On configuration spaces of higher-dimensional analogues of genus g surfaces

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Abstract

We study the unordered configuration spaces of higher-dimensional analogues of genus g surfaces; specifically, these analogues are the manifolds $W_g := \#_g\, S^n \times S^n$ where n is odd. This paper computes the Betti numbers of these configuration spaces.

1 Introduction

In this paper we study the unordered configuration spaces of manifolds. For a manifold M, this space is defined as the quotient by the symmetric group

$$B_k(M) = \{(x_1, \ldots, x_k) \in M^k \mid x_i \neq x_j \text{ if } i \neq j\}/\Sigma_k.$$

Our main result is the Betti numbers of unordered configuration spaces of manifolds

$$W_g := \#_g S^n \times S^n$$

for odd n.

The result in the case where W_g is a surface, namely when $n = 1$, is previously known. These Betti numbers were computed by Drummond-Cole and Knudsen in their paper on the configuration spaces of arbitrary surfaces of finite type in [3]. Their computation is based on a theorem of Knudsen which is the main result in [5]. It provides an identification of the rational homology of the disjoint union of unordered configuration spaces $\sqcup_k B_k(M)$ with the Lie algebra homology of a certain Lie algebra. When the cohomology of M is known, one can use a Chevalley-Eilenberg complex to compute the Lie algebra homology. The Chevalley-Eilenberg complex also features in previous works on the unordered configuration spaces of manifolds. Examples include the work of B odigheimer and Cohen in [1] and Félix and Thomas in [4]. In this paper, Félix and Thomas use a different approach than Drummond-Cole and Knudsen, and they compute the Betti numbers of configuration spaces of even-dimensional manifolds [4]. Schiessl employs their method to explicitly compute the Betti numbers when the background manifold is a torus [8]. The result for a torus also appears in [7].

Further directions for research would include finding the Betti numbers for $W_g := \#_g S^n \times S^n$ when n is even. Another course to pursue would be to find the Betti numbers for the punctured versions of W_g, which might be accomplished by generalizing the work in [3] on punctured surfaces. The study of configuration spaces also relates to the field of homological stability; stability phenomena is readily observed in the results for W_g. A next step would therefore be to generate tables of Betti numbers of the configuration spaces of W_g where n and g are fixed, much like the appendix of [7], and attempt to identify trends such as vanishing lines and stability lines. Church gives a stable range for when the background manifold is connected and has dimension greater than 2, which would be a useful point of comparison [2]. Also interesting would be to generate data which examines the dependence between any two of the parameters n, g, i and k determining one data point.

In this paper, Section 2 establishes conventions used and recalls pertinent material regarding Lie algebra homology. Section 3 computes the Chevalley-Eilenberg complex $\mathrm{CE}(\mathfrak{g}_{W_g})$ needed to apply the theorem of Knudsen. Section 4 gives the proof of the main result by emulating the work done in the case where W_g is a surface in [3].

2 Background

The proof of the main result is based on [3], so we have largely adopted their notational conventions, and we reproduce relevant definitions here for the reader's convenience. We first define the spaces of interest.

Definition 2.1. The *configuration space* of k ordered points in the topological space X is

$$\mathrm{Conf}_k(X) := \{(x_1, \ldots, x_k) \in X^k \mid x_i \neq x_j \text{ if } i \neq j\},$$

endowed with the subspace topology. The *unordered configuration space* is the quotient

$$B_k(X) := \mathrm{Conf}_k(X)/\Sigma_k.$$

The topology of these spaces is studied in detail in [6]. We will only be concerned with the case of X being a manifold.

We now clarify some conventions. Denote the n-th suspension of a graded vector space V by $V[n]$ such that $V[n]_i = V_{i-n}$. We write $\sigma^n x$ for the element in $V[n]$ corresponding to $x \in V$. Often, we will work with bigraded vector spaces which have an auxiliary grading; we refer to the primary grading as degree and the auxiliary grading as weight, and they are denoted by i and k, respectively.

In our computations we will need the compactly-supported rational cohomology of the background manifold M with negative degrees: $H_c^{-*}(M;\mathbb{Q})$. We can view the cohomology ring as a bigraded vector space concentrated in weight zero and with degree referring to the natural grading of the ring.

For graded vector spaces V which are finite-dimensional in each degree, it is useful to store the dimensions of its summands in a Poincaré series:

$$\mathcal{P}_V(t) = \sum_{n\in\mathbb{Z}} \dim_i(V)t^i.$$

Note that Poincaré series are additive under direct sum. They are also multiplicative under tensor product as long as there are no issues with convergence, which might occur with spaces that are nonzero in infinitely many degrees. For a graded vector space V and an operator $d : V \to V$ defined on it, we have

$$\mathcal{P}_{\ker d/\operatorname{im} d}(t) = \mathcal{P}_{\ker d}(t) + t^{|d|}(\mathcal{P}_{\ker d}(t) - \mathcal{P}_V(t)).$$

In case (V, d) is a complex, then the Poincaré series on the left side is that of the homology $H(V, d)$.

Next, we recall some relevant material regarding Lie algebras. Let $\mathfrak{g}$ be a graded Lie algebra. It is equipped with a Lie bracket $[-,-]$, which is a bilinear map satisfying the following:

1. graded antisymmetry: $[x, y] = (-1)^{|x||y|+1}[y, x]$
2. Jacobi identity: $(-1)^{|x||z|}[x,[y,z]] + (-1)^{|z||y|}[z,[x,y]] + (-1)^{|y||x|}[y,[z,x]] = 0.$

Here is an example of such an algebra.

Example 2.2. Let $\mathcal{L}(u_{n-1,1})$ be the free Lie algebra on one generator in bidegree $n-1,1$. If n is odd, then graded antisymmetry implies $[u,u]=0$. If n is even, then $[u,u]$ is nonzero but we have $[u,[u,u]]=0$ and $[[u,u],[u,u]]=0$ by the Jacobi identity and graded antisymmetry, respectively. Thus,

$$\mathcal{L}(u_{n-1,1}) = \begin{cases} \mathbb{Q}\langle u\rangle & n \text{ odd} \\ \mathbb{Q}\langle u\rangle \oplus \mathbb{Q}\langle [u,u]\rangle & n \text{ even.} \end{cases}$$

We can compute the homology of a Lie algebra by way of its Chevalley-Eilenberg complex.

Definition 2.3. Let $\mathfrak{g}$ be a graded Lie algebra. The *Chevalley-Eilenberg complex* of $\mathfrak{g}$ is the chain complex $\mathrm{CE}(\mathfrak{g})$ whose underlying graded vector space is $\mathrm{Sym}(\mathfrak{g}[1])$, which carries the structure of the cofree conilpotent cocommutative coalgebra on the graded vector space $\mathfrak{g}[1]$. The differential of $\mathrm{CE}(\mathfrak{g})$ is determined by the unique coderivation D of that coalgebra structure such that

$$D(\sigma x \cdot \sigma y) = (-1)^{|x|}\sigma[x,y].$$

The *Lie algebra homology* of $\mathfrak{g}$ is

$$H^{\mathcal{L}}(\mathfrak{g}) = H(\mathrm{CE}(\mathfrak{g})).$$

To elaborate on the above definition, we give an explicit formula for the differential. Also denoting it by D, its formula is

$$D(\sigma x_1 \ldots \sigma x_m) = \sum_{1\le i<j\le m} (-1)^{|x_i|}\epsilon(x_1,\ldots,x_m)\sigma[x_i,x_j]\sigma x_1 \cdots \widehat{\sigma x_i} \cdots \widehat{\sigma x_j} \cdots \sigma x_m,$$

with ϵ being a sign determined by

$$\sigma x_1 \cdots \sigma x_n = \epsilon(x_1,\ldots,x_n)\sigma x_i \cdot \sigma x_j \cdot \sigma x_1 \cdots \widehat{\sigma x_i} \cdots \widehat{\sigma x_j} \cdots \sigma x_m.$$

Note that D is zero when $m=1$. For more on Lie algebra homology, Chapter 7 of [9] is a useful reference.

We can now state the the key result underpinning our methodology.

Theorem 2.4 (Knudsen)**.** *For an orientable n-manifold M, there is an isomorphism of bigraded vector spaces*

$$\bigoplus_{k\ge 0} H_*(B_k(M);\mathbb{Q}) \cong H^{\mathcal{L}}(\mathfrak{g}_M)$$

where $\mathfrak{g}_M = H_c^{-*}(M;\mathbb{Q}) \otimes \mathcal{L}(u_{n-1,1})$.

In this paper we will arrive at the Betti numbers $\beta_i(B_k(W_g))$ by computing the Lie algebra homology on the right side [5]. Before we continue, we give some examples which further elucidate the behavior of the differential on the complexes $\mathrm{CE}(\mathfrak{g}_M)$.

Example 2.5. When M is an odd-dimensional manifold, from Example 2.2 we know that $[u,u]=0$, so the bracket of $\mathfrak{g}_M$ must be zero. We see from the definition of the coderivation D that it is identically zero, and so the differential it determines is trivial.

On the other hand, the even-dimensional case holds more complexity.

Example 2.6. Let M be an $2n$-dimensional manifold. Disregarding the weight grading, we can write

$$\mathfrak{g}_M \cong H_c^{-*}(M;\mathbb{Q})[2n-1] \oplus H_c^{-*}(M;\mathbb{Q})[4n-2].$$

The cup product determines the Lie bracket by the following:

$$[\sigma^{2n-1}\alpha, \sigma^{2n-1}\beta] = (-1)^{|\beta|}\sigma^{4n-1}(\alpha \smile \beta),$$

and all other brackets are zero. Then, the coderivation specifying the differential is given by

$$D(\sigma^{2n}\alpha \cdot \sigma^{2n}\beta) = (-1)^{|\alpha|+|\beta|+2n-1}\sigma^{4n-1}(\alpha \smile \beta).$$

Finally, we note that via a linear change of variable, it is possible to avoid the $(-1)^{|\alpha|+|\beta|+2n-1}$ sign. Multiplying specific generators by -1 gives the desired effect. Specifically, these are the generators of $\mathrm{CE}(\mathfrak{g}_M)$ which are in the form $\sigma^{4n-1}\alpha$ with $|\alpha|$ being even. This change of variable is assumed to have been done in the following computations, and so moving forward we can omit this sign.

3 Computation of Chevalley-Eilenberg complex $\mathrm{CE}(\mathfrak{g}_{W_g})$

To apply Theorem 2.4, we first compute the complex $CE(\mathfrak{g}_{W_g})$ where

$$\mathfrak{g}_{W_g} = H_c^{-*}(W_g;\mathbb{Q}) \otimes \mathcal{L}(u_{2n-1,1}).$$

The cohomology of W_g is

$$H^{-*}(W_g;\mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}^{2g}[-n] \oplus \mathbb{Q}[-2n].$$

Let the generators be γ in degree 0 and ρ in degree $-2n$. For $1 \leq i \leq g$, let α_i and β_i, be the two generators in degree $-n$ coming from the cohomology of the i-th summand of the connected sum that comprises W_g. Now we briefly outline how to determine the cup product structure needed for the computation. On the cohomology of $S^n \times S^n$, the only interesting cup products are those of the form $\alpha_i \smile \beta_i$ given by

$$\smile : H^{-n}(S^n \times S^n) \otimes H^{-n}(S^n \times S^n) \to H^{-2n}(S^n \times S^n).$$

We can compute these cup products using the representing submanifolds of the Poincaré duals of α_i and β_i. The product $\alpha_i \smile \beta_i$ is dual to the intersection of the oriented submanifolds. Lastly, we use the Mayer-Vietoris sequence to get the cup product on the entire connected sum. The result is stated below.

$$
\begin{array}{lllll}
\gamma \smile \gamma & = & \gamma & & \\
\gamma \smile \alpha_i & = & \alpha_i & = \quad \alpha_i \smile \gamma & \text{for } 1 \le i \le g \\
\gamma \smile \beta_i & = & \beta_i & = \quad \beta_i \smile \gamma & \text{for } 1 \le i \le g \\
\gamma \smile \rho & = & \rho & = \quad \rho \smile \gamma & \\
\alpha_i \smile \beta_i & = & \rho & = \quad (-1)^{n^2} \beta_i \smile \alpha_i & \text{for } 1 \le i \le g.
\end{array}
$$

Remark 3.1. Given what we have established about the cohomology of W_g, we can see from the definition of the graded Lie algebra $\mathfrak{g}_{W_g}$ that it has the same generators and relations for all n odd albeit with degrees shifted.

Now for the generators of $\mathfrak{g}_{W_g}[1]$, let v, a_i, b_i, p denote $\sigma(\gamma \otimes u)$, $\sigma(\alpha_i \otimes u)$, $\sigma(\beta_i \otimes u)$, $\sigma(\rho \otimes u)$, respectively, where $i = 1, \dots, g$. Furthermore, let $\tilde{v}$ be $\sigma(\gamma \otimes [u, u])$ and similarly for $\tilde{a_i}$, $\tilde{b_i}$ and $\tilde{p}$. The bidegrees of the generators are then:

	degree 0	degree n	degree $2n-1$	degree $2n$	degree $3n-1$	degree $4n-1$
weight 1	p	a_i, b_i		v		
weight 2			$\tilde{p}$		$\tilde{a_i}, \tilde{b_i}$	$\tilde{v}$

Since n is odd, we have

$$
\mathrm{Sym}(\mathfrak{g}_{W_g}[1]) = \mathbb{Q}[p, \tilde{a_1}, \dots, \tilde{a_g}, \tilde{b_1}, \dots, \tilde{b_g}, v] \otimes \Lambda[\tilde{p}, a_1, \dots, a_g, b_1, \dots, b_g, \tilde{v}].
$$

Based on the formulas given in Section 2, the coderivation D we get from the cup product structure above is $D(vx) = \tilde{x}$ for each generator x with weight 1 and $D(a_i b_i) = \tilde{p}$. We give an alternative formulation of the coderivation which is key to proving the main result. To this end we first establish two operators which play a major role throughout the rest of this paper.

Definition 3.2. We define two operators on $\mathrm{CE}(\mathfrak{g}_{W_g})$

$$
\Delta = \sum_j \partial_{b_j} \partial_{a_j}, \qquad \delta = \sum_{j; c \in \{a, b\}} \tilde{c_j} \partial_{c_j},
$$

where ∂_u is the degree $-n$ operator defined by formal differentiation with respect to u.

Note that Δ decreases degree by $2n$, δ increases degree by $2n-1$, and $\delta^2 = 0$. Furthermore, the two operators commute as stated in [3]. These facts will be significant later in our proof. Now, we can write

$$
D = \tilde{p}\Delta + \delta \partial_v + \frac{\tilde{v}}{2} \partial_v^2 + \tilde{p} \partial_p \partial_v.
$$

This formula comes from the discussion following Definition 4.1 in [3]; the identical expression works in the n odd case. In the next section, we compute the homology of $\mathrm{CE}(\mathfrak{g}_{W_g})$ by first constructing homotopies to spaces on which the second two summands of D are zero.

4 Betti numbers of configuration spaces $B_k(W_g)$

We are now ready to compute the Betti numbers of the configuration spaces $B_k(W_g)$ using Theorem 2.4. Computing the Lie algebra homology of $\mathfrak{g}_{W_g}$ via the Chevalley-Eilenberg complex $CE(\mathfrak{g}_{W_g})$ from Section 3 gives our main result. Throughout, the symbol n refers to the parameter in the definition

$$W_g := \#_g S^n \times S^n.$$

Our proof comes from mimicking the process of [3] in their computation of the Betti numbers of closed orientable surfaces. Doing so is motivated by the similarity in the graded Lie algebras $\mathfrak{g}_{W_g}$ for all n odd, as remarked upon in the previous section. The proof is carried out in two stages. Before anything else we define two subspaces of $\mathrm{CE}(\mathfrak{g}_{W_g})$:

$$\mathcal{X}_g := \mathbb{Q}[\tilde{a}_i, \tilde{b}_i] \otimes \Lambda[a_i, b_i]$$

$$\mathcal{K}_g := \ker \Delta\, |_{\mathcal{X}_g} \cap \ker \delta\, |_{\mathcal{X}_g} .$$

In the expression for $\mathcal{X}_g$, $\tilde{a}_i$ is taken to mean all of the generators $\tilde{a}_i$ for $1 \leq i \leq g$, and similarly for $\tilde{b}_i$, a_i and b_i. In case $g = 0$, $\mathcal{X}_0 = \mathbb{Q}$. We suppress the subscript when the genus g is unambiguous. Let X and K be the Poincaré series of $\mathcal{X}$ and $\mathcal{K}$. In the first stage of the proof we obtain Poincaré series encoding the the Betti numbers in terms of the series X and K.

We will see that the Betti numbers are zero when $i > \frac{3n-1}{2}k + \frac{n+1}{2}$. Furthermore, the Betti numbers are independent of k when $i < \frac{3n-1}{2}k$. We refer to these values of i, k as the stable range and we write $\beta_i^{\mathrm{st}}(B(W_g)) = \beta_i(B_k(W_g))$ for these i, k. Some stable Betti numbers are given in Figure 4. The series $\mathcal{P}_{\mathrm{st}}$, $\mathcal{P}_0$, $\mathcal{P}_1$ give the result in bidegrees where the Betti numbers are nonzero. We inherit this notation from [3] though in the n odd case their original meanings do not hold. When W_g is a surface S_g, the three series are exactly

$$\mathcal{P}_{\mathrm{st}}(t) = \sum_{i=1}^{\infty} \beta_i^{\mathrm{st}}(B(S_g))t^i$$

$$\mathcal{P}_0(t) = \sum_{i=0}^{\infty} \beta_i(B_i(S_g))t^i$$

$$\mathcal{P}_1(t) = \sum_{i=1}^{\infty} \beta_i(B_{i-1}(S_g))t^i.$$

In the n odd case, the values of i and k that each Poincaré series gives information about are no longer disjoint. However, they still give the ranks of the homology of certain subspaces of a deformation retract of $\mathrm{CE}(\mathfrak{g}_{W_g})$, and this correspondance will be elucidated in the proof carried out in Section 4.1.

$i \backslash g$	1	2	3	4	5	6
0	1	1	1	1	1	1
1	0	2	4	6	7	13
2	0	2	4	7	30	60
3	2	4	7	30	60	98
4	0	1	8	2	24	80
5	0	2	1	4	15	103
6	1	9	25	51	158	335
7	0	0	2	35	49	6
8	2	2	21	43	61	60
9	0	1	27	107	270	794
10	0	3	10	28	179	370
11	5	16	29	138	338	641
12	0	4	21	54	366	1173

Figure 1: Stable Betti numbers for $n = 3$ and low genus

Theorem 4.1. *Poincaré series for the Betti numbers of the unordered configuration spaces of W_g are given by*

$$\mathcal{P}_{\mathrm{st}}(t) = \frac{(1+t)(1+t^{4n-1})}{t^{2n}(1+t^{2n-1})}\left[(1+t^{2n-1})K + \frac{-t^{2n-1}+t^{4n-1}}{1+t}X - 1\right]$$
$$\mathcal{P}_0(t) = \frac{1}{t^{2n-1}}\left[(1+t^{4n-2}+t^{4n-1})K - \frac{1+t^{4n-1}}{1+t^{2n-1}} + \frac{-t^{6n-3}+t^{8n-3}}{1+t^{2n-1}}X\right]$$
$$\mathcal{P}_1(t) = t^{2n}K.$$

Next, finding the Poincaré series X_g is a relatively straightforward counting problem. To do so, define $\mathcal{S} = \mathbb{Q}[\tilde{a}, \tilde{b}]$ where $\tilde{a}, \tilde{b}$ are in bidegree $3n-1, 2$. We can decompose $\mathcal{X}_g$ as

$$\mathcal{X}_g \cong \mathcal{S}^g \otimes \Lambda[a_i, b_i].$$

We get the Poincaré series of $\mathcal{S}^g$ from page 22 of [3]. We then have

$$X_g = \sum_{j\geq 0; j=c(3n-1)} \binom{2g + \frac{j}{3n-1} - 1}{2g-1} t^j \cdot \sum_{l \geq 0; l=cn} \binom{2g}{\frac{l}{n}} t^l$$

where c is an integer. Obtaining K_g is much more difficult, and doing so constitutes the second stage of the proof which is carried out in Section **??**. In this section we find a description $\mathcal{K}_g$ which more easily allows us to calculate its Poincaré series, giving us the following theorem.

Theorem 4.2. *When $g \geq 1$ and $i \geq 2n+1$, we have*

$$\dim_i(\mathcal{K}_g) = \sum_{j=0}^{g-1}\sum_{l=0}^{j}(-1)^{g+j+1}\left(\binom{2j}{l} - \binom{2j}{l-2}\right) s(j, i+g-j-ln-1)$$

where $s : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ *is defined as:*

$$s(j,e) = \begin{cases} \binom{2j+c+1}{2j+1} & e = c(3n-1) \text{ or } e = c(3n-1)+4n-1 \text{ for } c \in \mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise.} \end{cases}$$

When $0 \leq i \leq 2n$*, we have*

$$\dim_i(\mathcal{K}_g) = \begin{cases} 1 & i = 0 \\ 2g & i = 2 \text{ and } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.3. For completeness, we note that for any n odd $\mathcal{K}_0 = \mathcal{X}_0 = \mathbb{Q}$, and so $K_0 = 1$.

4.1 Proof of Theorem 4.1

Our proof emulates the work of Section 5 of [3]. The first step is to eliminate the $\frac{\tilde{v}}{2}\delta_v^2$ term from D by constructing a contracting homotopy onto a smaller complex on which this term is zero. To this end, in addition to $\mathcal{X}_g$ and $\mathcal{K}_g$ we define the spaces

$$\mathcal{Y}_g := \mathbb{Q}[p, \tilde{a_i}, \tilde{b_i}] \otimes \Lambda[a_i, b_i]$$

$$\mathcal{Z}_g := \mathbb{Q}[p, \tilde{a_i}, \tilde{b_i}] \otimes \Lambda[\tilde{p}, a_i, b_i].$$

Lemma 4.4. *Define*

(1) a retraction $\mathrm{CE}(\mathfrak{g}_{W_g}) \to \mathcal{Z} \oplus v\mathcal{Z}$*, which, for* $q \in \mathcal{Z}$*, takes* $\tilde{v}q$ *to*

$$-2v\delta(q) - 2v\tilde{p}\partial_p q,$$

and

(2) a degree 1 *chain homotopy from* $\mathrm{CE}(\mathfrak{g}_{W_g})$ *to itself, which, for* $q \in \mathcal{Z}$*, takes* $v^m\tilde{v}q$ *to*

$$\frac{2m!}{(m+2)!}v^{m+2}q.$$

Then the inclusion of $\mathcal{Z} \oplus v\mathcal{Z}$ *into* $\mathrm{CE}(\mathfrak{g}_{W_g})$ *along with the retraction and chain homotopy constructed above constitute the data of a deformation retraction.*

Lemma 5.1 of [3] states this result for $n = 1$, and the proof is by direct computation. The same computation works for any n odd, and so the identical statement holds. By inspection of bidegrees of the generators of $\mathcal{Z} \oplus v\mathcal{Z}$, we make the subsequent observation.

Remark 4.5. It follows that $H_i(B_k(W_g); \mathbb{Q}) = 0$ for $i > \frac{3n-1}{2}k + \frac{n+1}{2}$.

Secondly, we eliminate $\tilde{p}\partial_p\partial_v$ by deforming $\mathcal{Z} \oplus v\mathcal{Z}$ onto a subspace $\mathcal{Y} \oplus v\tilde{p}\mathcal{Y} \oplus v\mathcal{X}$ equipped with a twisted differential.

Lemma 4.6. *Define*

(1) a degree -1 linear operator d on $\mathcal{Y} \oplus v\tilde{p}\mathcal{Y} \oplus v\mathcal{X}$ by

$$d|_{\mathcal{Y}}(q) = -\frac{p\Delta}{m+1}(v\tilde{p}\Delta + \delta)q$$
$$d|v\tilde{p}\mathcal{Y}(q) = -\frac{p\Delta}{m+1}\delta(q)$$
$$d|_{v\mathcal{X}}(q) = \left(\frac{1}{v}\delta + \tilde{p}\Delta\right) q$$

where $m = m(q)$ is the largest nonnegative integer such that $q = p^m q'$;

(2) a linear map $f : \mathcal{Y} \oplus v\tilde{p}\mathcal{Y} \oplus v\mathcal{X} \to \mathcal{Z} \oplus v\mathcal{Z}$ by

$$f|_{\mathcal{Y}}(q) = \left(1 - \frac{p}{m+1}v\Delta\right) q$$
$$f|_{v\tilde{p}\mathcal{Y}}(q) = \left(1 - \frac{p}{\tilde{p}(n+1)}\delta\right) q$$
$$f|_{v\mathcal{X}}(q) = q;$$

(3) a linear map $g : \mathcal{Z} \oplus v\mathcal{Z} \to \mathcal{Y} \oplus v\tilde{p}\mathcal{Y} \oplus v\mathcal{X}$ by

$$g|_{\mathcal{Y}\oplus v\tilde{p}\mathcal{Y}}(q) = q$$
$$g|_{v\mathcal{Y}}(q) = \begin{cases} q & m = 0 \\ 0 & else \end{cases}$$
$$g|_{\tilde{p}\mathcal{Y}}(q) = -\frac{p}{m+1}\left(v\Delta - \frac{1}{\tilde{p}}\delta\right) q;$$

(4) a degree 1 linear operator on $\mathcal{Z} \oplus v\mathcal{Z}$ by stipulating that

$$F\ |_{\tilde{p}\mathcal{Y}} = \frac{p}{\tilde{p}(m+1)}v$$

and extending by zero.

Then d is a differential, f and g are chain maps, and F is a chain homotopy $\mathrm{id} \sim fg$.

The same proof as that of Lemma 5.2 of [3] works here. The next lemma concerns the stable Betti numbers.

Corollary 4.7. *For $i < \frac{3n-1}{2}k$,*

$$H_i(B_k(W_g); \mathbb{Q}) \cong H_i(\mathcal{X} \oplus \mathcal{X}[4n-1], d_{\mathrm{st}})$$

where d_{st} has the components $\Delta^2[4n-1] : \mathcal{X} \to \mathcal{X}[4n-1]$, $\delta\Delta : \mathcal{X} \to \mathcal{X}$, and $-\delta\Delta : \mathcal{X}[4n-1] \to \mathcal{X}[4n-1]$. In particular, the dimension of this vector space is independent of k.

Proof. This corollary is the generalized n odd version of Corollary 5.3 in [3]. The statement is the same except for adjustments in grading, which we elucidate below. Apart from the grading, the proof identical to that of Corollary 5.3.

The proof gives an isomorphism of chain complexes

$$(p\mathcal{Y} \oplus pv\tilde{p}\mathcal{Y}, d) \cong (\mathcal{X} \oplus \mathcal{X}[4n-1], d_{\mathrm{st}})$$

where the complex of the left side is a subcomplex of the one in Lemma 4.6. This subcomplex can only be nonzero in bidegrees satisfying $i < \frac{2}{3n-1}k$. Lastly, the degree of $v\tilde{p}$ is $4n-1$, hence why we take the $(4n-1)$-th suspension of $\mathcal{X}$. □

Next, let $\mathcal{X}_{>0} \subset \mathcal{X}$ denote the subspace of polynomials with no constant term.

Lemma 4.8. *When δ is restricted to $\mathcal{X}_{>0}$, we have $\ker \delta|_{\mathcal{X}_{>0}} = \operatorname{im} \delta|_{\mathcal{X}_{>0}}$. Furthermore, there is an operator H on $\mathcal{X}$ that has the property that $\Delta H \Delta H$ and $H \Delta H \Delta$ differ by an invertible linear transformation.*

Proof. This statement echoes Lemma 5.4 of [3], but for n odd the same proof only gives this weaker statement. A detail to be mindful of is that we have to account for the dependence of the degrees of the generators on n. We observe that the commutation relation is now $H\Delta H\Delta = 0 = \Delta H \Delta H$ on elements in $\mathcal{X}$ of degree at most $4n$ (rather than 4), and in polynomial degree greater than 4, applying $\Delta H \Delta H$ to a monomial of degree m gives $\frac{m-4n}{m} H\Delta H \Delta$. □

Note that this version of the lemma being weaker does not hinder the rest of the proof of the main theorem. Our next lemma mirrors Lemma 5.5 of [3].

Lemma 4.9. *There is an isomorphism $\ker d_{\mathrm{st}} \cong \ker(\delta\Delta \oplus \delta\Delta[4n-1])$ of bigraded vector spaces.*

Again, the identical proof works. The final piece required for the proof of 4.1 is the following.

Lemma 4.10. *The Poincaré series for $\ker \delta\Delta$ on $\mathcal{X}$ satisfies*

$$\mathcal{P}_{\ker \delta\Delta}(t) = \frac{1}{t^{2n-1}(1+t^{2n-1})}[(1+t^{2n-1})K - 1 + t^{4n-2}X].$$

Proof. The proof of Lemma 5.6 in [3] implies that

$$\mathcal{P}_{\ker \delta\Delta}(t) = \frac{1}{t^{2n-1}}(K-1) + \mathcal{P}_{\ker \delta}(t).$$

Next, lemma 4.8 implies that $\mathcal{P}_{\ker \delta / \operatorname{im} \delta}(t) = 1$, where the kernel and image in the subscript are those when δ is restricted to $\mathcal{X}$. Applying the formula from Section 2 for a Poincaré series of this form, we have

$$\mathcal{P}_{\ker \delta}(t) = \frac{1 + t^{2n-1}X}{1+t^{2n-1}}.$$

The lemma now follows. □

Proof of Theorem 4.1. The same proof as that of Theorem 4.2 from [3] gives

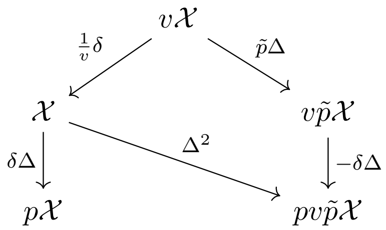
$$\mathcal{P}_{\mathrm{st}}(t) = (1 + t^{4n-1})\mathcal{P}_{H(\mathcal{X},\delta\Delta)}(t)$$

and

$$\mathcal{P}_{H(\mathcal{X},\delta\Delta)}(t) = \frac{1+t}{t}\mathcal{P}_{\ker\delta\Delta}(t) - \frac{1}{t}X.$$

Combining the above with Lemma 4.10 gives the formula for $\mathcal{P}_{\mathrm{st}}(t)$.

Next, $\mathcal{P}_0(t)$ comes from taking the kernel modulo image of the maps in the diagram below. These are the kernels and images residing in $\mathcal{X} \oplus v\tilde{p}\mathcal{X}$.



We therefore have

$$\begin{aligned}
\mathcal{P}_0(t) &= \mathcal{P}_{\ker d_{\mathrm{st}}}(t) - \mathcal{P}_{\mathrm{im}(\frac{1}{v}\delta\oplus\tilde{p}\Delta)}(t) \\
&= \mathcal{P}_{\ker\delta\Delta\oplus\delta\Delta[4n-1]}(t) - \frac{1}{t}(t^{2n}X - t^{2n}K) \\
&= (1+t^{4n-1})\mathcal{P}_{\ker\delta\Delta}(t) - t^{2n-1}(X-K)
\end{aligned}$$

with the second line coming from Lemma 4.9 and subtracting the Poincaré series of $\ker\frac{1}{v}\delta \oplus \tilde{p}\Delta$ from that of $v\mathcal{X}$. Expanding the expression gives the desired result.

Finally, $\mathcal{P}_0(t)$ is the Poincaré series of the kernel of $\frac{1}{v}\delta \oplus \tilde{p}\Delta$, which we determined above. □

We close this subsection with a remark which makes explicit how $\mathcal{P}_{\mathrm{st}}$, $\mathcal{P}_0$, $\mathcal{P}_1$ function to encode the desired Betti numbers.

Remark 4.11. Our proof implies that

$$\begin{aligned}
\mathcal{P}_{\mathrm{st}}(t) &= \mathcal{P}_{H(p^m\mathcal{X}\oplus p^m v\tilde{p}\mathcal{X},d)}(t) \\
\mathcal{P}_1(t) &= \mathcal{P}_{H(\mathcal{X}\oplus v\tilde{p}\mathcal{X},d)}(t) \\
\mathcal{P}_0(t) &= \mathcal{P}_{H(v\mathcal{X},d)}(t)
\end{aligned}$$

where m is a fixed positive integer and

$$p\mathcal{Y} \oplus pv\tilde{p}\mathcal{Y} \cong \bigoplus_{m=1}^{\infty} p^m\mathcal{X} \oplus p^m v\tilde{p}\mathcal{X}.$$

The subspaces $p\mathcal{Y} \oplus pv\tilde{p}\mathcal{Y}$, $\mathcal{X} \oplus v\tilde{p}\mathcal{X}$ and $v\mathcal{X}$ partition the complex $(\mathcal{Y} \oplus v\tilde{p}\mathcal{Y} \oplus v\mathcal{X}, d)$ given in Lemma 4.6. Generating Betti number data explicitly can be done by recovering the bidegrees of the relevant subspaces.

4.2 Proof of Theorem 4.2

We obtain a new description of $\mathcal{K}_g$ in large part by mimicking the work of Section 6 of [3]. To begin, for $g \geq 0$ and $m \geq 1$ define

$$\mathcal{V}(g,m) = \{(q,r) \in \mathcal{X}_g \oplus \mathcal{X}_g[4n-1] \mid \Delta^m q = \delta\Delta^{m-1} r, \Delta^m r = 0\}.$$

As a convention, we let $\mathcal{V}(g,0) = 0$. Next, recall that we previously defined $\mathcal{S} = \mathbb{Q}[\tilde{a}, \tilde{b}]$ where $\tilde{a}$ and $\tilde{b}$ are in bidegree $3n-1, 2$. We also introduce some useful notation. We have the following decomposition: $\mathcal{X}_{g+1} \cong \Lambda[a,b] \otimes \mathbb{Q}[\tilde{a}, \tilde{b}] \otimes \mathcal{X}_g$. Then, for a generic homogeneous element q of $\mathcal{X}_{g+1}$, we can write

$$\begin{array}{rcrrl} q & = & & \tilde{a}^i\tilde{b}^j & q_{1,ij} \\ & + & a & \tilde{a}^{i-1}\tilde{b}^j & (q_{2,ij} - q_{3,ij}) \\ & + & b & \tilde{a}^i\tilde{b}^{j-1} & (q_{2,ij} - q_{3,ij}) \\ & + & ab & \tilde{a}^{i-1}\tilde{b}^{j-1} & q_{4,ij} \end{array}$$

where we implicitly sum over (i,j), and each q_{kij} is a homogeneous element of $\mathcal{X}_g$. Here, $q_{1,ij}$ is defined for nonnegative i and j, $q_{2,ij}$ and $q_{3,ij}$ are defined when at least one of i and j is positive, and $q_{4,ij}$ is defined when i and j are both positive. We extend by zero in all other cases such that q_{kij} is defined for arbitrary (i,j). Lastly, we establish a new bigrading, separate from the previous bigrading of the generators of $\mathrm{CE}(\mathfrak{g}_{W_g})$, for q_{kij}: $|q_{1,ij}| = |q_{2,ij}| = (i,j)$ and $|q_{3,ij}| = |q_{4,ij}| = (i-1, j-1)$. A more detailed description of this decomposition can be found on page 24 of [3].

We can now specify the mapping

$$q \mapsto \sum_{i,j} \tilde{a}^i\tilde{b}^j \otimes (q_{1,ij}, -q_{3,i+1,j+1}).$$

Lemma 6.6 of [3] shows that the above is a valid map $\mathcal{K}_{g+1} \to \mathcal{S} \otimes \mathcal{V}(g,1)$. We also define a map $\mathcal{S} \otimes \mathcal{V}(g,1)[2n-1] \to \mathcal{K}_g$ by the composite

$$\mathcal{S} \otimes \mathcal{V}(g,1)[2n-1] \xrightarrow{\tilde{a},\tilde{b}=0} \mathcal{V}(g,1)[2n-1] \xrightarrow{(q,r)\mapsto\delta q} \mathcal{K}_g.$$

Note that since the operator δ shifts degree by $2n-1$, the vector space $\mathcal{S} \otimes \mathcal{V}(g,1)$ is suspended $2n-1$ times of to ensure that the map to $\mathcal{K}_g$ respects grading. The subsequent lemma now describes how the spaces $\mathcal{V}(g,m)$ and $\mathcal{S}$ relate to $\mathcal{K}_g$.

Lemma 4.12. *For $g \geq 0$, the sequence of graded vector spaces*

$$0 \longrightarrow \mathcal{K}_{g+1}[2n-1] \longrightarrow \mathcal{S} \otimes \mathcal{V}(g,1)[2n-1] \longrightarrow \mathcal{K}_g$$

is exact.

Proof. Lemma 6.1 in [3] gives this result for $n = 1$, and the identical proof works in the n odd case. ☐

The above lemma relates the dimensions of the spaces in each degree. This result gives part of the information for establishing a recursive formula for the relevant Poincaré series; the following lemma completes the work. We use S and $V_{g,m}$ to denote the Poincaré series of $\mathcal{S}$ and $\mathcal{V}(g,m)$.

Lemma 4.13. *For $g \geq 1$, the Poincaré series K_{g+1} satisfies the recurrence relation*

$$t^{2n-1}K_{g+1} = 1 + t^{4n-1} + t^{2n-1}SV_{g,1} - K_g$$

with base case

$$K_1 = SV_{0,1}.$$

Proof. Given Lemma 4.12, we only need to determine in which degrees

$$\mathcal{V}(g,1)[2n-1] \xrightarrow{(q,r)\mapsto\delta q} \mathcal{K}_g$$

is surjective and ascertain the necessary correction terms for the Poincaré series when it is not. First, the same method as [3] in the proof of Lemma 6.2 to show surjectivity in all degrees that are at least 4 works to show surjectivity in the n odd case in all degrees greater than or equal to $4n$. We note that the argument relies on an operator H from Lemma 4.8. Though it is no longer a cochain nullhomotopy, it nevertheless satisfies the necessary relations for reasoning the hold.

Next, we consider degrees less than $4n$. The arguments made in degrees 0, 2 and 3 in the $n = 1$ case give the analogous results in degrees 0, $3n-1$, and $4n-1$ for any n odd. They are the following. In degree zero, $\mathcal{K}_g$ is generated by 1 and the vector space $\mathcal{S} \otimes \mathcal{V}(g,1)[2n-1]$ is clearly zero. In degree $3n-1$, $\mathcal{K}_g$ is generated by elements of the form $\tilde{c_i}$ with $c \in \{a,b\}$ and $1 \leq i \leq g$. Every such element has $(c_i, 0) \in \mathcal{V}(g,1)$ in its preimage. Next, the identical proof as in the $n = 1$ case shows that the modified exact sequence

$$0 \longrightarrow \mathcal{K}_{g+1,1}[2n-1] \longrightarrow \mathcal{S} \otimes \mathcal{V}(g,1)[2n-1] \longrightarrow \mathcal{K}_g/(\tilde{a_1}b_1 - a_1\tilde{b_1})$$

is surjective in degree $4n-1$; here, $|\tilde{a_1}b_1 - a_1\tilde{b_1}| = 4n-1$. Finally, we can see by inspection that $\mathcal{K}_g$ is zero in all other degrees less than $4n$. The recurrence relation now follows.

For the base case we consider the exact sequence from Lemma 4.12 with $g = 0$. The δ operator is now a zero map and $\mathcal{K}_0 = \mathbb{Q}$, so we have surjectivity of the rightmost map in every nonzero degree. Thus,

$$t^{2n-1}K_1 = 1 + t^{2n-1}SV_{0,1} - K_0$$

which simplifies to the desired equality. □

The next lemma lets us compute the Poincaré series $SV_{g,1}$ which appears in the formula of above.

Lemma 4.14. *For $g \geq 0$ and $m \geq 1$, we have the following isomorphism of graded vectorspaces:*

$$\mathcal{V}(g+1,m) \cong \mathcal{S} \otimes \big(\mathcal{V}(g,m+1) \oplus \mathcal{V}(g,m)^2[n] \oplus \mathcal{V}(g,m-1)[2n]\big)\,.$$

Proof. A generic homogeneous element of $\mathcal{S}\otimes\big(\mathcal{V}(g,m+1) \oplus \mathcal{V}(g,m)^2[n] \oplus \mathcal{V}(g,m-1)[2n]\big)$ can be written as

$$\tilde{a}^i\tilde{b}^j \otimes ((p_{ij},s_{ij}),(e_{ij},f_{ij}),(g_{ij},h_{ij}),(t_{ij},w_{ij}))$$

where we implicitly sum over (i,j). If i or j is negative, we declare p_{ij}, s_{ij}, e_{ij}, f_{ij}, g_{ij}, h_{ij}, t_{ij}, w_{ij} to be zero.

Define a map

$$\phi : \mathcal{V}(g+1,m) \to \mathcal{S} \otimes \big(\mathcal{V}(g,m+1) \oplus \mathcal{V}(g,m)^2[n] \oplus \mathcal{V}(g,m-1)[2n]\big)$$

which takes (q,r) to

$$\begin{aligned}
\tilde{a}^i\tilde{b}^j\otimes(&(q_{1,i,j}, \frac{n+1}{n} r_{1,i,j}),\\
&(q_{2,i+1,j} - q_{3,i+1,j} - \frac{1}{n} r_{1,i,j-1}, r_{2,i+1,j} + r_{3,i+1,j}),\\
&(q_{2,i,j+1} + q_{3,i,j+1} + \frac{1}{n} r_{1,i-1,j}, r_{2,i,j+1} - r_{3,i,j+1}),\\
&(nq_{4,i+1,j+1} + \Delta q_{1,i,j} - \frac{\delta}{n} r_{1,i,j} - 2r_{2,i,j}, \frac{n-1}{n}(\Delta r_{1,i,j} + nr_{4,i+1,j+1})))
\end{aligned}$$

with q_{kij} and r_{kij} given by the polarizations of q and r. The four tuples are valid elements of $\mathcal{V}(g,m+1)$, $\mathcal{V}(g,m)$, $\mathcal{V}(g,m)$ and $\mathcal{V}(g,m-1)$, respectively, due to the membership relations given in the proof of Lemma 6.3 in [3] and since $\mathcal{V}(g,m)$ is closed with respect to entry-wise addition. Now define

$$\psi : \mathcal{S} \otimes \big(\mathcal{V}(g,m+1) \oplus \mathcal{V}(g,m)^2[n] \oplus \mathcal{V}(g,m-1)[2n]\big) \to \mathcal{V}(g+1,m)$$

as the map which takes a homogeneous element to $(q,r) \in \mathcal{V}(g+1,m)$ specified by

$$\begin{aligned}
q_{1,i,j} &= p_{i,j}\\
q_{2,i,j} &= \frac{1}{2}(e_{i-1,j} + g_{i,j-1})\\
q_{3,i,j} &= \frac{1}{2}(g_{i,j-1} - e_{i-1,j}) - \frac{1}{n+1} s_{i-1,j-1}\\
q_{4,i,j} &= \frac{1}{n}(t_{i-1,j-1} - \Delta p_{i-1,j-1} - \frac{\delta}{n+1} s_{i-1,j-1} - f_{i-2,j-1} - h_{i-1,j-2})\\
r_{1,i,j} &= \frac{n}{n+1} s_{i,j}\\
r_{2,i,j} &= \frac{1}{2}(f_{i-1,j} + h_{i,j-1})\\
r_{3,i,j} &= \frac{1}{2}(f_{i-1,j} - h_{i,j-1})\\
r_{4,i,j} &= \frac{1}{n-1} w_{i-1,j-1} - \frac{1}{n+1}\Delta s_{i-1,j-1}
\end{aligned}$$

A direct computation shows that $\phi\psi$ and $\psi\phi$ are identities. □

The isomorphism gives the recursive formula

$$V_{g+1,m} = S(V_{g,m+1} + 2t^n V_{g,m} + t^{2n} V_{g,m-1})$$

for $g \geq 0$ and $m \geq 1$. The same proof as that of Corollary 6.4 in [3] gives the formula in closed form.

Corollary 4.15. *For nonnegative g and m, we have*

$$V_{g,m} = S^g(1 + t^{4n-1}) \left(\sum_{j=0}^{g+m-1} t^{jn} \left(\binom{2g}{j} - \binom{2g}{j-2m} \right) \right)$$

Proof of Theorem 4.2. We repeatedly apply the recurrence from Lemma 4.13. The first two iterations are:

$$\begin{aligned} K_g &= \frac{1 + t^{4n-1}}{t^{2n-1}} + SV_{g-1,1} - \frac{K_{g-1}}{t^{2n-1}} \\ &= \frac{1 + t^{4n-1}}{t^{2n-1}} + SV_{g-1,1} - \frac{1}{t^{2n-1}} \left(\frac{1 + t^{4n-1}}{t^{2n-1}} + SV_{g-2,1} - \frac{K_{g-2}}{t^{2n-1}} \right). \end{aligned}$$

The recursion ends at the base case $K_1 = SV_{0,1}$. Using this fact and combining terms that have factors of $1 + t^{4n-1}$, we get

$$K_g = \frac{1 + t^{4n-1}}{1 + t^{2n-1}} (1 - (-t^{2n-1})^{1-g}) + \sum_{j=0}^{g-1} \frac{SV_{j,1}}{(-t^{2n-1})^{g-1-j}}$$

for $g \geq 1$. The first summand has degree at most $2n$, so for $i \geq 2n+1$, $\dim_i(\mathcal{K}_g)$ is the coefficient of the degree i term of

$$\sum_{j=0}^{g-1} \frac{SV_{j,1}}{(-t^{2n-1})^{g-1-j}}.$$

Substituting for $V_{j,1}$ using Corollary 4.15, this sum becomes

$$\sum_{j=0}^{g-1} \sum_{l=0}^{j} (-1)^{g+j+1} S^{j+1} (1 + t^{4n-1})^{j+ln+1-g} \left(\binom{2j}{l} - \binom{2j}{l-2} \right).$$

For a fixed degree i, the contribution of the (j, l) summand to $\dim_i(\mathcal{K}_g)$ is $(-1)^{g+j+1} \left(\binom{2j}{l} - \binom{2j}{l-2} \right)$ multiplied by the degree $i + g - j - ln - 1$ coefficient of $(1 + t^{4n-1})S^{j+1}$. Based on the discussion following Theorem 4.1, we have the following expression for S^{j+1}:

$$S^{j+1} = \sum_{i \geq 0; i = c(3n-1)} \binom{2j + \frac{i}{3n-1} + 1}{2j+1} t^i.$$

We then compute that

$$(1+t^{4n-1})S^{j+1} = \sum_{\substack{i\geq 0\\ i=c(3n-1)}} \binom{2j+\frac{i}{3n-1}+1}{2j+1} t^i + \sum_{\substack{i\geq 4n-1\\ i=c(3n-1)+4n-1}} \binom{2j+\frac{i-4n+1}{3n-1}+1}{2j+1} t^i$$
$$= \sum_{\substack{i\geq 0\\ i=c(3n-1) \text{ or } i=c(3n-1)+4n-1}} \binom{2j+c+1}{2j+1} t^i$$

Substituting now gives the first part of Theorem 4.2. Next, when in degrees less than or equal to $2n$, we can see by inspection that $\mathcal{K}_g$ is generated by 1 in degree zero and is zero in all other degrees. An exception is when $n = 1$, in which case $\mathcal{K}_g$ is generated by elements of the form c_i, with $c \in \{a, b\}$ and $1 \leq i \leq g$. These observations give the rest of Theorem 4.2.

□

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