A COMBINATORIAL PROOF FOR A GENERALIZED RECIPROCITY THEOREM
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ALEXANDRA HOEY AND WANYI XIAO
MENTOR PAKAWUT JIRADILOK
PROJECT SUGGESTED BY ALEXANDER POSTNIKOV

Abstract. We study "proper pairings" for finite simple graphs. These are combinatorial objects that Huang and Postnikov used to give a bijective proof of Pak and Postnikov's reciprocity formula for the spanning forest polynomial f_G. We find that by introducing the "component graph" – a combinatorial object related to proper pairings – we are able to see new enumerative properties for these graph objects. As a result of our study, we give another combinatorial proof of the reciprocity theorem similar to Huang and Postnikov's. Furthermore, we generalize f_G to a polynomial $f_{G,H}$ that records the spanning trees of a graph that contain a fixed subgraph, and we show that these generalized polynomials exhibit a similar reciprocity property. As an application, we deduce a generalization of Cayley's formula from our generalized reciprocity.

1. Introduction

A spanning tree of a graph is a connected acyclic subgraph whose vertex set is that of the graph. Counting the number of labeled spanning trees of a simple finite graph is a well-studied problem in classical combinatorics. A. Cayley's formula states that there are n^{n-2} spanning trees of the complete graph K_n on n vertices [3], and the Matrix-Tree Theorem expresses the number of spanning trees of any connected graph G as the determinant of a matrix [2]. While these results can provide nice formulas for the number of spanning trees with labeled vertices of a graph, neither offers a way to list the spanning trees.

A famous algorithm for listing labeled spanning trees is due to H. Prüfer [6], whose coding provides a bijection between labeled spanning trees of K_n and length-$(n-2)$ sequences of vertices from the vertex set $[n] = \{1, 2, \ldots, n\}$. The Prüfer sequence for a given tree T contains each vertex v with multiplicity $\deg_T(v) - 1$, and A. Rényi introduced a graph polynomial t_G which records these multiplicities for each spanning tree of a given graph [7].

I. Pak and A. Postnikov modified Rényi's polynomial to record the degree sequences of spanning rooted forests of a graph G, rather than those of spanning trees [5]. This graph polynomial f_G exhibits a remarkable reciprocity property, which relates the spanning trees of a graph to those of its complement. Pak and Postnikov initially proved this reciprocity formula by an inductive algebraic argument.

S. Huang and A. Postnikov later gave a combinatorial proof of this reciprocity property by constructing a bijection which specializes to the Prüfer code [4]. This bijection allows each term in the reciprocity formula to be interpreted as a unique "proper pairing," a combinatorial object that defines a new spanning tree – the "replacement graph" – by specifying edges to add and delete in an initial spanning tree. Inclusion and exclusion of these replacement graphs yields the correct spanning trees of the complement graph.

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In this paper, we generalize the spanning forest polynomial f_G to a polynomial $f_{G,H}$, which records the spanning forests of a graph G that contain all edges in H. We then show that this polynomial exhibits a reciprocity property (Theorem 3.3) similar to the one that Pak and Postnikov proved for f_G. Our generalization of the spanning forest polynomial allows us to prove the generalized form of Cayley's formula (Corollary 3.4). In our proof of the generalized reciprocity property, we introduce a directed multigraph called the "component graph" as an alternative way to interpret the relationship between proper pairings and terms in the spanning tree polynomial. More specifically, we establish a constant-to-one map from a subset of proper pairings to each component graph (Proposition 5.5) and then compute the number of component graphs with labeled vertices (Proposition 5.4) to show that the terms in the reciprocity formula are exactly the monomials of the "replacement graphs" obtained from the operation specified by the proper pairings. Our interpretation of the terms in the generalized spanning forest polynomial $f_{G,H}$ may provide a way to give further insight into the geometry of spanning tree polytopes.

The structure of this paper is as follows: in section 2, we review some definitions and the properties of the graph polynomial f_G as discussed in [5]; in section 3, we give our generalized reciprocity formula; in section 4, we define and discuss proper pairings; in section 5, we define the component graph and use it to prove enumerative properties of proper pairings; in section 6, we give the combinatorial proof of the generalized reciprocity formula.

2. f_G and the Reciprocity Formula

We use standard graph definitions and notation, most of which can be found in [2]. In brief, the *degree* of a vertex v in a graph G (i.e. the number of edges adjacent to v in G) is denoted $\deg_G(v)$; the number of connected components of a graph G will be denoted $k(G)$; the *complete* graph on n vertices is denoted K_n and the *empty* graph E_n; and the *complement* of a graph G is denoted $\overline{G}$. Furthermore, a *spanning tree of a graph G* is a connected acyclic subgraph whose vertex set is that of the graph, and a *spanning tree on a set of n vertices* is a spanning tree of the complete graph on those vertices. From now on, we reserve the symbol G for a finite simple graph, and T for a spanning tree.

We now review the spanning forest polynomial f_G, its properties, and the reciprocity formula as presented in [5]. Consider a simple graph G with vertex set $V(G) = [n]$ and edge set $E(G)$. We define the *extended graph* $\widetilde{G}$ of G to be the simple graph on vertices $V(\widetilde{G}) = \{0\} \cup V(G)$ with edge set $E(\widetilde{G}) = E(G) \cup \{0i : i \in V(G)\}$. We associate a variable x_i to each vertex i (x is used in place of x_0) and assign each spanning tree T of $\widetilde{G}$ a monomial

$$m(T) := \prod_{i \in V(T)} x_i^{\deg_T(i)-1},$$

as is done in [5]. We now define the spanning forest polynomial f_G.

Definition 2.1. [5] For a graph G on $[n]$, the *spanning forest polynomial*

$$f_G(x; x_1, x_2, \ldots, x_n) := \sum_T m(T),$$

where the sum is over all spanning trees T of the extended graph $\widetilde{G}$.

Remark 2.2. Notice that, for a spanning tree T on vertices $\{0, 1, \ldots, n\}$, we can construct a spanning rooted forest F_T by deleting vertex 0 from T and designating all vertices adjacent to 0 in T as *roots* of F_T. From now on, we use the symbol F_T for the spanning rooted forest constructed from T. Thus, f_G is also a sum over all the spanning rooted forests of G.

This technique of constructing a rooted forest from a tree also appears in [2].

The graph polynomial f_G exhibits several useful properties. One relates the spanning forests of a disjoint union of two graphs to the spanning trees of each graph. Let G_1 and G_2 be two graphs such that $V(G_1) \cap V(G_2) = \varnothing$. We associate the variables $y_1, y_2, \dots, y_{n_1}$ and $z_1, z_2, \dots, z_{n_2}$ to the vertices of G_1 and G_2 respectively, and we associate the variable x to vertex 0 in $\widetilde{G_1}$, $\widetilde{G_2}$, and $\widetilde{G_1 \cup G_2}$. Then the following formula holds:

$$f_{G_1 \cup G_2}(x; y_1, \dots, y_{n_1}, z_1, \dots, z_{n_2}) = x \cdot f_{G_1}(x; y_1, \dots, y_{n_1}) \cdot f_{G_2}(x; z_1, \dots, z_{n_2}).$$

The proof can be found in [5].

Another important property of f_G is the reciprocity property between a graph and its complement, which was discovered by S. D. Bedrosian [1] in the case $x_1 = x_2 = \cdots = x_n = 1$.

Theorem 2.3 (Reciprocity)**.** [5] *Let G be a graph on $[n]$. Then*

$$f_G(x; x_1, \dots, x_n) = (-1)^{n-1} f_{\overline{G}}(-x - Y; x_1, \dots, x_n),$$

where $Y = x_1 + \cdots + x_n$.

One can find an algebraic proof for Theorem 2.3 in [5]. Additionally, Huang and Postnikov gave a combinatorial proof of Theorem 2.3 in [4], which we discuss in Remark 4.3.

3. $f_{G,H}$ and the Generalized Reciprocity Formula

We generalize the definition of f_G from [5] by introducing a polynomial $f_{G,H}$, which records the spannning trees of G that contain H as a subgraph.

Definition 3.1. For graphs G, H on $[n]$ with $E(H) \subseteq E(G)$, the *generalized spanning forest polynomial*

$$f_{G,H}(x; x_1, \dots, x_n) := \sum_T m(T),$$

where the sum is over all spanning trees T of $\widetilde{G}$ with $E(H) \subseteq E(T)$.

Remark 3.2. Notice that when H is not a forest, $f_{G,H}$ is zero. In the case $H = E_n$, f_{G,E_n} equals f_G, the spanning rooted forest polynomial in [5].

As with f_G, the generalized spanning forest polynomial $f_{G,H}$ exhibits a disjoint union property. Let G_1 and G_2 be two graphs such that $V(G_1) \cap V(G_2) = \varnothing$ and H_1, H_2 be graphs such that $V(H_1) = V(G_1)$, $E(H_1) \subseteq E(G_1)$, and $V(H_2) = V(G_2)$, $E(H_2) \subseteq E(G_2)$. We again associate the variables $y_1, y_2, \dots, y_{n_1}$ and $z_1, z_2, \dots, z_{n_2}$ to the vertices of G_1 and G_2 respectively, and we associate the variable x to vertex 0 in $\widetilde{G_1}$, $\widetilde{G_2}$, and $\widetilde{G_1 \cup G_2}$. Then the following formula holds:

$$f_{G_1 \cup G_2, H_1 \cup H_2}(x; y_1, \dots, y_{n_1}, z_1, \dots, z_{n_2}) = x \cdot f_{G_1,H_1}(x; y_1, \dots, y_{n_1}) \cdot f_{G_2,H_2}(x; z_1, \dots, z_{n_2}).$$

Notice that every spanning tree T such that $E(H_1 \cup H_2) \subseteq E(T)$ in graph $\widetilde{G_1 \cup G_2}$ splits into two spanning trees T_1 and T_2 in graphs $\widetilde{G_1}$ and $\widetilde{G_2}$ respectively. Since $\deg_T(0) - 1 = (\deg_{T_1}(0) - 1) + (\deg_{T_2}(0) - 1) + 1$, we need one additional x on the right hand side.

Furthermore, the polynomial $f_{G,H}$ exhibits a reciprocity property similar to the reciprocity formula in [5]. The following is the main result of this paper.

Theorem 3.3 (Generalized Reciprocity)**.** *Let G_1, G_2, H be graphs on $[n]$ with $G_1 \cup G_2 = K_n$ and $G_1 \cap G_2 = H$. Then*

$$f_{G_2,H}(x; x_1, \dots, x_n) = (-1)^{n-|E(H)|-1} f_{G_1,H}(-x - Y; x_1, \dots, x_n),$$

where $Y = x_1 + \cdots + x_n$.

We prove Theorem 3.3 in section 5. In the specialization $G_1 = G$ and $G_2 = \overline{G}$ (so that $H = E_n$), this result becomes Theorem 2.3. Additionally, our proof gives another combinatorial proof of Theorem 2.3.

A mild generalization of Cayley's formula [3] follows easily from Theorem 3.3.

Corollary 3.4. *Let F be an unrooted spanning forest on the vertex set $[n]$ with k components $c_1, \ldots, c_k$, and suppose component c_i contains n_i vertices. Then the number of spanning trees T of the complete graph K_n on $[n]$ such that $E(F) \subseteq E(T)$ is*

$$n^{k-2} \prod_{i=1}^{k} n_i.$$

Proof of Corollary 3.4. First, notice that

$$f_{G,H}(0; x_1, \ldots, x_n) = (x_1 + \cdots + x_n) \left(\sum_{T} m(T) \right),$$

where the sum is over all spanning trees T of G such that $E(H) \subseteq E(T)$. Consider a monomial of a spanning tree T' of $\overline{G}$, and note that the degree of x will be 0 in $m(T')$ if and only if the degree of vertex 0 is 1 in T'. Deleting this edge and vertex 0 will define a spanning tree T of G. Since 0 can be adjacent to any of vertices $\{1, \ldots, n\}$, we have the $(x_1 + \cdots + x_n)$ on the right hand side.

Consider an unrooted forest F on the vertex set $[n]$. Then the family of spanning trees T of $\widetilde{F}$ with $E(F) \subseteq E(T)$ is in bijective correspondence with the family of rooted forests F_T with underlying unrooted forest F. Therefore,

$$f_{F,F}(x; x_1, \ldots, x_n) = \sum_{(r_1, \ldots, r_k)} x^{k-1} \prod_{i=1}^{k} x_{r_i},$$

where, for each i, r_i is the root of component c_i. By Theorem 3.3 with $G_1 = F$ and $G_2 = K_n$, we see that

$$\begin{aligned} f_{K_n,F}(x; x_1, \ldots, x_n) &= (-1)^{|E(F)|+n-1} f_{F,F}(-x - Y; x_1, \ldots, x_n) \\ &= (x + Y)^{k-1} \sum_{(r_1, \ldots, r_k)} \prod_{i=1}^{k} x_{r_i}. \end{aligned}$$

where $Y = x_1 + \cdots + x_n$. Thus, we have

$$f_{K_n,F}(0; 1, \ldots, 1) = n^{k-1} \prod_{i=1}^{k} n_i.$$

Then the number of spanning trees T of K_n with $E(F) \subseteq E(T)$ is

$$n^{k-2} \prod_{i=1}^{k} n_i.$$

□

We note that Corollary 3.4 reduces to Cayley's formula when F is the empty graph on n vertices.

4. Proper Pairings

In this section, we discuss a combinatorial object called a proper pairing. In later sections, we will classify proper pairings into families $\mathcal{P}_{(T,\mathcal{V})}$, which will allow us to interpret the terms in $(-1)^{n-|E(H)|-1} f_{G_1,H}(-x-Y; x_1, \ldots, x_n)$ in Theorem 3.3.

We first define proper pairings. Consider a spanning tree T on vertices $\{0, 1, \ldots, n\}$.

Definition 4.1. Let n be a fixed positive integer.

(1) A *pairing* is a triple $P = (T, \mathcal{V}, \mathcal{S})$, where T is a spanning tree on vertices $\{0, 1, \ldots, n\}$, $\mathcal{V}$ is a multiset of $\rho := |\mathcal{V}|$ vertices from $[n]$, and $\mathcal{S} = \{(v_1, r_1), (v_2, r_2), \ldots, (v_\rho, r_\rho)\}$ is a set of ordered pairs of vertices such that $r_1, r_2, \ldots, r_\rho$ are distinct roots of F_T and $\{v_1, v_2, \ldots, v_\rho\} = \mathcal{V}$ as a multiset.
(2) Given a pairing $P = (T, \mathcal{V}, \mathcal{S})$, we call T its *tree*, $\mathcal{V}$ its *multiset*, and $\mathcal{S}$ its *pair set*. We define $\rho := |\mathcal{V}| = |\mathcal{S}|$ to be the *size* of the pairing.
(3) The *replacement graph* R_P of a pairing $P = (T, \mathcal{V}, \{(v_1, r_1), (v_2, r_2), \ldots, (v_\rho, r_\rho)\})$ is a multigraph obtained from T by, for each $i \in [\rho]$, deleting the edge $0r_i$ and adding $v_i r_i$.
(4) A pairing P is called a *proper pairing* if its replacement graph R_P is a spanning tree on $\{0, 1, \ldots, n\}$.

The following are some intuitive remarks about Definition 4.1.

Remark 4.2.

(1) Since the roots $r_1, \ldots, r_\rho$ are required to be distinct in the pair set $\{(v_1, r_1), \ldots, (v_\rho, r_\rho)\}$, the ρ elements of the pair set will be distinct even if some $v_i = v_j$ for $i \neq j$.
(2) The size ρ of a pairing $P = (T, \mathcal{V}, \mathcal{S})$ cannot exceed the number of roots of F_T because the roots $r_1, \ldots, r_\rho$ in $\mathcal{S}$ must be distinct.
(3) The replacement graph R_P will have the same number of edges, counted with multiplicity, as T.

Figure 1 gives an example of two pairings with the same tree T and multiset $\mathcal{V}$.

We discuss the bijection that Huang and Postnikov constructed in [4] in the following remark.

Remark 4.3. In this remark, we will use our notation instead of that used in [4]. Huang and Postnikov use an algorithm similar to the Prüfer coding. Their algorithm essentially associates each proper pairing $P = (T, \mathcal{V}, \mathcal{S})$ to an ordered pair (T, W). Here, W is a permutation of the set $\mathcal{V} \cup \{0\}_{k(F_T)-1-\rho}$, where $\{0\}_{k(F_T)-1-\rho}$ denotes the multiset of $k(F_T)-1-\rho$ zeros. Using this bijection, Huang and Postnikov were able to interpret each monomial in $(-1)^{n-1} f_{\overline{G}}(-x-Y; x_1, \ldots, x_n)$ as a proper pairing. More details about the bijection are provided in Appendix A.

5. Classification of Proper Pairings by Component Graphs

In this section, we classify proper pairings by their trees and multisets, which will allow us to interpret terms in $(-1)^{n-|E(H)|-1} f_{G_1,H}(-x-Y; x_1, \ldots, x_n)$.

Let $\mathcal{P}_{(T,\mathcal{V})}$ denote the family of all proper pairings with tree T and multiset $\mathcal{V}$. Theorem 5.1 gives the key enumerative property needed for our interpretation of $(-1)^{n-|E(H)|-1} f_{G_1,H}(-x-Y; x_1, \ldots, x_n)$.

Theorem 5.1. *Suppose T is a spanning tree on $\{0, 1, \ldots, n\}$. Let $k := k(F_T)$, and let $\mathcal{V}$ be a multiset of ρ vertices from $[n]$ so that, for each $i \in [n]$, vertex i is listed with multiplicity ρ_i. Then*

$$|\mathcal{P}_{(T,\mathcal{V})}| = \binom{k-1}{\rho} \cdot \binom{\rho}{\rho_1, \rho_2, \ldots, \rho_n}.$$

A spanning tree T on vertices $\{0, 1, \ldots, 4\}$
and a multiset $\mathcal{V} = \{2, 3\}$

Ex. 1. $P_1 = (T, \mathcal{V}, \{(2, 1), (3, 4)\})$ has R_{P_1}

Ex. 2. $P_2 = (T, \mathcal{V}, \{(2, 4), (3, 2)\})$ has R_{P_2}

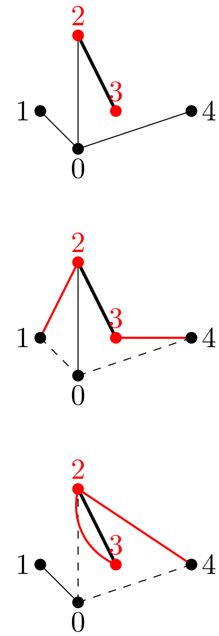


FIGURE 1. For a spanning tree T and a multiset $\mathcal{V} = \{2, 3\}$, we give two examples of pairings with their replacement graphs. The dashed lines in the replacement graphs indicate deleted edges, and the red lines indicate added edges. The red vertices are the vertices in $\mathcal{V}$. Note that P_1 is a proper pairing; P_2 is not.

This theorem follows from Propositions 5.4 and 5.5 below.

Consider a spanning tree T on vertices $\{0, 1, \ldots, n\}$. The connected components of F_T canonically induce a set partition of $[n]$, which we will denote by $B(F_T)$. Let the map $h : [n] \to B(F_T)$ send each vertex $v \in [n]$ to the component of v in F_T. Instead of counting $|\mathcal{P}_{(T,\mathcal{V})}|$ directly, we introduce a new graph called the component graph for each pairing P.

Definition 5.2. Suppose $P = (T, \mathcal{V}, \{(v_1, r_1), (v_2, r_2), \ldots, (v_\rho, r_\rho)\})$ is a pairing. Then the *component graph* C_P of P is the directed multigraph whose

(1) vertex set $V(C_P) = B(F_T)$, and
(2) directed edge set $E(C_P) = \{h(v_i)h(r_i) : i \in [\rho]\}$.

Figure 2 gives the component graphs for the proper pairing examples in Figure 1. Note that the red edges in the replacement graphs, which are the edges added due to the pair set, become directed edges in the component graphs. However, the black edges, which are the edges in F_T, are no longer present in the component graph.

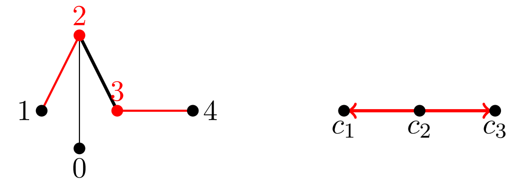
Let $\mathcal{C}_{(T,\mathcal{V})}$ denote the family of component graphs for all proper pairings in $\mathcal{P}_{(T,\mathcal{V})}$. We will count $|\mathcal{C}_{(T,\mathcal{V})}|$ and present a numerical relationship between $|\mathcal{C}_{(T,\mathcal{V})}|$ and $|\mathcal{P}_{(T,\mathcal{V})}|$ to prove Theorem 5.1.

We define a *cycle* in a (directed) multigraph to be any closed trail, regardless of edge direction, with exactly one repeated vertex (i.e. the start and finish vertex). In particular, a loop will be considered a cycle.

Proposition 5.3. *Suppose T is a spanning tree on $\{0, 1, \ldots, n\}$, and let $\mathcal{V}$ be a multiset of ρ vertices from $[n]$ so that, for each $i \in [n]$, vertex i is listed with multiplicity ρ_i.*

Consider a directed multigraph H on vertex set $B(F_T)$. Then $H \in \mathcal{C}_{(T,\mathcal{V})}$ if and only if all of the the following conditions are satisfied:

Ex. 1. $P_1 = (T, \mathcal{V}, \{(2,1),(3,4)\})$ has R_{P_1} and C_{P_1}



Ex. 2. $P_2 = (T, \mathcal{V}, \{(2,4),(3,2)\})$ has R_{P_2} and C_{P_2}

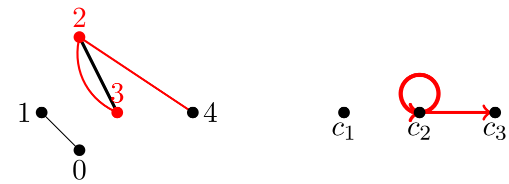


FIGURE 2. We construct the component graphs for the pairings P_1 and P_2 shown in Figure 1. The vertices c_1, c_2, c_3 of the component graph are $\{1\}, \{2,3\}, \{4\}$ respectively.

(1) *Exactly ρ vertices in H have indegree* 1*, and all other vertices have indegree* 0*.*
(2) *For each $b \in B(F_T)$, the outdegree of vertex b in H is* $\operatorname{outdeg}_H(b) = \sum_{i \in b} \rho_i$.
(3) *H contains no cycles.*

Proof. Suppose $H \in \mathcal{C}_{(T,\mathcal{V})}$, i.e. H is the component graph of some proper pairing of the form $P = (T, \mathcal{V}, \{(v_1, r_1), \ldots, (v_\rho, r_\rho)\})$. Points (1) and (2) are straightforward. For (3), suppose to the contrary that H contained a cycle $b_1 b_2 \cdots b_m$ of length $m \geq 1$, where $b_1, \ldots, b_m \in B(F_T)$ are distinct components of F_T. Note that the induced subgraph of T on b_i is a tree on $|b_i|$ vertices and therefore has $|b_i| - 1$ edges. The subgraph of R_P induced on $\bigcup_{i=1}^m b_i$ has "original edges" from T and "new edges" from P. We know that there are at least m new edges: those which appear as edges in the cycle $b_1 \cdots b_m$. Therefore, the subgraph has at least

$$\left(\sum_{i=1}^{m} |b_i| - 1 \right) + m = \sum_{i=1}^{m} |b_i| = \left| \bigcup_{i=1}^{m} b_i \right|$$

edges, which contradicts the fact that R is a tree.

Conversely, suppose that all three properties hold for some directed graph H. We construct a proper pairing $P = (T, \mathcal{V}, \mathcal{S})$ with component graph H by assigning an ordered pair $(v(e), r(e))$ of vertices in $[n]$ to each edge $e \in E(H)$ as follows:

(1) $v(e)$: For each edge e from the set of $\operatorname{outdeg}_H(b)$ edges leaving $b \in V(H)$, we assign $v(e)$ to be a vertex in component b so that, after some $v(e)$ has been assigned to each edge, the set $\{e' : v(e') = j\}$ contains ρ_j elements.
(2) $r(e)$: To each directed edge $e = h(v(e))h(r_i)$ in H, we assign the root r_i as $r(e)$.

(We note that, although in general there are many ways to execute step 1, there is exactly one way to execute step 2.) If we let $\mathcal{S} := \{(v(e), r(e)) : e \in E(H)\}$, then this assignment process defines a pairing $P = (T, \mathcal{V}, \mathcal{S})$ whose component graph is H.

We now show that P is a proper pairing, i.e. that its replacement graph R_P is a tree. Because R_P contains the same number of edges as T, it suffices to check that R_P is acyclic. Suppose to the contrary that R_P had a cycle. Since the rooted forest F_T contained no cycles, the cycle must include at least one edge in $E(R_P - \{0\}) - E(F_T)$. When all components of F_T are contracted within R_P to form the underlying undirected graph of H, this cycle will become a closed trail with at least one edge, which contradicts condition (3). Hence, P is a proper pairing, so $H \in \mathcal{C}_{(T,\mathcal{V})}$. □

As we see in Figure 2, C_{P_1} satisfies all conditions in Proposition 5.3, and C_{P_2} does not.

Using the characterization of $\mathcal{C}_{(T,\mathcal{V})}$ from Proposition 5.3, we now count $|\mathcal{C}_{(T,\mathcal{V})}|$ in the following proposition.

Proposition 5.4. *Let T, $\mathcal{V} = \{v_1, \ldots, v_\rho\}$, ρ, ρ_i, and k be defined as in the previous proposition. Label the k components of F_T as $c_1, \ldots, c_k$ (so that $\{c_1, \ldots, c_k\} = B(F_T)$). For each $i \in [k]$, let $q_i := \sum_{j \in c_i} \rho_j$.*

Then $|\mathcal{C}_{(T,\mathcal{V})}| = \frac{(k-1)!}{(k-\rho-1)!} \cdot \frac{1}{q_1! \cdots q_k!}$.

Proof. It suffices to count the number of directed graphs H on $\{c_1, \ldots, c_k\}$ that satisfy the three requirements in Proposition 5.3. Let H_0 denote the empty graph on vertices $\{c_1, \ldots, c_k\}$. For each vertex v_i in $\mathcal{V}$, we construct a new directed multigraph H_i by adding a directed edge $h(v_i)F(v_i)$ to H_{i-1}, where $F(v_i)$ is some other vertex in $\{c_1, \ldots, c_k\}$, so that H_i is acyclic and each vertex of H_i has indegree at most 1. For a given $i \in [\rho]$, we call $F(v_i)$ the "finish vertex," and we will count the number of ways to assign distinct finish vertices $F(v_i)$ for $i \in [\rho]$ to create H_i. (Figure 3 gives an example of this assignment process.)

We claim that, after $F(v_1), \ldots, F(v_{i-1})$ have been designated (so that H_{i-1} has been constructed), there are exactly $k - i$ legal choices for $F(v_i)$: all k vertices $c_1, \ldots, c_k$ except

(1) $F(v_1), \ldots, F(v_{i-1})$, which were already used as finish vertices, and
(2) One additional vertex $h(v_{\gamma_m})$, where γ_m is the last value on the list defined by the algorithm
   (a) Start with $\gamma = i$ (so that $\gamma_0 = i$). Set a counter $m = 0$.
   (b) Loop:
      (i) Store the current value of γ as γ_m.
      (ii) If $h(v_\gamma) \notin \{F(v_1), \ldots, F(v_{i-1})\}$, RETURN the list $\gamma_0, \gamma_1, \ldots, \gamma_m$, and TERMINATE.
      (iii) Else, $h(v_\gamma) = F(v_{\gamma'})$ for some $\gamma' \in [i-1]$. Increment m, reassign γ to be γ', and repeat the loop.

   (Assuming there were no cycles formed from edges in $\{h(v_1)F(v_1), \ldots, h(v_{i-1})F(v_{i-1})\}$, the algorithm will terminate.)

   More intuitively, $h(v_{\gamma_m})$ is the vertex "at the start" of the component of $h(v_i)$ in H_{i-1}. If $h(v_{\gamma_m})$ were used as $F(v_i)$, then the edge $h(v_{\gamma_0})h(v_{\gamma_m})$ would close a cycle with other edges $h(v_{\gamma_0})F(v_{\gamma_0}), h(v_{\gamma_1})F(v_{\gamma_1}), \ldots, h(v_{\gamma_m})F(v_{\gamma_m})$.

   By construction, $h(v_{k_m})$ is not on the list in (1).

We claim that, for any of the $k-i$ choices for $F(v_i)$ described above, adding the edge $h(v_i)F(v_i)$ to H_{i-1} will not create a cycle in H_i. Assume that H_{i-1} was acyclic, and suppose to the contrary that adding edge $h(v_i)F(v_i)$ created a cycle in H_i. Then the cycle would contain the new edge $h(v_i)F(v_i)$, so $F(v_i) = h(v_j)$ for some edge $h(v_j)F(v_j)$ ($j \leq i$) in the component of $h(v_i)$ in H_{i-1}, where $h(v_j)$ was not previously used as a finish vertex. The only such vertex $h(v_j)$ is $h(v_{k_m})$, which is forbidden by the above algorithm. We have shown that there are $k - i$ legal choices for $F(v_i)$, so there are $\frac{(k-1)!}{(k-\rho-1)!}$ ways to assign finish vertices to the edges $h(v_1)F(v_1), \ldots, h(v_\rho)F(v_\rho)$ in that order. To count $|\mathcal{C}_{(T,\mathcal{V})}|$, we need to divide by $\prod_{i \in [n]} q_i!$ since, for each i, the q_i edges leaving vertex c_i are indistinguishable. Therefore, $|\mathcal{C}_{(T,\mathcal{V})}| = \frac{(k-1)!}{(k-\rho-1)!} \cdot \frac{1}{q_1! \cdots q_k!}$ as desired. □

Now that we have computed $|\mathcal{C}_{(T,\mathcal{V})}|$, we compute $|\mathcal{P}_{(T,\mathcal{V})}|$ by defining a map $g : \mathcal{P}_{(T,\mathcal{V})} \to \mathcal{C}_{(T,\mathcal{V})}$ that sends each proper pairing $P \mapsto C_P$.

Proposition 5.5. *Let k, ρ_i, and q_i be defined as in the previous proposition. The map $g : \mathcal{P}_{(T,\mathcal{V})} \to \mathcal{C}_{(T,\mathcal{V})}$ is a surjective $\frac{q_1! q_2! \cdots q_k!}{\rho_1! \rho_2! \cdots \rho_n!}$-to-1 map.*

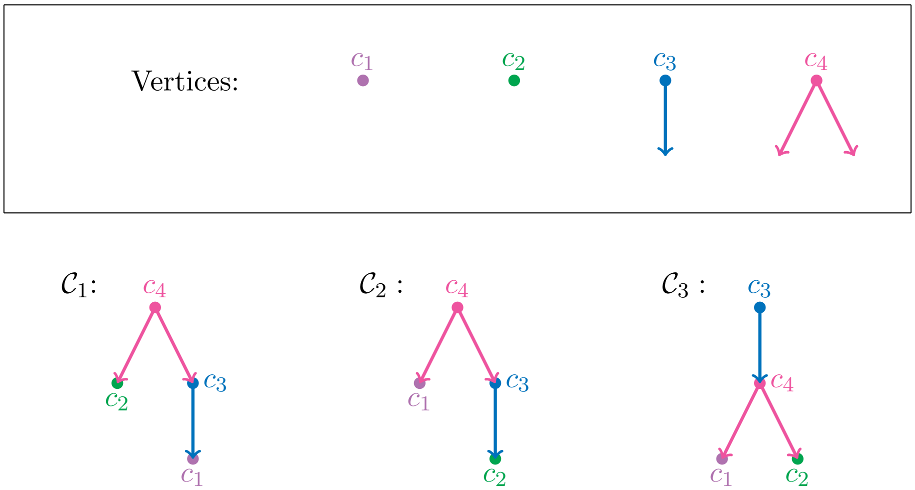


FIGURE 3. As in the proof of Proposition 5.4, we construct the three component graphs $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ for the $(T, \mathcal{V})$ shown in Figure 4 by assigning "finish vertices" to vertices c_3, c_4. Note that the two edges leaving c_4 are considered indistinguishable.

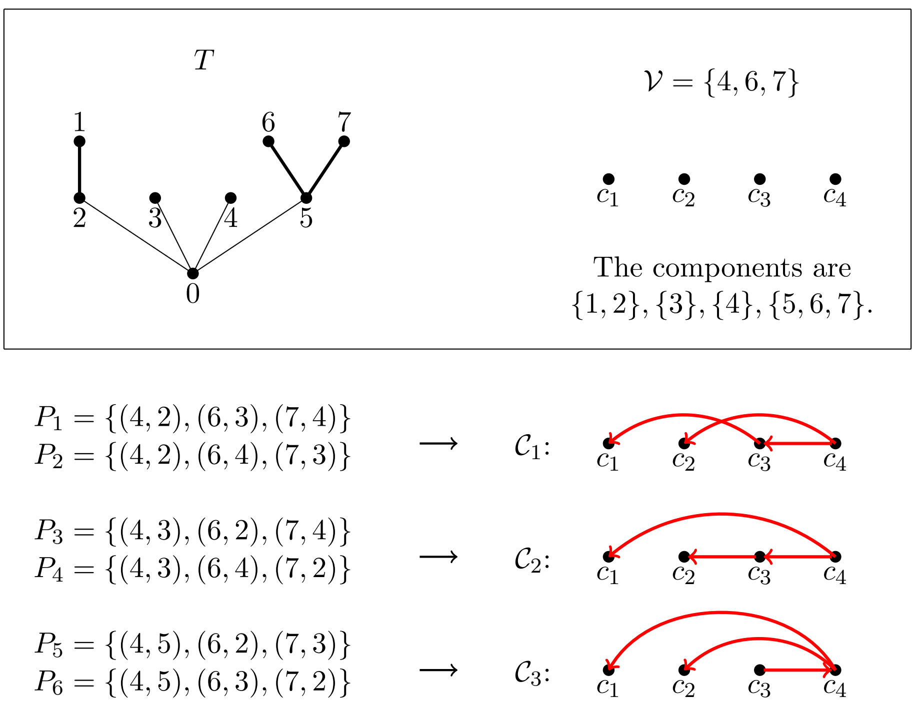


FIGURE 4. $k = 4$, $\rho = 3$, $(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7) = (0, 0, 0, 1, 0, 1, 1)$, $(q_1, q_2, q_3, q_4) = (0, 0, 1, 2)$. The map g is a surjective 2-to-1 map, and there are 3 component graphs in $\mathcal{C}_{(T,\mathcal{V})}$.

Proof. Consider a component graph $C_P \in \mathcal{C}_{(T,\mathcal{V})}$. We label the components of F_T as $\{c_1, \ldots, c_k\}$ as before. We construct all $\frac{q_1! q_2! \cdots q_k!}{\rho_1! \rho_2! \cdots \rho_n!}$ proper pairings $P = (T, \mathcal{V}, \mathcal{S})$ in $g^{-1}(C_P)$ by assigning an ordered pair (v_i, r_i) to each directed edge $c_j c_i$ in C_P so that $h(v_i) = c_j$ and the multiset $\{v_1, \ldots, v_\rho\} = \mathcal{V}$ (in some order). We note that each ordered pair's second element r_i is determined by the corresponding edge's endpoint c_i (because there is only one root in each component of F_T), so it suffices to assign only the first vertex v_i to each edge.

For each $j \in [k]$, the outdegree of c_j is q_j. Since all directed edges in C_P have distinct finish vertices, there are

$$\frac{q_j!}{\prod_{i \in c_j} \rho_i!}$$

ways to assign the q_j edges leaving the vertex $c_j \in C_P$ to the q_j vertices in the component C_j of r_j in F_T, noting that the ρ_i copies of the same vertex $i \in \mathcal{V}$ should not be distinguished during the assignment process. It follows that there are $\frac{q_1!q_2!\cdots q_k!}{\rho_1!\rho_2!\cdots\rho_n!}$ such assignments possible, so there are $\frac{q_1!q_2!\cdots q_k!}{\rho_1!\rho_2!\cdots\rho_n!}$ pairings with component graph C_P. Because we began with a component graph C_P, Proposition 5.3 shows that each of these pairings is proper. □

Figure 4 shows an example of the map g. Since $\frac{q_1!q_2!q_3!q_4!}{\rho_1!\rho_2!\rho_3!\rho_4!\rho_5!\rho_6!\rho_7!} = \frac{0!0!1!2!}{0!0!0!1!0!1!1!} = 2$, we see that g is a 2-to-1 map. Furthermore, since $\frac{(k-1)!}{(k-\rho-1)!} \cdot \frac{1}{q_1!q_2!q_3!q_4!} = \frac{(4-1)!}{(4-3-1)!} \cdot \frac{1}{0!0!1!2!} = 3$, there are 3 component graphs in $\mathcal{C}_{(T,\mathcal{V})}$. As Theorem 5.1 asserts, there are 6 proper pairings in $\mathcal{P}_{(T,\mathcal{V})}$.

In the next section, we show how Theorem 5.1 allows us to interpret the terms in Theorem 3.3.

6. Proof of Generalized Reciprocity

Definition 6.1. For a graph G on $[n]$ and a spanning subgraph H of G,

1. Let $S_i(G,H)$ denote the family of spanning trees T of $\widetilde{G}$ with $H \subseteq T$ and $\deg_T(0) = i+1$.
2. Define a degree-$(n-i-1)$ polynomial in $x_1, \ldots, x_n$ by
$$a_i(G,H) = x^{-i} \sum_{T \in S_i(G,H)} m(T).$$
Then $f_{G,H}(x; x_1, \ldots, x_n) = a_{n-1}(G,H)x^{n-1} + a_{n-2}(G,H)x^{n-2} + \ldots + a_0(G,H)$.
3. Let $Y := x_1 + \cdots + x_n$.

We note that if H is not a forest, both $S_i(G,H)$ and $a_i(G,H)$ are zero.

In the following proposition, we describe the monomials of the replacement graph of the proper pairings in $\mathcal{P}_{(T,\mathcal{V})}$.

Proposition 6.2. *Suppose i,j are integers such that $0 \le j \le i \le n-1$.*

1. *For $T \in S_i(G,H)$ and $\mathcal{V}$ such that $|\mathcal{V}| = i-j$,*
$$\sum_{P \in \mathcal{P}_{(T,\mathcal{V})}} m(R_P) = \binom{i}{j}\binom{i-j}{\rho_1, \ldots, \rho_n} x^{-(i-j)} m(T) \prod_{v \in [n]} x_v^{\rho_v},$$
where $\rho_1, \rho_2, \ldots, \rho_n$ are the multiplicities of vertices $1, 2, \ldots, n$ in $\mathcal{V}$.
2. *Then*
$$x^j a_i(G,H)\binom{i}{j} Y^{i-j} = \sum_{T \in S_i(G,H)} \sum_{\substack{\mathcal{V} \\ |\mathcal{V}|=i-j}} \sum_{P \in \mathcal{P}_{(T,\mathcal{V})}} m(R_P).$$

Proof. Fix i,j such that $0 \le j \le i \le n-1$, and let $T \in S_i(G,H)$. First consider a fixed multiset $\mathcal{V}$ of $i-j$ vertices, in which each vertex $v \in [n]$ is listed with multiplicity ρ_v. From the definition of the replacement graph, we observe the following:

1. The degree of x in $\frac{m(R_P)}{m(T)}$ is $-|\mathcal{V}| = -(i-j)$.

(2) The degree of x_i in $\frac{m(R_P)}{m(T)}$ is ρ_i, the multiplicity of vertex i in $\mathcal{V}$.

Then, we see that the monomial of R_P for any proper pairing $P \in \mathcal{P}_{(T,\mathcal{V})}$ will be

$$m(R_P) = x^{-(i-j)} m(T) \prod_{v\in[n]} x_v^{\rho_v}.$$

By Theorem 5.1, there are $\binom{i}{i-j}\binom{i-j}{\rho_1,\dots,\rho_n} = \binom{i}{j}\binom{i-j}{\rho_1,\dots,\rho_n}$ proper pairings in $\mathcal{P}_{(T,\mathcal{V})}$, and the replacement graph of each has the same monomial. This proves (1). Figure 5 is an example of (1).

For (2), we sum over all $T \in S_i(G,H)$ and $\mathcal{V}$ with $|\mathcal{V}| = i-j$:

$$\begin{aligned}
\sum_{T\in S_i(G,H)} \sum_{\substack{\mathcal{V} \\ |\mathcal{V}|=i-j}} \sum_{P\in\mathcal{P}_{(T,\mathcal{V})}} m(R_P) &= \sum_{T\in S_i(G,H)} \left[\sum_{\mathcal{V}:|\mathcal{V}|=i-j} \binom{i}{j}\binom{i-j}{\rho_1,\dots,\rho_n} x^{-(i-j)} m(T) \prod_{v\in[n]} x_v^{\rho_v} \right] \\
&= x^{-(i-j)} \binom{i}{j} \sum_{T\in S_i(G,H)} \left[m(T) \sum_{\rho_1+\dots+\rho_n=i-j} \binom{i-j}{\rho_1,\dots,\rho_n} \prod_{v\in[n]} x_v^{\rho_v} \right] \\
&= x^j \left[x^{-i} \sum_{T\in S_i(G,H)} m(T) \right] \binom{i}{j} \sum_{\rho_1+\dots+\rho_n=i-j} \binom{i-j}{\rho_1,\dots,\rho_n} \prod_{v\in[n]} x_v^{\rho_v} \\
&= x^j a_i(G,H) \binom{i}{j} \sum_{\rho_1+\dots+\rho_n=i-j} \binom{i-j}{\rho_1,\dots,\rho_n} \prod_{v\in[n]} x_v^{\rho_v} \\
&= x^j a_i(G,H) \binom{i}{j} Y^{i-j}.
\end{aligned}$$

□

Proposition 6.2 provides an interpretation for the monomials of the replacement graphs. We now prove Theorem 3.3.

The proof of Theorem 3.3 relies on the following lemma.

Lemma 6.3. *Given a spanning tree T' on vertices $\{0, 1, \dots, n\}$ and an unrooted forest F in G such that $E(F) \subseteq E(T')$, there is exactly one proper pairing $P = (T, \mathcal{V}, \mathcal{S})$ such that*

(1) *F is the underlying unrooted forest of the rooted forest F_T.*
(2) *The replacement graph R_P is T'.*

Proof. Suppose T' is a spanning tree on $\{0, 1, \dots, n\}$ and F an unrooted forest in G such that $E(F) \subseteq E(T')$. Define the set $E(T', F) := E(F_{T'}) - E(F)$.

We prove by induction on $|E(T', F)|$ that, given T' and F, exactly one proper pairing satisfies properties (1) and (2) in the lemma. If $E(T', F) = \varnothing$, then the only proper pairing satisfying (1) and (2) is $(T', \varnothing, \varnothing)$. Now suppose that the result holds whenever $|E(T', F)| < j$, and consider $|E(T', F)| = j$. We claim that there exists a component $\hat{C}$ of F that contains a vertex x incident to an edge in $E(T', F)$ and a vertex y adjacent to 0 in T'. If there were no such component $\hat{C}$, then each component of F would be one of two types:

(1) No vertex in the component is adjacent to 0.
(2) No vertex in the component is incident to any edge in $E(T', F)$.

Note that not all components are in type 1 because 0 cannot be an isolated point in T', and not all components are in type 2 because $E(T', F) \neq \varnothing$. However, there are no edges in F between any

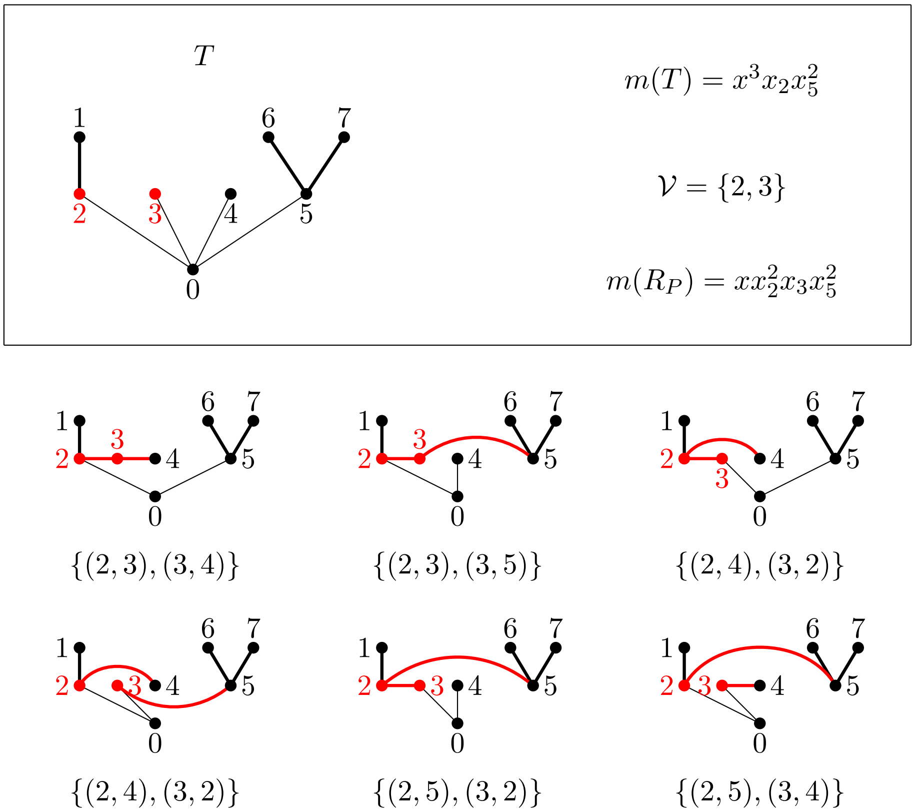


FIGURE 5. This figure shows the replacement graphs of proper pairings in $\mathcal{P}_{(T,\mathcal{V})}$. Each replacement graph has the monomial $x^{-2}m(T)x_2x_3 = xx_2^2x_3x_5^2$, and there are $\binom{3}{2}\binom{2}{0,1,1,0,0,0,0} = 6$ of them in $\mathcal{P}_{(T,\mathcal{V})}$.

component in type 1 and any component in type 2, so the components in type 2 are not connected to 0 in T', a contradiction.

We will now identify the unique proper pairing $P = (T, \mathcal{V}, \mathcal{S})$ for T' and F. Select a component $\hat{C}$ of F such that $x \in \hat{C}$ is incident to an edge $xx' \in E(T', F)$ and $y \in \hat{C}$ is adjacent to 0 in T'. Note that $x' \notin \mathcal{V}$ because then $0x \in E(T)$, which would create a 3-cycle $0yx$ in T. Thus $x \in \mathcal{V}$, and the ordered pair $(x, x') \in \mathcal{S}$.

We delete edge xx' in T' and add edge $0x'$ to get a new tree T'° whose set $E(T'^{\circ}, F) := E(F_{T'^{\circ}}) - E(F)$ has order $|E(T'^{\circ}, F)| = |E(T', F)| - 1$. By the inductive hypothesis, there is exactly one proper pairing $P^{\circ} = (T^{\circ}, \mathcal{V}^{\circ}, \mathcal{S}^{\circ})$ such that

(1) F is the underlying unrooted forest of $F_{T^{\circ}}$.
(2) The replacement graph $R_{P^{\circ}} = T'^{\circ}$.

Then the unique proper pairing for T' and F must be $P = (T^{\circ}, \mathcal{V}^{\circ} \cup \{x\}, \mathcal{S}^{\circ} \cup \{(x, x')\})$. $\square$

We use this lemma to prove Theorem 3.3.

Proof of Theorem 3.3. We expand each $(x+Y)^i$ term in $(-1)^{n-|E(H)|-1}f_{G_1,H}(-x-Y;x_1,\dots,x_n)$ and apply Proposition 6.2 to get

$$\begin{aligned}(-1)^{n-|E(H)|-1}&f_{G_1,H}(-x-Y;x_1,\dots,x_n)\\ &=(-1)^{-|E(H)|}\Big[a_{n-1}(G_1,H)(x+Y)^{n-1}-\cdots\pm a_0(G_1,H)\Big]\\ &=(-1)^{-|E(H)|}\sum_{j=0}^{n-1}x^j\sum_{i=j}^{n-1}(-1)^{n-i-1}a_i(G_1,H)\binom{i}{j}Y^{i-j}\\ &=\sum_{j=0}^{n-1}\sum_{i=j}^{n-1}(-1)^{n-i-1-|E(H)|}\left[\sum_{T\in S_i(G_1,H)}\sum_{\substack{\mathcal{V}\\|\mathcal{V}|=i-j}}\sum_{P\in\mathcal{P}_{(T,\mathcal{V})}}m(R_P)\right].\end{aligned}$$

We note that a spanning tree $T\in S_i(G,H)$ has $i+1$ edges incident to vertex 0 and $n-i-1$ edges not incident to 0. Since $|E(H)|$ of the edges not incident to 0 are edges of H, we see that $a_i(G_1,H)\neq 0$ iff $n-i-1\geq |E(H)|$. In particular, $a_i(G_1,H)=0$ for all $i>n-|E(H)|-1$, so the previous expression can be rewritten as

$$(*)\qquad \begin{aligned}(-1)^{n-|E(H)|-1}&f_{G_1,H}(-x-Y;x_1,\dots,x_n)\\ &=\sum_{j=0}^{n-1}\sum_{i=j}^{n-|E(H)|-1}(-1)^{n-i-1-|E(H)|}\left[\sum_{T\in S_i(G,H)}\sum_{\substack{\mathcal{V}\\|\mathcal{V}|=i-j}}\sum_{P\in\mathcal{P}_{(T,\mathcal{V})}}m(R_P)\right].\end{aligned}$$

For a given replacement graph R_P, we show using Lemma 6.3 that the coefficient of $m(R_P)$ in the above expression is 1 if every edge of R_P is in G_2 (i.e. no edge of R_P is in $G_1-E(H)$) and 0 otherwise.

By Lemma 6.3, the underlying unrooted forest F of F_T and a replacement graph T' together determine the proper pairing P, so we can reindex the triple sum from equation $(*)$ as

$$\sum_{T\in S_i(G_1,H)}\sum_{\substack{\mathcal{V}\\|\mathcal{V}|=i-j}}\sum_{P\in\mathcal{P}_{(T,\mathcal{V})}}m(R_P)=\sum_F\sum_{T'}m(T'),$$

where F is the underlying unrooted forest of F_T for $T\in S_i(G_1,H)$ and T' is a spanning tree on $\{0,1,\dots,n\}$ such that $E(F)\subseteq E(T')$ with exactly $j+1$ edges incident to 0. Consider a particular spanning tree T' with $j+1$ edges incident to 0 and $b(T')$ edges in G_1-H. Then F must have had $n-i-1-|E(H)|$ edges in G_1-H, so there are $\binom{b(T')}{n-i-1-|E(H)|}$ choices of F that give replacement tree T'. It follows that

$$\begin{aligned}(-1)^{n-i-1-|E(H)|}&\sum_{T\in S_i(G_1,H)}\sum_{\substack{\mathcal{V}\\|\mathcal{V}|=i-j}}\sum_{P\in\mathcal{P}_{(T,\mathcal{V})}}m(R_P)\\ &=\sum_{T'}(-1)^{n-i-1-|E(H)|}\binom{b(T')}{n-i-1-|E(H)|}m(T'),\end{aligned}$$

where the sum on the right hand side is over all spanning trees T' on $\{0,1,\dots,n\}$ containing all the edges in H and exactly $j+1$ edges incident to 0.

From the above discussion, we see that $(-1)^{n-|E(H)|-1} f_{G_1,H}(-x-Y; x_1, \ldots, x_n)$ can be expressed as

$$\sum_{j=0}^{n-1} \sum_{i=j}^{n-|E(H)|-1} \sum_{T'} \binom{b(T')}{n-i-1-|E(H)|} m(T'),$$

where T' is again a spanning tree on $\{0, 1, \ldots, n\}$ such that $E(H) \subseteq E(T')$ with exactly $j+1$ edges incident to 0. Exchanging the first two summations and summing over all T' such that $E(H) \subseteq E(T')$ (note that the number of edges incident to 0 is no longer fixed to be $j+1$), we see that

$$\begin{aligned}
&(-1)^{n-|E(H)|-1} f_{G_1,H}(-x-Y; x_1, \ldots, x_n) \\
&\quad = \sum_{i=0}^{n-1-|E(H)|} \sum_{T'} (-1)^{n-i-1-|E(H)|} \binom{b(T')}{n-i-1-|E(H)|} m(T') \\
&\quad = \sum_{T'} \left[\sum_{i=0}^{n-1-|E(H)|} (-1)^{n-i-1-|E(H)|} \binom{b(T')}{n-i-1-|E(H)|} \right] m(T') \\
&\quad = \sum_{\substack{T'\text{: spanning} \\ \text{tree of } G_2 \\ \text{s.t. } E(H) \subseteq E(T')}} m(T').
\end{aligned}$$

Notice that the coefficient of $m(T')$ for spanning trees $T' \notin G_2$ becomes 0 due to cancellation. Thus, we are left with the monomials of the spanning trees of G_2 that contains H as a subgraph, which is by definition $f_{G_2,H}(x; x_1, \ldots, x_n)$. This finishes the proof of Theorem 3.3. □

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References

[1] S. D. Bedrosian, *Generating formulas for the number of trees in a graph*, J. Franklin Inst. **227** (1964), no. 4, 313–326.

[2] B. Bollobás, *Modern graph theory.* Springer-Verlag, New York, 1998. MR1633290

[3] A. Cayley, *A theorem on trees.* Quart. J. Pure Appl. Math. **23** (1889), 376–378.

[4] S. Huang and A. Postnikov, *A Bijective Proof for Reciprocity Theorem.* (2009). Available at `https://arxiv.org/abs/0909.2508`.

[5] I. Pak and A. Postnikov, *Enumeration of spanning trees of graphs.* (1994). Available at `http://math.mit.edu/~apost/papers/tree.ps`.

[6] H. Prüfer, *Neuer Beweis eines Satzes über Permutationen.* Arch. Math. Phys. **27** (1918), 742–744.

[7] A. Rényi, J. Mayar Tud. Akd. Mat. Fiz. Oszt. Közl. **12** (1966), 77-105.

Appendix A. Bijection by Huang and Postnikov

We present the bijection found by Huang and Postnikov in [4]. We will use our notation instead of that used in [4].

Let $\mathcal{A}$ be the set of all proper pairings $P = (T, \mathcal{V}, \mathcal{S})$, and let $\mathcal{B}$ be the set of all (T, W), where W is a sequence of $k(F_T) - 1$ vertices chosen from $\{0, 1, \ldots, n\}$. The bijection between $\mathcal{A}$ and $\mathcal{B}$ works as follows:

I. $\varphi : \mathcal{A} \to \mathcal{B}$ (see Definition 4.1 from [4])

Consider some proper pairing $P = (T, \mathcal{V}, \mathcal{S}) \in \mathcal{A}$. Start with a set $\mathcal{R} = \{r \in [n] : r \text{ is a root of } F_T\}$ and an empty sequence $W = ()$.

WHILE $|\mathcal{R}| > 1$:

(1) Delete leaves from $V(R_P) - \mathcal{R}$ until all remaining leaves are in $\mathcal{R}$.
(2) Let $\ell_{\max}$ be the leaf of maximum index remaining in R_P.
    (a) Append $\ell_{\max}$'s neighbor to the end of W.
    (b) Delete $\ell_{\max}$ from both R_P and $\mathcal{R}$.

RETURN $(T, W) \in \mathcal{B}$.

We note that φ specializes to the Prüfer code when $F_T = E_n$.

II. $\varphi^{-1} : \mathcal{B} \to \mathcal{A}$ (see Definition 4.3 from [4])

Consider some weight sequence $W \in \mathcal{B}$. Again, let $\mathcal{R}$ be the set of roots of F_T. Start with a copy T^* of the tree T and an empty set $\mathcal{S} = \{\}$.

WHILE $|W| > 0$:

(1) Let v be the first element in the remaining sequence W, and let r be the root of maximum index in $\mathcal{R}$ that is not in v's component of F_{T^*}. (In the case $v = 0$, let r be the root of maximum index, regardless of component.)
(2) Delete v from W and r from $\mathcal{R}$.
(3) If $v \neq 0$:
    (a) Add edge vr and remove $0r$ in T^*.
    (b) Add (v, r) to $\mathcal{S}$.

RETURN $(T, \mathcal{V}, \mathcal{S}) \in \mathcal{A}$.