)^{\beta}$ of the $S\delta$-balls inside $B_w$, which would violate (12) by a factor of $S^{\beta}$. To resolve this issue, we partition the $S\delta$-balls into about $S^{\beta}$ groups, each of which can satisfy (12). We do a similar process with tubes, and we may now apply induction.

We do not know of a direct partitioning argument that preserves the full dimension condition. Instead, we will define $\epsilon$-weak $\beta$-dimensional balls (and similarly $\epsilon$-weak $\alpha$-dimensional tubes), and prove a partitioning argument that works for this weaker version of dimension.

Recall that the $\delta$-tubes and $\delta$-balls are located in a $2 \times 2$ square, $[0, 2]^2$.

**Definition 4.** Fix $w \leq 1$. The canonical ball $w$-tiling is the covering of the $2 \times 2$ square by $w$-balls whose centers are spaced in a $w$-lattice $(w\mathbb{Z})^2$. The canonical tube $w$-tiling is a covering of the $2 \times 2$ square by $w \times 2$ tubes, defined as follows. For each direction $d \in \{0, \frac{\pi}{2}, w, \frac{3\pi}{2}, \ldots, \pi\}$, let $L_d$ be the lattice generated by $\frac{\pi}{2}e_1$ and $\hat{e}_2$, where $e_1$ is a unit vector in the direction of $d$ and $\hat{e}_2$ is a unit normal to $d$. Place a $w \times 2$ tube with direction $d$ and centered at each point in $L_d$, and keep only those tubes that intersect $[0, 2]^2$.

From now on, we will refer to $w \times 2$ tubes as $w$-tubes. Referring to Figure 9, the left figure shows the canonical ball $w$-tiling. The right figure shows a portion of the canonical tube $w$-tiling. Note that the “midpoint” tubes were omitted from the figure, so a typical $2w \times 4$ rectangle should contain nine $w$-tubes, not just four.

We observe that for the canonical ball $w$-tiling,

- for every $\delta$-ball $q$, at most $50$ $w$-balls of the tiling contain $q$;
- every $w$-ball $p$ is strongly contained in the union of at most $50$ $w$-balls $q_1, \ldots, q_{50}$ of the tiling, in the sense that if a $\delta$-ball is contained inside $p$, then it must be contained inside some $q_i$, $1 \leq i \leq 50$.

Likewise for the canonical tube $w$-tiling (although this is less obvious to see):

- for every $\delta$-tube $t$, at most $50$ $w$-tubes of the tiling contain $t$;
- every $\delta$-tube $t$ is strongly contained in the union of at most $50$ $w$-tubes $v_1, \ldots, v_{50}$ of the tiling, in the sense that if a $\delta$-tube is contained inside $t$, then it must be contained inside some $v_i$, $1 \leq i \leq 50$. 

The constant 50 is not optimal, but the important thing is that it’s a constant. We now define a weaker notion of dimension using the canonical tiling.

**Definition 5.** A set of essentially distinct $\delta$-balls $P$ is called $\varepsilon$-weak $\beta$-dimensional if for every $n \in \mathbb{N}$ with $w = \delta^{(1-\varepsilon/20)^n} < \frac{1}{2}$, every ball $B_w$ in the canonical ball $w$-tiling satisfies

$$\# \{ q \in P \mid q \subset B_w \} < 100 \cdot \left( \frac{w}{\delta} \right)^{\beta}.$$ 

Likewise, a set of essentially distinct $\delta$-tubes $T$ is called $\varepsilon$-weak $\alpha$-dimensional if for every $n \in \mathbb{N}$ with $w = \delta^{(1-\varepsilon/20)^n} < \frac{1}{2}$, every tube $T_w$ in the canonical tube $w$-tiling satisfies

$$\# \{ T \in T \mid T \subset T_w \} < 100 \cdot \left( \frac{w}{\delta} \right)^{\alpha}.$$ 

A few remarks are helpful. First, the reason why we choose $w$ from the set $A = \{ \delta^{(1-\varepsilon/20)^n} \mid n \in \mathbb{N} \}$ is that when we change scale from $\delta$ to $S\delta = \delta^{1-\varepsilon/20}$, we can re-write $A$ as

$$A = \{ (S\delta)^{(1-\varepsilon/20)^{n-1}} \mid n \in \mathbb{N} \}.$$ 

Thus, the structure of $A$ is preserved when we change the scale. Second, note that there are $\sim \log \log D$ choices for $n$ to satisfy $w = \delta^{(1-\varepsilon/20)^n} < \frac{1}{2}$. Third, note that there is no condition imposing a lower bound on $|P|$ or $|T|$. We are ready to state our main partitioning results.

**Proposition 2.** Let $\varepsilon > 0$, $D = \delta^{-1}$, and $S = D^{\varepsilon/20}$. Given a set $P$ of $\varepsilon$-weak $\beta$-dimensional $\delta$-balls, form $Q$ by changing all $\delta$-balls in $P$ to $S\delta$-balls with the same center. Then we can partition $Q$ into $\lesssim S^3 \log \log D \varepsilon$-weak $\beta$-dimensional sets of $S\delta$-balls.

**Proposition 3.** Let $\varepsilon > 0$, $D = \delta^{-1}$, and $S = D^{\varepsilon/20}$. Given a set $T$ of $\delta$-tubes, form $V$ by changing all $\delta$-tubes in $T$ to $S\delta$-tubes with the same center and direction. Then we can partition $V$ into $\log \log D \varepsilon$-weak 1-dimensional sets of $S\delta$-tubes.

**Proof.** We will only prove Proposition 2. The proof of Proposition 3 is essentially the same, except change all balls to tubes, and use the properties for the canonical tube $w$-tiling instead of the canonical ball $w$-tiling.
Form a graph where the vertices are balls in $P$. We will add edges to represent two balls being in different partitions. We form the edge set following these two steps:

**Step A.** For each $w = \delta^{(1-\varepsilon/20)n}$, and for each $w$-ball $B_w$ in the canonical ball $w$-tiling, let $\mathcal{A}$ be the set of balls in $P$ that lie in $B_w$. Divide the balls in $\mathcal{A}$ into groups of size $2S^\beta$, with the remaining group of size at most $2S^\beta$. In each group, draw all edges between members of the group, forming a complete graph. (See Figure 10.)

**Step B.** For each $S\delta$-ball $q \in Q$, let $\mathcal{A}$ be the set of balls in $P$ that lie in $q$. Draw an edge between every ball in $\mathcal{A}$.

![Figure 10. Forming complete graphs.](image)

Now we will calculate the maximum degree of the graph. Fix $p \in P$. For any $w$, $p$ will lie in $\lesssim 1$ balls $B_w$ of the canonical ball $w$-tiling. For each $B_w$, $p$ is connected to at most $\lesssim S^\beta$ neighbors. Finally, there are $\sim \log \log D$ choices for $n$, and thus $w$, to satisfy $\delta^{(1-\varepsilon/20)n} < \frac{1}{2}$. Thus, the maximum degree of $p$ in Step A is $d \lesssim S^\beta \log \log D$. For Step B, we get $\lesssim S^\beta$ more edges, since any $(S\delta)$-ball is strongly contained in the union of 50 $(S\delta)$-balls of the canonical $S\delta$-tiling, and each $(S\delta)$-ball contains $\lesssim S^\beta$ of the $\delta$-balls. We will now employ the following well-known lemma from graph theory.

**Lemma 5.** Any graph with maximum degree $n$ admits a coloring of the vertices with $n + 1$ colors such that no two adjacent vertices share the same color.

Thus, we may color the graph with $d + 1$ colors. The coloring induces a partition of $P$ into $d + 1$ sets, $P_1, P_2, \ldots, P_{d+1}$. For each $i$, we thicken each $\delta$-ball in $P_i$ to a $S\delta$-ball, forming $Q_i$. Now, we will show each set is $\varepsilon$-weak $\beta$-dimensional. It suffices to check $i = 1$.

Consider a ball $B_w$ of the canonical ball $w$-tiling. Since $P$ is $\varepsilon$-weak $\beta$-dimensional, it contains at most $b \lesssim \left(\frac{w}{\delta}\right)^\beta$ balls in $P$. There are $\left\lceil \frac{b}{2S^\beta} \right\rceil \leq \max(1, \frac{b}{S^\beta})$ groups, and each group contains at most one member of $P_1$ by construction. Thus, $B_w$ contains at most

$$\max(1, \frac{b}{S^\beta}) \leq 100 \cdot \left(\frac{w}{S^\beta}\right)^\beta$$

elements of $P_1$. This bound still holds for $Q_1$. Finally, $Q_1$ is essentially distinct by Step B, and so $Q_1$ is an $\varepsilon$-weak $\beta$-dimensional set of $S\delta$-balls, as desired. 

4.4. The upper bound for $I'$. Using the balls and tube partitioning results, we can now prove an upper bound on $I'$ using the inductive argument mentioned at the start of Section 4.3.

**Theorem 4.** Fix $\varepsilon > 0$. Let $P$ be a set of $\varepsilon$-weak $\alpha$-dimensional $\delta$-balls and $T$ be a set of $\varepsilon$-weak $\beta$-dimensional $\delta$-tubes. Let $D = \delta^{-1}$. Let $c = \max \left( \frac{1}{2}, \frac{\alpha+\beta}{2} - 1 \right)$. Then for any $\varepsilon > 0$, there exists a constant $C_\varepsilon$ such that

$$I'(P, T) \leq C_\varepsilon D^{c+\varepsilon} |P|^{1/2} |T|^{1/2}.$$

**Proof.** First, we may clearly assume $\varepsilon < \frac{1}{20}$. Induct on $\delta$. Base case is $\delta > \delta_0$, which is true if we choose $C_\varepsilon > C_0$ for a sufficiently large $C_0(\delta_0)$. We will choose the exact value of $\delta_0$ at the end of the proof. For the inductive step, assume the result is true for $\delta > \delta$. We will use Proposition 1. Assuming $D^3 \geq 2$ and $D^{\varepsilon/20} \geq 2$, we have for some constant $C_1$,

$$I'(P, T) \leq C_1 S \cdot D^{1/2} |P|^{1/2} |T|^{1/2} + C_\varepsilon S^{-1} D^3 \sum_{q \in P} W_S(q). \tag{13}$$

It suffices to show each term is bounded above by $\frac{1}{2} C_\varepsilon D^{c+\varepsilon} |P|^{1/2} |T|^{1/2}$.

This is clear for the first term, since $c < \frac{1}{2}$ and $\varepsilon \geq \frac{1}{20}$. We can choose $C_\varepsilon \geq 2C_1$.

For the second term, we can thicken the $\delta$-tubes to $S\delta$-tubes and thicken the $\delta$-balls to $S\delta$-balls. Let the set of tubes $\mathcal{V}$ and set of balls $Q$ be the results of this thickening. By Proposition 2 and 3 we can partition $\mathcal{V}$ into $\alpha$-dimensional sets $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_{K_2}$ and $P$ into $\beta$-dimensional sets $P_1, P_2, \ldots, P_{K_1}$, where $K_1 = S^{\alpha} \log \log D$ and $K_2 = S^{\beta} \log \log D$. Note that by partition property, $\sum_{i=1}^{K_1} |Q_i| = |P|$ and $\sum_{i=1}^{K_2} |V_i| = |T|$. Thus, using the inductive hypothesis and Cauchy-Schwarz, we get

$$\sum_{q \in P} W_S(q) \leq I(Q, \mathcal{V}) = \sum_{i=1}^{K_1} \sum_{j=1}^{K_2} I(Q_i, V_j)$$

$$\leq \sum_{i=1}^{K_1} \sum_{j=1}^{K_2} 2C_\varepsilon (S^{-1} D)^{1/2+\varepsilon} |Q_i|^{1/2} |V_j|^{1/2}$$

$$= 2C_\varepsilon (S^{-1} D)^{c+\varepsilon} \sum_{i=1}^{K_1} |Q_i|^{1/2} \sum_{j=1}^{K_2} |V_j|^{1/2}$$

$$\leq 2C_\varepsilon (S^{-1} D)^{c+\varepsilon} (K_1 |P|)^{1/2} (K_2 |T|)^{1/2}$$

$$\leq 2C_\varepsilon (S^{-1} D)^{c+\varepsilon} S^{(\alpha+\beta)/2} \log \log D \cdot |P|^{1/2} |T|^{1/2}$$

$$= \left(2C_\varepsilon S^{(\alpha+\beta)/2 - c - \varepsilon} \log \log D \right) D^{c+\varepsilon} |P|^{1/2} |T|^{1/2}.$$ 

Note that $\frac{\alpha+\beta}{2} \leq c + 1$. Thus,

$$C_1 S^{-1} D^{3} \sum_{q \in P} W_S(q) \leq \left(2C_1 S^{-\varepsilon} D^3 \log \log D \right) C_\varepsilon D^{c+\varepsilon} |P|^{1/2} |T|^{1/2}.$$ 

Assume that $\varepsilon < \frac{1}{40}$. Since $S = D^{\varepsilon/20}$, there exists $D_0$ such that for $D > D_0$, we have $2C_1 S^{-\varepsilon} D^3 \log \log D < \frac{1}{2}$. Now choose $D_1 > D_0$ such that $D_1^{3} > 2$ and $D_1^{\varepsilon/20} > 2$, and let $\delta_0 = D_1^{-1}$. This $\delta_0$ allows us to choose $C_0(\delta_0)$, and lastly we can
choose $C_\varepsilon = \max(C_0, 2C_1)$. Thus, for $\delta < \delta_0$, we obtain the desired bound for the second term of (13). This completes the inductive step and thus the proof of the theorem. \hfill \Box

4.5. **Finding the bound for $I$.** We will now show how the bound for $I'$ in Theorem 4 implies a bound for $I$. Form $Q$ by tripling the radius of every $\delta$-ball in $P$, and form $V$ by making each $\delta$-tube in $T$ into a $3\delta$-tube with the same center and direction. In this way, we effectively changed the scale from $\delta$ to $3\delta$. Suppose $p \in P, t \in T$, and $q, v$ are the thickened versions of $p, t$ respectively. Then $p$ intersects $t$ translates into $|q \cap v| \geq \frac{1}{2}|q|$. Hence, $I(P, T) \leq I'(Q, V)$.

It is possible that some of the balls in $Q$ are not essentially distinct. To remedy this, we will use a partitioning argument similar to the proof of Proposition 2. Form a graph on the balls of $P$, with two balls connected by an edge if their centers are at most distance $3\delta$. Since there are at most 99 essentially distinct balls that are distance $\leq 3\delta$ from any given ball $p$, the graph has maximum degree 99. Thus, by Lemma 6 we may partition $P$ into 100 groups $P_1, P_2, \ldots, P_{100}$ such that any two balls in each group are separated by distance at least $3\delta$. Form $Q_i$ by tripling the radius of each ball in $P_i$. Using the dimension property of $P$, we can form a dimension property for $Q_i$:

$$
\# \{q \in Q_i \mid q \subset B_w \} \leq \# \{p \in P \mid p \subset B_w \} \leq 100 \left( \frac{w}{\delta} \right)^\beta = 100 \cdot 2^\beta \left( \frac{w}{2\delta} \right)^\beta.
$$

Thus, the dimension property holds for $Q_i$ with a different constant $100 \cdot 2^\beta$. We can similarly partition $V$ into $V_1, V_2, \ldots, V_{100}$ that satisfy the dimension property for constant $100 \cdot 2^\alpha$.

We will also update the dimension constants for $\varepsilon$-weak $\alpha$-dimensional tubes and $\varepsilon$-weak $\beta$-dimensional balls. Then each $Q_i$ is an $\varepsilon$-weak $\beta$-dimensional collection of balls, and each $V_i$ is an $\varepsilon$-weak $\alpha$-dimensional collection of tubes. We modify the proof of Theorem 4 to use the different dimension constants. Now we have, where $c = \max \left( \frac{1}{2}, \frac{\alpha + \beta - 1}{2} \right)$:

$$
I(P, T) \leq I'(Q, V) = \sum_{i=1}^{100} \sum_{j=1}^{100} I'(Q_i, V_j) \leq \sum_{i=1}^{100} \sum_{j=1}^{100} 2C_\varepsilon D^{c+\varepsilon} |Q_i|^{1/2} |V_j|^{1/2}
\leq \sum_{i=1}^{100} \sum_{j=1}^{100} 2C_\varepsilon D^{c+\varepsilon} |P|^{1/2} |T|^{1/2} = 20000 C_\varepsilon D^{c+\varepsilon} |P|^{1/2} |T|^{1/2}.
$$

Thus, Theorem 4 also holds (with a different constant) when $I'$ is replaced by $I$.

5. **Proof of Theorem 4**

It’s time for the grand finale. We restate Theorem 4 here:

**Theorem 5.** Suppose $\alpha, \beta$ satisfy $0 \leq \alpha, \beta \leq 2$. For every $\varepsilon > 0$, there exists $C_\varepsilon = C(\varepsilon, \alpha, \beta)$ with the following property: for every $\beta$-dimensional set of balls $P$ and $\alpha$-dimensional set of tubes $T$, the following bound holds:

$$
I(P, T) \leq C_\varepsilon \delta^{-f(\alpha, \beta)-\varepsilon},
$$

where $f(\alpha, \beta)$ is defined as in Figure 4. These bounds are sharp up to $\delta^\varepsilon$. 

Proof. The sharpness of these bounds was proved in Section 2, with the constructed examples. First, we have $|P| \lesssim D^\beta$ and $|T| \lesssim D^\alpha$ by dimension property (take $w = 2$). We will split into cases.

- If $1 \geq \alpha \geq \beta$ or $\alpha \geq 1 \geq \beta \geq \alpha - 1$, then we use Theorem 3 to get
  \[ I(P, T)^{\alpha + \beta} \lesssim D^{\alpha \beta + \epsilon} D^{\beta^2} D^{\alpha \alpha}. \]

- If $1 \geq \beta \geq \alpha$ or $\beta \geq 1 \geq \alpha \geq \beta - 1$, then we use Theorem 2 to get
  \[ I(P, T)^{\alpha + b} \lesssim D^{\alpha b + \epsilon} D^b D^{\alpha^2}. \]

- If $\alpha \geq 1$ and $\beta \geq 1$, then we use Theorem 4 and Section 4.5 to get
  \[ I(P, T) \lesssim I'(P, T) \lesssim D^{c+\epsilon} D^{\alpha/2} D^{\beta/2}, \]
  where $c = \max \left( \frac{1}{2}, \frac{\alpha + \beta}{2} - 1 \right)$.

- If $\alpha \geq \beta + 1$, then since each ball intersects $D$ tubes, we get $I(P, T) \lesssim D^\beta \cdot D$.

- If $\beta \geq \alpha + 1$, then since each tube intersects $D$ balls, we get $I(P, T) \lesssim D^\alpha \cdot D$.

Combining these results proves Theorem 1. □

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References