THE BRACKET IN THE BAR SPECTRAL SEQUENCE FOR A FINITE-FOLD LOOP SPACE

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Abstract. When X is an associative H-space, the bar spectral sequence computes the homology of the delooping, $H_*(BX)$. If X is an n-fold loop space for $n \geq 2$ this is a spectral sequence of Hopf algebras. Using machinery by Sugawara and Clark, we show that the spectral sequence filtration respects the Browder bracket structure on $H_*(BX)$, and so it is moreover a spectral sequence of Poisson algebras. Through the bracket on the spectral sequence, we establish a connection between the degree $n-1$ bracket on $H_*(X)$ and the degree $n-2$ bracket on $H_*(BX)$. This generalizes a result of Browder and puts it in a computation-ready context.

1. Introduction

For an associative H-space G, the Milnor-Dold-Lashof construction builds the delooping BG. The algebraic analogue is the bar construction, and this relationship gives rise to the bar spectral sequence (also known as the Rothenberg-Steenrod spectral sequence) which computes $H_*(BG)$ from $H_*(G)$. The case where G is an infinite loop space has been extensively studied: working at a prime p, Ligaard and Madsen [**5**] showed that there is a Dyer-Lashof action on the spectral sequence compatible with the ones on $H_*(G)$ and $H_*(BG)$.

Here, we will instead consider spaces $\Omega^n X$ where n is finite, and we will work over an arbitrary field. S^{n-1} parametrizes the multiplications on $\Omega^n X$, producing Browder's bracket in homology—this process is reviewed in §2.3. This bracket shifts degree by $n-1$ and is an obstruction to $\Omega^n X$ being an $(n+1)$-fold loop space. Likewise, $H_*(\Omega^{n-1}X)$ has a bracket of degree $n-2$.

Our main result is that the bar spectral sequence relating $H_*(\Omega^n X)$ and $H_*(\Omega^{n-1}X)$ respects these bracket structures.

Theorem. *The bar spectral sequence*

$$E^2_{**} \cong \mathrm{Tor}^{H_*(\Omega^n X)}_{*,*}(k,k) \Rightarrow H_*(\Omega^{n-1}X)$$

is a spectral sequence of Poisson algebras, converging to its target as a Poisson algebra. The bracket in the spectral sequence shifts bidegree by $(-1, n-1)$.

Moreover, we will explicitly describe the bracket in the spectral sequence in terms of the one on $H_*(\Omega^n X)$. It is responsible for the vertical shift of $n-1$. Because of the horizontal shift of -1, the bracket has total degree $n-2$, which matches the one on $H_*(\Omega^{n-1}X)$.

In §2 we review the relevant definitions and constructions. Then, in §3 we set up and prove the theorem stated above, relating the brackets on $H_*(\Omega^n X)$ and $H_*(\Omega^{n-1}X)$ through the bar spectral sequence. We obtain Theorem 2-1 of [**1**] as a special case. For technical reasons, we consider $n = 2$ separately from $n \geq 3$.

This machinery can be used to infer properties of the bracket on $H_*(\Omega^n X)$ from the one on $H_*(\Omega^{n-1}X)$. In some cases it determines it completely; $H_*(\Omega^n S^k; \mathbb{Q})$ is such an example, and the reader is invited to compute it. There also appears to be related phenomena in the topological Hochschild homology of E_n ring spectra.

2. Background

Given a space X, the loop space ΩX is typically modeled as the space of all paths $I \to X$ sending $\{0,1\} \mapsto * \in X$, the basepoint. This model has a deficit: the multiplication (given by concatenation and doubling the speed) is only unital and associative up to homotopy. To remedy this, we instead adopt the following homotopy equivalent model, called the "Moore loops" of X after J. C. Moore [**8**].

Definition 2.1 (Moore Loops)**.** Given a space X with basepoint $*$, let ΩX be the subspace of $X^{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ consisting of pairs (α, l) such that $\alpha(t) = *$ if $t \leq 0$ or $t \geq l$. That is, ΩX consists of loops in X together with their lengths. ΩX is itself a pointed space, with basepoint the constant path c_* at $* \in X$ of length 0, and moreover it is an H-space with the multiplication $(\alpha, l_\alpha)(\beta, l_\beta) = (\omega, l_\alpha + l_\beta)$ where

$$\omega(t) = \begin{cases} \alpha(t) & \text{if } t \leq l_\alpha, \text{ and} \\ \beta(t - l_\alpha) & \text{if } l_\alpha \leq t. \end{cases}$$

The benefit of this model is that ΩX is a strictly associative H-space with unit c_*. Let C_* denote the normalized singular chain complex, which is the singular chain complex modded out by degenerate chains. The complex $C_*(\Omega X)$ is a differential graded algebra (henceforth abbreviated DGA) with the multiplication

$$C_*(\Omega X) \otimes C_*(\Omega X) \xrightarrow{\text{EZ}} C_*(\Omega X \times \Omega X) \to C_*(\Omega X)$$

where the first map is the Eilenberg-Zilber map and the latter is induced by the multiplication on ΩX. In turn, this induces a multiplication on $H_*(\Omega X)$—the Pontryagin product.

Remark (Notation). Throughout, we will consider only connected spaces. Coefficients are taken in a field k to ensure that the Künneth map is an isomorphism. As such, this condition can be relaxed to k a commutative ring and $H_*(\Omega^n X; k)$ flat.

If x is an element of a (bi)graded object, then we take $|x|$ to mean its total degree. "Commutative" will mean "commutative in the graded sense" so that $xy = (-1)^{|x||y|} yx$.

2.1. **The bar construction.** The following is a brief summary of the bar construction. For more details than what is provided here, see [**4**] and [**6**]. We begin in a purely algebraic context and then specialize to the case of topological interest at the end of §2.2. Consider a DGA $(\mathcal{A}, d_{\mathcal{A}})$ with augmentation $\epsilon \colon \mathcal{A} \to k$.

Definition 2.2. Let $\overline{\mathcal{A}} = \ker \epsilon$. The (normalized) *bar construction* $\mathrm{B}_{*,*}(\mathcal{A})$ is a bigraded k-module with

$$\mathrm{B}_{s,*}(\mathcal{A}) = \underbrace{\overline{\mathcal{A}} \otimes \cdots \otimes \overline{\mathcal{A}}}_{s \text{ times}}$$

of which the component in degree t is denoted $\mathrm{B}_{s,t}(\mathcal{A})$. We call s the *external degree*, and t the *internal degree*, and $s + t$ the *(total) degree*. It is conventional to write $\alpha_1 \otimes \cdots \otimes \alpha_s \in \mathrm{B}_{s,*}(\mathcal{A})$ as $[\alpha_1 | \cdots | \alpha_s]$. The bar construction is a double complex with an *internal differential* d of bidegree $(0, -1)$ and an *external differential* δ of bidegree $(-1, 0)$, defined as

$$d[\alpha_1 | \cdots | \alpha_s] = \sum_{i=1}^{s} (-1)^{\sigma(i-1)} [\alpha_1 | \cdots | d_{\mathcal{A}} \alpha_i | \cdots | \alpha_s],$$

$$\delta[\alpha_1 | \cdots | \alpha_s] = \sum_{i=1}^{s-1} (-1)^{\sigma(i)} [\alpha_1 | \cdots | \alpha_i \alpha_{i+1} | \cdots | \alpha_s],$$

where the sign is given by $\sigma(i) = \deg[\alpha_1 | \cdots | \alpha_i]$. We will often use d to denote both the differential on $\mathcal{A}$ and the internal differential in $\mathrm{B}_{*,*}(\mathcal{A})$, as there is no risk of ambiguity.

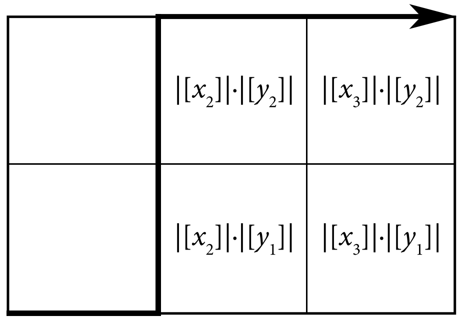

FIGURE 2.1. A pictorial representation of the term $\pm[x_1\otimes 1|1\otimes y_1|1\otimes y_2|x_2\otimes 1|x_3\otimes 1]$ in the shuffle of $[x_1|x_2|x_3]$ and $[y_1|y_2]$. The sign is determined by the blocks under the walk.

The homology of the total complex with differential $D = d+\delta$ computes $\mathrm{Tor}^{\mathcal{A}}_*(k,k)$:

$$H_*(\mathrm{tot}\,\mathrm{B}_{*,*}(\mathcal{A})) \cong \mathrm{Tor}^{\mathcal{A}}_*(k,k). \tag{2.1}$$

The bar construction has a natural comultiplication $\Delta\colon \mathrm{B}_{*,*}(\mathcal{A}) \to \mathrm{B}_{*,*}(\mathcal{A}) \otimes \mathrm{B}_{*,*}(\mathcal{A})$ sending

$$\Delta[\alpha_1|\cdots|\alpha_s] = \sum_{i=0}^{s} [\alpha_1|\cdots|\alpha_i] \otimes [\alpha_{i+1}|\cdots|\alpha_s]$$

which gives $\mathrm{Tor}^{\mathcal{A}}_*(k,k)$ a coalgebra structure.

Definition 2.3. Call φ a (p,q)-*shuffle* if it is a permutation of $\{1,\dots,p+q\}$ such that $\varphi(a)<\varphi(b)$ if $1\le a<b\le p$ or if $p+1\le a<b\le p+q$. If $\mathcal{A}$ and $\mathcal{A}'$ are DGAs, the *shuffle product* maps $\mathrm{B}_{*,*}(\mathcal{A})\otimes \mathrm{B}_{*,*}(\mathcal{A}') \to \mathrm{B}_{*,*}(\mathcal{A}\otimes\mathcal{A}')$, and is defined as

$$[\alpha_1|\cdots|\alpha_p]\otimes[\alpha'_1|\cdots|\alpha'_q] \mapsto \sum_{(p,q)\text{-shuffles } \varphi} (-1)^{\sigma(\varphi)}[a_{\varphi^{-1}(1)}|\cdots|a_{\varphi^{-1}(p+q)}],$$

where

$$a_i = \begin{cases} \alpha_i\otimes 1 & \text{if } i\le p,\\ 1\otimes \alpha'_{i-p} & \text{if } i>q, \end{cases}$$

$$\sigma(\varphi) = \sum_{\varphi(i)>\varphi(j+p)} (|\alpha_i|+1)(|\alpha'_j|+1).$$

The shuffle product was introduced by Eilenberg and Mac Lane in [**4**].

A shuffle can be thought of as a walk from $(0,0)$ to (p,q) that goes rightwards or upwards at each step—see Figure 2.1. The identity shuffle is the walk that goes through $(p,0)$, and deviation from this shuffle incurs the usual signs from moving elements past each other. Note that $[\alpha_i]$ has bidegree $(1,|\alpha_i|)$ and thus total degree $|\alpha_i|+1$.

In the case $\mathcal{A}$ is commutative, its multiplication is an algebra map $\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$, and the bar construction receives a commutative multiplication via the composite

$$\mathrm{B}_{*,*}(\mathcal{A})\otimes\mathrm{B}_{*,*}(\mathcal{A}) \xrightarrow{\mathrm{EZ}} \mathrm{B}_{*,*}(\mathcal{A}\otimes\mathcal{A}) \to \mathrm{B}_{*,*}(\mathcal{A})$$

where the first map is the shuffle product and the second is induced by the multiplication on $\mathcal{A}$.

2.2. **The bar spectral sequence.** Since the preceding gives $\mathrm{Tor}_*^{\mathcal{A}}(k,k)$ as the total homology of a first-quadrant double complex, we receive a strongly convergent spectral sequence by filtering the total complex by external degree s:

$$F_s\left(\operatorname{tot}\mathrm{B}_{*,*}(\mathcal{A})\right)_n = \bigoplus_{\substack{p+q=n\\ p\leq s}} \mathrm{B}_{p,q}(\mathcal{A}).$$

The associated graded of this filtration is

$$E^0_{s,t} = \mathrm{gr}_s\left(\operatorname{tot}\mathrm{B}_{*,*}(\mathcal{A})\right)_{s+t} = \mathrm{B}_{s,t}(\mathcal{A})$$

and the differentials on the first two pages are given by $d_0 = d$, $d_1 = \delta$ as in Definition 2.2. By the Künneth formula, $H_*(\mathrm{B}_{s,*}(\mathcal{A})) \cong \mathrm{B}_{s,*}(H_*(\mathcal{A}))$ and the $E^1_{*,*}$ page is the bar complex on $H_*(\mathcal{A})$ if we treat it as a DGA with trivial differential. Therefore our spectral sequence has the form

$$E^2_{*,*} \cong \mathrm{Tor}_{*,*}^{H_*(\mathcal{A})}(k,k) \Rightarrow \mathrm{Tor}_*^{\mathcal{A}}(k,k). \tag{2.2}$$

The topological significance of this construction stems from the fact that, if G is an associative H-space, there is a chain equivalence

$$\operatorname{tot}\mathrm{B}_{*,*}(C_*(G)) \xrightarrow{\simeq} C_*(BG) \tag{2.3}$$

inducing the homology isomorphism

$$\mathrm{Tor}_*^{C_*(G)}(k,k) \cong H_*(BG;k). \tag{2.4}$$

In general, the spectral sequence and (2.4) are of coalgebras [**7**]. Even if $\mathcal{A}$ itself is not strictly commutative, it may still be possible to grant the total complex $\operatorname{tot}\mathrm{B}_{*,*}(\mathcal{A})$ a multiplication—this is the case when $\mathcal{A} = C_*(\Omega G)$. We will return to this point in §3.1. In this case, the spectral sequence and (2.4) are of Hopf algebras [**2**].

2.3. **Browder's bracket.** Browder [**1**] inductively defines[1] a map $\tilde{\phi}\colon S^{n-1}\times\Omega^n X\times\Omega^n X \to \Omega^n X$ where S^{n-1} parametrizes the choice of multiplication on $\Omega^n X$.

The map $\tilde{\phi}_1\colon S^0\times\Omega X\times\Omega X\to\Omega X$ sends $\tilde{\phi}(1,a,b) = ab$ and $\tilde{\phi}(-1,a,b) = ba$, where multiplication is concatenation of loops.

Given $\tilde{\phi}_n\colon S^{n-1}\times G\times G\to G$, define a map $S^{n-1}\times[-1,1]\times\Omega G\times\Omega G\to\Omega G$ by

$$(u,t,a,b)\mapsto\begin{cases}\tilde{\phi}_n(u, c(l(b)t)a, b) & \text{if } t\geq 0,\\ \tilde{\phi}_n(u,a,c(l(a)\,|t|)b) & \text{if } t\leq 0,\end{cases}$$

where $l(a)$ denotes the length of a and $c(t)$ is a constant loop of length t at the basepoint. This map factors through

$$\frac{S^{n-1}\times[-1,1]}{S^{n-1}\times\{-1\}\cup S^{n-1}\times\{1\}}\times\Omega G\times\Omega G \cong S^n\times\Omega G\times\Omega G\to\Omega G$$

which we take to be $\tilde{\phi}_{n+1}$.

Instead of $\tilde{\phi}\colon S^{n-1}\times\Omega^n X\times\Omega^n X\to\Omega^n X$, we will write the map as

$$\phi\colon \Omega^n X\times S^{n-1}\times\Omega^n X\to\Omega^n X$$

to avoid signs in the following definition.

[1]We have opted to use the Moore loops model of ΩX for its benefit of strict associativity. If one instead uses the non-associative model where elements of ΩX are maps $(I,\partial I)\to(X,*)$, the little cubes operad gives a map

$$\tilde{\phi}\colon \mathcal{C}_n(2)\times\Omega^n X\times\Omega^n X\to\Omega^n X$$

where $\mathcal{C}_n(2)\simeq S^{n-1}$.

Definition 2.4. Let $\gamma \in H_{n-1}(S^{n-1})$ be a generator. Then $H_*(\Omega^n X)$ has a *bracket*[2] of degree $n-1$

$$[-,-]\colon H_p(\Omega^n X) \otimes H_q(\Omega^n X) \to H_{p+q+n-1}(\Omega^n X),$$
$$[x, y] = \phi_*(x \otimes \gamma \otimes y).$$

We will write ϕ_* both for the map induced on chains and for the map induced on homology.

The sign difference between our bracket and Browder's ψ is given by $[x, y] = (-1)^{(n-1)|x|}\psi(x, y)$.

In [**3**], Cohen showed that $H_*(\Omega^n X)$ is a Poisson n-algebra with this bracket. That is to say, the bracket is:

- antisymmetric:
$$[x, y] = -(-1)^{(|x|+n-1)(|y|+n-1)}[y, x],$$
- (Poisson identity) a derivation with respect to the multiplication:
$$[x, yz] = [x, y]z + (-1)^{|y|(|x|+n-1)}y[x, z],$$
- (Jacobi identity) and a derivation with respect to itself:
$$[x, [y, z]] = [[x, y], z] + (-1)^{(|x|+n-1)(|y|+n-1)}[y, [x, z]].$$

3. The bracket on the bar spectral sequence

3.1. **Further structure in the total bar complex.** As mentioned in §2.1, $\mathrm{B}_{*,*}(\mathcal{A})$ has a multiplication when $\mathcal{A}$ is commutative. However, $\mathcal{A} = C_*(\Omega G)$ is not commutative, so the multiplication $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is not an algebra map. Nonetheless, the multiplication on G induces a multiplication on $B\Omega G$ which translates through (2.3) to a multiplication on $\operatorname{tot} \mathrm{B}_{*,*}(\mathcal{A})$, as follows.

Looping the multiplication $G \times G \to G$ gives a map $\Omega(G \times G) \to \Omega G$. Note that with the Moore loops model of ΩG, we have $\Omega G \times \Omega G \not\cong \Omega(G \times G)$. To get a multiplication on $B\Omega G$, we would like to deloop the composite

$$M_0\colon \Omega G \times \Omega G \xrightarrow{\zeta} \Omega(G \times G) \to \Omega G$$

where $\zeta((\omega_1, l_1), (\omega_2, l_2)) = ((\omega_1, \omega_2), \max\{l_1, l_2\})$. In general, Sugawara [**9**] shows that a map $M_0\colon Y_1 \to Y_2$ of associative H-spaces induces a map $BY_1 \to BY_2$ if for $n \geq 1$ there exist homotopies

$$M_n\colon \underbrace{Y_1 \times I \times Y_1 \times I \times \cdots \times I \times Y_1}_{n \text{ copies of } I} \to Y_2$$

satisfying

$$M_n(y_1, t_1, \ldots, t_n, y_{n+1}) = \begin{cases} M_{n-1}(y_1, t_1, \ldots, t_{i-1}, y_i \times_1 y_{i+1}, t_{i+1}, \ldots, t_n, y_{n+1}) & \text{if } t_i = 0, \\ M_{i-1}(y_1, t_1, \ldots, t_{i-1}, y_i) \times_2 M_{n-i}(y_{i+1}, t_{i+1}, \ldots, t_n, y_{n+1}) & \text{if } t_i = 1, \end{cases}$$

where $\times_1$ and $\times_2$ denote the multiplications on Y_1 and Y_2 respectively.

Clark [**2**] shows that ζ satisfies the above conditions, so our M_0 can be delooped. We give a summary below.

Since G is an H-space, $X = \Omega G$ has an outer multiplication $\times_1$ which concatenates two loops and an inner "pointwise" multiplication $\times_2$ using the multiplication on G. If $l(a)$ denotes the length of a and $c(t)$ denotes the constant path of length t at the basepoint of G, then explicitly M_1 is defined as

$$M_1((a_1, b_1), t, (a_2, b_2)) = (a_1 \times_1 c\,(t(\max\{l(a_1), l(b_1)\} - l(a_1))) \times_1 a_2)$$
$$\times_2 (b_1 \times_1 c\,(t(\max\{l(a_1), l(b_1)\} - l(b_1))) \times_1 b_2)\,.$$

This is illustrated in Figure 3.1, where the dotted segments represent constant paths at the basepoint, and the total upper and lower loops are multiplied pointwise.

[2]Unfortunately, there are a multitude of square brackets appearing in this work—and all of them are conventional notation.

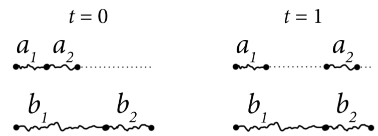

FIGURE 3.1. The homotopy M_1 on $((a_1, b_1), t, (a_2, b_2))$ at $t = 0, 1$.

The reader is referred to Proposition 1.6 in [**2**] for inductive definitions of higher M_n, but we will only need M_1 for now. These M_n are used to construct the delooped map $B(\Omega G \times \Omega G) \to B\Omega G$.

We describe the corresponding map $\operatorname{tot} B_{*,*}(\mathcal{A} \otimes \mathcal{A}) \to \operatorname{tot} B_{*,*}(\mathcal{A})$. At the chain level, M_n induces a map

$$h_n \colon (\mathcal{A} \otimes \mathcal{A})^{\otimes(n+1)} \to \mathcal{A}$$

by taking the 1-chain given by the identity on I in each appearance of $C_*(I)$. Thus the map h_n has degree n. These assemble into the map

$$\begin{aligned}[x_1 \otimes y_1 | \cdots | x_n \otimes y_n] \mapsto \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k \geq 0}} & [h_{i_1-1}(x_1 \otimes y_1 | \cdots | x_{i_1} \otimes y_{i_1})| \\ & h_{i_2-1}(x_{i_1+1} \otimes y_{i_1+1} | \cdots | x_{i_1+i_2} \otimes y_{i_1+i_2})| \cdots \\ & \cdots | h_{i_k-1}(x_{n-i_k+1} \otimes y_{n-i_k+1} | \cdots | x_n \otimes y_n)]\end{aligned}$$

which defines the second map in the composite

$$\begin{aligned}\mathrm{B}_{p,q}(\mathcal{A}) \otimes \mathrm{B}_{s,t}(\mathcal{A}) \xrightarrow{\mathrm{EZ}} \mathrm{B}_{p+s,q+t}(\mathcal{A} \otimes \mathcal{A}) \to & \bigoplus_{\substack{m+n=p+q+s+t \\ m \leq p+s}} \mathrm{B}_{m,n}(\mathcal{A}) \\ & = F_{p+s}(\operatorname{tot} \mathrm{B}_{*,*}(\mathcal{A}))_{p+q+s+t}\end{aligned} \tag{3.1}$$

where the first is the shuffle product. This extends to a multiplication on $\operatorname{tot} \mathrm{B}_{*,*}(\mathcal{A})$.

3.2. **The case $n = 2$.** Browder's bracket on $H_*(\Omega X)$ has degree 0; it is just the commutator $[x, y] = xy - (-1)^{|x||y|} yx$. We consider the commutator on the bar complex and show that it induces a bracket on the spectral sequence, converging to the one on $H_*(\Omega X)$ by (2.4). The portion of (3.1) landing in $\mathrm{B}_{p+s,q+t}(\mathcal{A})$ is the shuffle product, which is commutative. Hence that portion vanishes in the commutator, which is therefore a map with a -1 shift in filtration degree.

Proposition 3.1. *The bar spectral sequence*

$$E^2_{*,*} \cong \operatorname{Tor}^{H_*(\Omega^2 X)}_{*,*}(k, k) \Rightarrow \operatorname{Tor}^{C_*(\Omega^2 X)}_{*}(k, k)$$

has a bracket of bidegree $(-1, 1)$.

Proof. Because the multiplication respects the differential D on the total complex, the commutator does also:

$$D[x, y] = [Dx, y] + (-1)^{|x|}[x, Dy]. \tag{3.2}$$

Abbreviate $\operatorname{tot} \mathrm{B}_{*,*}(\mathcal{A})$ as A. The bracket is a map $F_p A \otimes F_s A \to F_{p+s-1} A$. We check that it induces a bracket on the spectral sequence of the form $E^*_{p,q} \otimes E^*_{s,t} \to E^*_{p+s-1,q+t+1}$. Take

$$x \in Z^r_{p,q} = F_p A_{p+q} \cap D^{-1}(F_{p-r} A_{p+q-1})$$
$$y \in Z^r_{s,t} = F_s A_{s+t} \cap D^{-1}(F_{s-r} A_{s+t-1})$$

representing classes in $E^r_{p,q}$ and $E^r_{s,t}$, respectively. Then (3.2) implies

$$[x, y] \in F_{p+s-1} A_{p+q+s+t} \cap D^{-1}(F_{p+s-r-1} A_{p+q+s+t-1}) = Z^r_{p+s-1,q+t+1}.$$

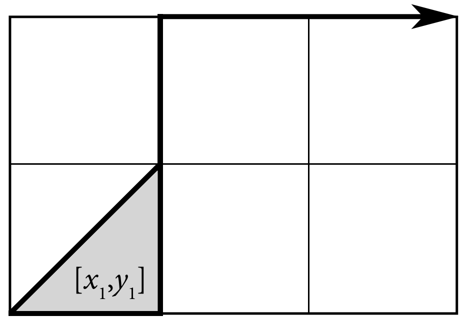

FIGURE 3.2. The term $\pm[[x_1, y_1]|y_2|x_2|x_3]$ in the bracket $[[x_1|x_2|x_3], [y_1|y_2]]$.

Moreover, bracketing an element $Dz \in B^r_{p,q} = F_pA_{p+q} \cap D(F_{p+r}A_{p+q+1})$ with y gives an element of $B^r_{p+s-1,q+t+1}$, again by (3.2): $D[z, y] = [Dz, y]$, where $[z, y] \in F_{p+s+r-1}A_{p+q+s+t+1}$. Thus we get a bracket of the form $E^*_{p,q} \otimes E^*_{s,t} \to E^*_{p+s-1,q+t+1}$ as desired.

The bracket on each page of the spectral sequence is induced by the one on A, so they are always compatible. In particular, on the $E^\infty_{*,*}$ page, the bracket is the one induced on $\operatorname{gr}_* H_*(A)$. □

Now that we know the bracket is well-behaved in the spectral sequence, we explicitly compute it on the $E^1_{*,*}$ page to see how it relates to the degree 1 bracket on $H_*(\Omega^2X)$.

Theorem 3.1. *For $x, y \in H_*(\Omega^2X)$, let $[x, y]$ denote their degree 1 bracket. Then on $E^1_{*,*}$ in the bar spectral sequence converging to $H_*(\Omega X)$, the bracket $[[x_1|\cdots|x_p], [y_1|\cdots|y_q]]$ is given by (using the notation of Definition 2.3)*

$$\sum_{(p,q)\text{-shuffles } \varphi} \sum_{\substack{\varphi^{-1}(i)\le p\\ \varphi^{-1}(i+1)>p}} (-1)^{\sigma(\varphi)}[a_{\varphi^{-1}(1)}|\cdots|[a_{\varphi^{-1}(i)}, a_{\varphi^{-1}(i+1)}]|\cdots|a_{\varphi^{-1}(p+q)}]$$

$$a_i = \begin{cases} x_i & \text{if } i \le p, \\ y_{i-p} & \text{if } i > p. \end{cases}$$

If one thinks of a shuffle as a path traveling up and right from $(0, 0)$ to (p, q), then the terms in the above are such paths with a single bracket inserted on a lower-right corner. See Figure 3.2.

Proof. Let $\mathbf{x} = [x_1|\cdots|x_p], \mathbf{y} = [y_1|\cdots|y_q] \in E^0_{*,*}$. In the spectral sequence, only the portion of the bracket lying in filtration F_{m+n-1} is visible. Consider the terms contributed by $\mathbf{xy}$ in the commutator—by (3.1), these are shuffles with one h_1 operation thrown in. They have one of four forms:

(1) $[\cdots|h_1((x \otimes 1) \otimes (x' \otimes 1))|\cdots]$
(2) $[\cdots|h_1((x \otimes 1) \otimes (1 \otimes y))|\cdots]$
(3) $[\cdots|h_1((1 \otimes y) \otimes (x \otimes 1))|\cdots]$
(4) $[\cdots|h_1((1 \otimes y) \otimes (1 \otimes y'))|\cdots]$

where x, x' are arbitrary x_i and likewise for y, y'. However, $h_1((x \otimes 1) \otimes (x' \otimes 1))$ and $h_1((1 \otimes y) \otimes (1 \otimes y'))$ are degenerate, and thus terms of the first and fourth types vanish.

Fix $x \in \{x_i\}$ and $y \in \{y_i\}$, as well as the shuffle hidden by the "$\cdots$" in the bar expression, and consider the term of the second form $[\cdots|h_1((x \otimes 1) \otimes (1 \otimes y))|\cdots]$. Without loss of generality, assume that this shuffle has positive sign in $\mathbf{xy}$. In the following, we refer to Figure 3.3. Let $\text{in}_{\mathbf{AB}}: I \to S^1$ cover the quarter-circle sending $0 \mapsto \mathbf{A}$ and $1 \mapsto \mathbf{B}$. We then have the commutative

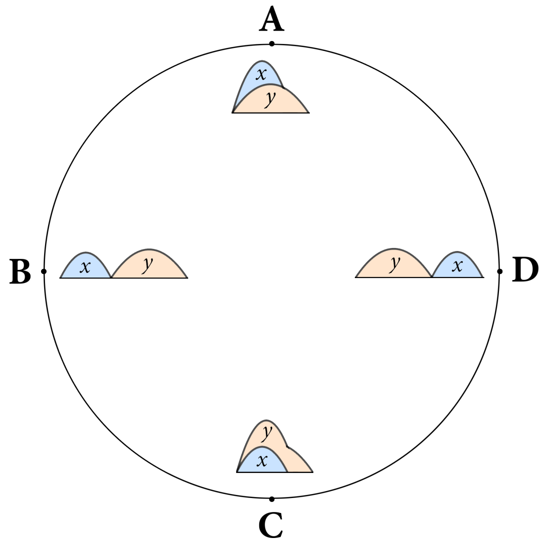

FIGURE 3.3. The map $\phi\colon \Omega^2X \times S^1 \times \Omega^2X \to \Omega^2X$, illustrated for (x,t,y) as $t \in S^1$ varies. x and y are visualized as "mounds" of points in X, where their outlines are mapped to the basepoint. The outer concatenation on Ω^2X is depicted horizontally and the inner (pointwise) concatenation is depicted vertically.

diagram

$$\begin{array}{ccccc} & & \Omega^2X \times * \times I \times * \times \Omega^2X & & \\ & \swarrow & \downarrow & \searrow^{\mathrm{in}_{\mathbf{AB}}} & \\ \Omega^2X \times \Omega^2X \times I \times \Omega^2X \times \Omega^2X & & & & \Omega^2X \times S^1 \times \Omega^2X \\ & \searrow_{M_1} & & \swarrow_{\phi} & \\ & & \Omega^2X & & \end{array}$$

If we let $i_{\mathbf{AB}} \in C_1(S^1)$ denote the 1-chain given by $\mathrm{in}_{\mathbf{AB}}$, it follows that

$$\phi_*(x \otimes i_{\mathbf{AB}} \otimes y) = h_1((x \otimes 1) \otimes (1 \otimes y)).$$

Now consider $[\cdots|h_1((1 \otimes y) \otimes (x \otimes 1))|\cdots]$, which has sign $(-1)^{(|x|+1)(|y|+1)}$ in $\mathbf{xy}$. One similarly finds that

$$\begin{aligned} h_1((1 \otimes y) \otimes (x \otimes 1)) &= (-1)^{|x||y|+|x|+|y|}\phi_*(x \otimes i_{\mathbf{AD}} \otimes y) \\ &= (-1)^{(|x|+1)(|y|+1)}\phi_*(x \otimes -i_{\mathbf{AD}} \otimes y) \end{aligned}$$

where the sign $(-1)^{|x||y|+|x|+|y|}$ comes from interchanging x and y across a 1-chain. Hence this contributes the term $[\cdots|\phi_*(x \otimes -i_{\mathbf{AD}} \otimes y)|\cdots]$.

Likewise, we consider the terms that arise in $-(-1)^{|\mathbf{x}||\mathbf{y}|}\mathbf{xy}$. One is $[\cdots|h_1((1 \otimes x) \otimes (y \otimes 1))|\cdots]$, which has sign $(-1)^{|\mathbf{x}||\mathbf{y}|}$ in $\mathbf{yx}$ and thus negative sign in the commutator. But

$$h_1((1 \otimes x) \otimes (y \otimes 1)) = \phi_*(x \otimes i_{\mathbf{CB}} \otimes y)$$

so it ultimately contributes the term $[\cdots|\phi_*(x \otimes -i_{\mathbf{CB}} \otimes y)|\cdots]$. Lastly there is the term $[\cdots|h_1((y \otimes 1) \otimes (1 \otimes x))|\cdots]$ with sign $-(-1)^{(|x|+1)(|y|+1)}$, and

$$h_1((y \otimes 1) \otimes (1 \otimes x)) = (-1)^{|x||y|+|x|+|y|}\phi_*(x \otimes i_{\mathbf{CD}} \otimes y),$$

so it results in the term $[\cdots|\phi_*(x \otimes i_{\mathbf{CD}} \otimes y)|\cdots]$.

The sum $\gamma = i_{\mathbf{AB}} - i_{\mathbf{CB}} + i_{\mathbf{CD}} - i_{\mathbf{AD}}$ represents a generator of $H_1(S^1)$, and altogether the set of four terms considered above in the commutator combine to give $[\cdots|\phi_*(x\otimes\gamma\otimes y)|\cdots]$. When we pass to the $E^1_{*,*}$ page, this becomes $[\cdots|[x,y]|\cdots]$ where $[x,y]$ is the bracket on $H_*(\Omega^2X)$ as defined in §2.3. □

3.3. **The case $n\geq 3$.** We now turn to the case $n\geq 3$, with the aim of relating the degree $n-1$ bracket on $H_*(\Omega^nX)$ to the degree $n-2$ bracket on $H_*(\Omega^{n-1}X)$ through the bar spectral sequence. Our treatment will not be as explicit as in the $n=2$ case.

We proceed similarly as in §3.1, except now we begin with $\phi_{n-1}\colon \Omega^{n-1}X\times S^{n-2}\times\Omega^{n-1}X\to\Omega^{n-1}X$. Looping gives a map

$$M_0\colon \Omega^nX\times\Omega S^{n-2}\times\Omega^nX\xrightarrow{\zeta}\Omega(\Omega^{n-1}X\times S^{n-2}\times\Omega^{n-1}X)\to\Omega^nX$$

which operates "pointwise" on loops, and there are homotopies

$$M_n\colon \underbrace{(\Omega^nX\times\Omega S^{n-2}\times\Omega^nX)\times I\times\cdots\times I\times(\Omega^nX\times\Omega S^{n-2}\times\Omega^nX)}_{n\text{ copies of }I}\to\Omega^nX$$

which are used to deloop M_0. By taking the 1-chain given by the identity on each occurrence of I, we obtain chain maps

$$h_n\colon (C_*(\Omega^nX)\otimes\Omega S^{n-2}\otimes C_*(\Omega^nX))^{\otimes(n+1)}\to C_*(\Omega^nX)$$

which have degree n.

In turn, we obtain a map

$$\begin{aligned}\mathrm{B}_{*,*}(C_*(\Omega^nX))\otimes\mathrm{B}_{*,*}(C_*(\Omega S^{n-2}))\otimes\mathrm{B}_{*,*}(C_*(\Omega^nX))&\xrightarrow{\mathrm{EZ}}\mathrm{B}_{*,*}(C_*(\Omega^nX)\otimes C_*(\Omega S^{n-2})\otimes C_*(\Omega^nX))\\&\to\operatorname{tot}\mathrm{B}_{*,*}(C_*(\Omega^nX))\end{aligned}$$

where the first map is the shuffle product on three terms and the second is as described in (3.1). The bracket on the total bar complex is given by fixing $[\xi]\in\mathrm{B}_{*,*}(C_*(\Omega S^{n-2}))$ in the above, where $\xi\in C_{n-3}(\Omega S^{n-2})$ represents the generator in homology. If $\beta\in C_{n-3}(S^{n-3})$ represents a generator of $H_{n-3}(S^{n-3})$ and $\eta\colon S^{n-3}\to\Omega S^{n-2}$ is the unit of the loop-suspension adjunction (see Figure 3.4), then we can take $\xi=\eta_*(\beta)$. Note that this only makes sense for $n\geq 3$, which is why we considered $n=2$ separately.

Using η, define $\hat{M}_n$ to be the composite

$$\begin{aligned}\hat{M}_n\colon &(\Omega^nX\times S^{n-3}\times\Omega^nX)\times I\times\cdots\times I\times(\Omega^nX\times S^{n-3}\times\Omega^nX)\\&\to(\Omega^nX\times\Omega S^{n-2}\times\Omega^nX)\times I\times\cdots\times I\times(\Omega^nX\times\Omega S^{n-2}\times\Omega^nX)\to\Omega^nX\end{aligned}$$

which at the chain level induces the degree n map

$$\hat{h}_n\colon (C_*(\Omega^nX)\otimes C_*(S^{n-3})\otimes C_*(\Omega^nX))^{\otimes(n+1)}\to C_*(\Omega^nX). \tag{3.3}$$

Proposition 3.2. *Let $n\geq 2$. The bar spectral sequence*

$$E^2_{*,*}\cong\operatorname{Tor}^{H_*(\Omega^nX)}_{*,*}(k,k)\Rightarrow\operatorname{Tor}^{C_*(\Omega^nX)}_{*}(k,k)$$

has a bracket of bidegree $(-1,n-1)$.

Proof. We have already proved it for $n=2$ so let $n\geq 3$. A priori, the degree $n-2$ bracket on $A=\bar{B}_*(C_*(X))$ maps $[-,-]\colon F_pA\otimes F_qA\to F_{p+q+1}A$. The task is to show that the bracket actually lands in $F_{p+q-1}A$.

Each shuffle of $[x_1|\cdots|x_p]$, $[\xi]$, and $[y_1|\cdots|y_q]$ contains terms $x_i\otimes 1\otimes 1$, $1\otimes 1\otimes y_j$, and one instance of $1\otimes\xi\otimes 1$. However, note that $h_0(1\otimes\xi\otimes 1)$ gives a degenerate chain, thus vanishing in $C_*(X)$. Hence the part of the bracket lying in F_{p+q+1} vanishes.

The part of the bracket lying in F_{p+q} is given by inserting one h_1 into each shuffle. If the h_1 does not touch $1\otimes\xi\otimes 1$, then the result vanishes by the preceding. But even if the h_1 is inserted as

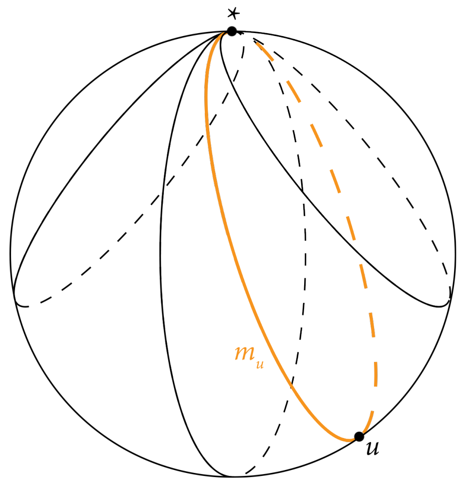

FIGURE 3.4. The map $\eta\colon S^{n-3} \to \Omega S^{n-2}$ sending u to the loop m_u, depicted for $n = 4$.

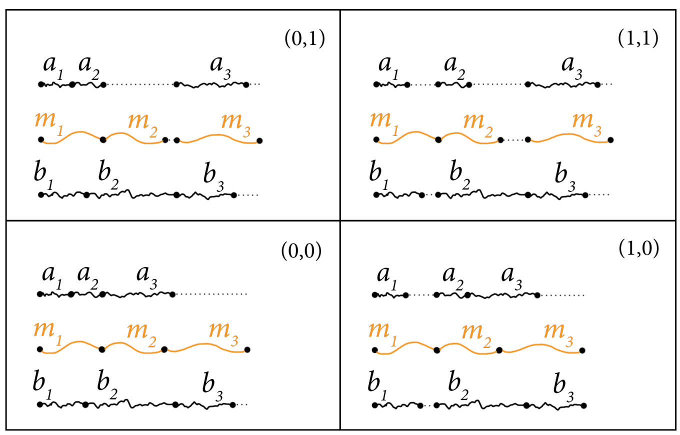

FIGURE 3.5. The homotopy M_2 on $((a_1, m_1, b_1), t_1, (a_2, m_2, b_2), t_2, (a_3, m_3, b_3))$ as $(t_1, t_2) \in I^2$ varies.

$h_1((x \otimes 1 \otimes 1) \otimes (1 \otimes \xi \otimes 1))$ for example, the result is *still* degenerate, since the role of ΩS^{n-2} is to control the multiplication on elements of $\Omega^n X$. Thus the part of the bracket residing in F_{p+q} also vanishes.

The remainder proceeds as in the proof of Proposition 3.1. □

The terms of the bracket landing in F_{p+q-1} are given by shuffles with either one h_2 inserted or two h_1s inserted, but by the same reasoning as in the above proof, shuffles with two h_1s inserted will vanish. As such, an understanding of the homotopy M_2 is necessary. It is similar in spirit to M_1. In Figure 3.5, we offer a visual description of M_2 which the reader should compare to Figure 3.1. An explicit definition can be constructed from Proposition 1.6 in [**2**].

Now we give a characterization of the bracket in the spectral sequence, which is our main result.

Theorem 3.2. *Let $n \geq 2$. For $x, y \in H_*(\Omega^n X)$, let $[x, y]$ denote their degree $n-1$ bracket. Then on $E^1_{*,*}$ in the bar spectral sequence converging to $H_*(\Omega^{n-1} X)$, the bracket* $[[x_1|\cdots|x_p], [y_1|\cdots|y_q]]$

is given by (using the notation of Definition 2.3)

$$\sum_{(p,q)\text{-shuffles } \varphi} \sum_{\substack{\varphi^{-1}(i)\leq p \\ \varphi^{-1}(i+1)>p}} (-1)^{\sigma(\varphi)}[a_{\varphi^{-1}(1)}|\cdots|(-1)^{\sigma'(\varphi,i)}[a_{\varphi^{-1}(i)}, a_{\varphi^{-1}(i+1)}]|\cdots|a_{\varphi^{-1}(p+q)}]$$

$$a_i = \begin{cases} x_i & \text{if } i \leq p, \\ y_{i-p} & \text{if } i > p, \end{cases}$$

where the sign $(-1)^{\sigma'(\varphi,i)}$ is defined as

$$\sigma'(\varphi, i) = n\left(\sum_{\varphi(j)>i+1}(|x_j|+1) + \sum_{\varphi(j)<i}(|y_j|+1)\right).$$

Proof. The sign $(-1)^{\sigma'(\varphi,i)}$ appears only when n is odd, so when $n = 2$ this reduces to Theorem 3.1.

Let $n \geq 3$. First we consider $E^0_{*,*}$. The h_2 inserted into a shuffle must take $1 \otimes \xi \otimes 1$ as one of its arguments, or the shuffle will vanish. If the other two arguments are both of the form $x \otimes 1 \otimes 1$, or both of the form $1 \otimes 1 \otimes y$, then the result is again degenerate and vanishes. In light of these considersations, if we fix the surrounding shuffle denoted by "$\cdots$" below, the bracket appears in sets of 6 pieces (with β and $\hat{h}_2$ as in the discussion preceding (3.3)):

(1) $[\cdots|\hat{h}_2((1\otimes\beta\otimes 1)\otimes(x\otimes 1\otimes 1)\otimes(1\otimes 1\otimes y))|\cdots]$
(2) $[\cdots|\hat{h}_2((x\otimes 1\otimes 1)\otimes(1\otimes\beta\otimes 1)\otimes(1\otimes 1\otimes y))|\cdots]$
(3) $[\cdots|\hat{h}_2((x\otimes 1\otimes 1)\otimes(1\otimes 1\otimes y)\otimes(1\otimes\beta\otimes 1))|\cdots]$
(4) $[\cdots|\hat{h}_2((1\otimes\beta\otimes 1)\otimes(y\otimes 1\otimes 1)\otimes(1\otimes 1\otimes x))|\cdots]$
(5) $[\cdots|\hat{h}_2((y\otimes 1\otimes 1)\otimes(1\otimes\beta\otimes 1)\otimes(1\otimes 1\otimes x))|\cdots]$
(6) $[\cdots|\hat{h}_2((y\otimes 1\otimes 1)\otimes(1\otimes 1\otimes x)\otimes(1\otimes\beta\otimes 1))|\cdots]$

Each of the above $\hat{h}_2$ expressions can be induced from a restriction of $\hat{M}_2$. For example, terms of the second form are induced by the restriction of $\hat{M}_2$ to

$$F_2 \colon (\Omega^n X \times * \times *) \times I \times (* \times S^{n-3} \times *) \times I \times (* \times * \times \Omega^n X) \to \Omega^n X.$$

Define the restrictions F_i for $i = 1, \ldots, 6$ analogously. These can be stitched together to be parts of a larger map $F \colon \Omega^n X \times S^{n-3} \times I^2 \times \Omega^n X \to \Omega^n X$, as illustrated in Figure 3.6. The table beneath it shows how F acts at specific points. The loop $m_u \in \Omega S^{n-2}$ controls the pointwise multiplication of loops $a, b \in \Omega^n X = \Omega(\Omega^{n-1} X)$. The dotted segments denote constant paths at the basepoint.

Note that where m_u does not interact with a or b, it produces constant paths at the basepoint. For instance, evaluating F at the point labeled **A** gives: b, followed by a, followed by a constant path (of length equal to that of m_u). By removing all such extraneous constant paths, we obtain another map $\tilde{F} \colon \Omega^n X \times S^{n-3} \times I^2 \times \Omega^n X \to \Omega^n X$ which is homotopic to F. However, $\tilde{F}$ on the segment $\overline{\mathbf{AH}}$ is the same as on $\overline{\mathbf{DK}}$, and $\tilde{F}$ is constant on $\overline{\mathbf{AD}}$ as well as on $\overline{\mathbf{HK}}$. Making these identifications, the depicted rectangle **ADKH** becomes S^2. Moreover, the value of $\tilde{F}$ on $\overline{\mathbf{AH}} = \overline{\mathbf{DK}}$ is independent of m_u, and when $u = * \in S^{n-3}$, the value of $\tilde{F}$ does not depend on vertical position in the rectangle **ADKH** (i.e. the picture in Figure 3.6 can be flattened to just the line $\overline{\mathbf{AH}} = \overline{\mathbf{DK}}$ for $u = *$). Thus $\tilde{F}$ factors through $\Omega^n X \times S^{n-3} \wedge S^2 \times \Omega^n X \cong \Omega^n X \times S^{n-1} \times \Omega^n X$.

Now we have a map $\tilde{F} \colon \Omega^n X \times S^{n-1} \times \Omega^n X \to \Omega^n X$, and in homology (the $E^1_{*,*}$ page) the total of the six terms listed at the beginning is $[\cdots|\tilde{F}_*(x, \gamma, y)|\cdots]$ for a generator $\gamma \in H_{n-1}(S^{n-1})$.

The map $\tilde{F}$ involves an unwanted "twisting" of a and b according to m_u. This is remedied by the "untwisting" homotopy H_t in Figure 3.7. Let

$$L = \frac{L_0 + l(m_u)}{L_h + l(m_u)} L_h.$$

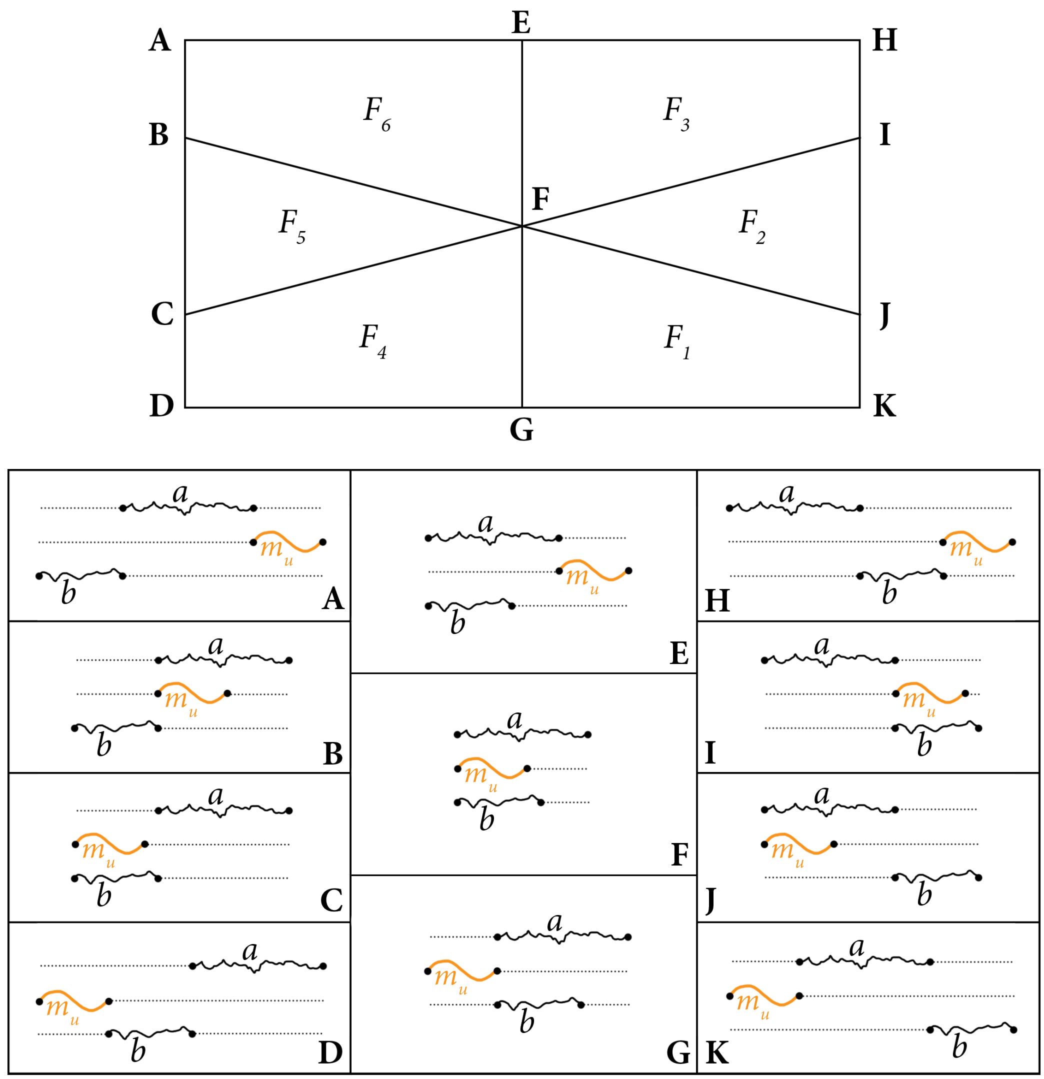

FIGURE 3.6. The pieces F_i assembled into a single map $F\colon \Omega^n X \times S^{n-3} \times I^2 \times \Omega^n X \to \Omega^n X$, with $a, b \in \Omega^n X$ (thought of as loops in $\Omega^{n-1}X$) and $u \in S^{n-3}$. The map $u \mapsto m_u$ is as in Figure 3.4.

The homotopy multiplies a and b according to the bold path, which coincides with m_u on the interval $[Lt, L_h(1-t) + Lt]$ and is extended by constant paths on both sides.

At $t = 1$ we obtain the desired $\phi\colon \Omega^n X \times S^{n-1} \times \Omega^n X \to \Omega^n X$, and $\tilde{F}_*(x, \gamma, y) = \phi_*(x, \gamma, y) = [x, y]$.

Lastly, the sign $(-1)^{\sigma'(\varphi, i)}$ as defined in the theorem statement results from terms $x_j \otimes 1 \otimes 1$ being shuffled after $1 \otimes \xi \otimes 1$ (which has total degree $n-2 = n \bmod 2$), and terms $1 \otimes 1 \otimes y_j$ being shuffled before it—a sign which is not accounted for in the shuffle of $[x_1, \ldots, x_p]$ only with $[y_1, \ldots, y_q]$. $\square$

There is a point we have yet to address. To recap, we considered a bracket on $\operatorname{tot} \mathrm{B}_{*,*}(C_*(\Omega^n X))$ corresponding to the one on $C_*(\Omega^{n-1}X)$. As it respects the differential and the filtration, it induces a bracket on the spectral sequence, which converges to the one on $H_*(\Omega^{n-1}X)$. However, while the bracket on $H_*(\Omega^{n-1}X)$ makes it a Poisson algebra, we have not yet shown that the bracket on the spectral sequence has this property.

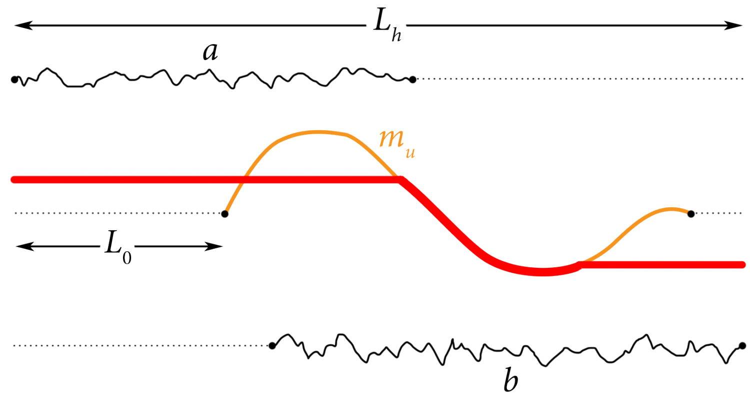

FIGURE 3.7. The homotopy $H_t\colon \Omega^n X \times S^{n-1} \times \Omega^n X \to \Omega^n X$, $t \in I$. (Although L_0 as drawn is positive, it can be in the interval $[-l(m_u), L_h]$.)

Theorem 3.3. *The bracket as described in Theorem 3.2 endows the bar spectral sequence with a Poisson algebra structure from the E^1_{**} page onwards.*

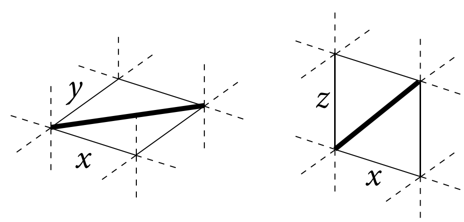

FIGURE 3.8. Brackets appearing in terms in (3.4).

Proof. The axioms listed at the end of §2.3 can be checked combinatorially using Theorem 3.2 and Definition 2.3. We check the Poisson identity as an example, in the case n is even. Since the bracket has even total degree $n-2$, the Poisson identity states

$$[\mathbf{x}, \mathbf{yz}] - [\mathbf{x}, \mathbf{y}]\mathbf{z} - (-1)^{|\mathbf{y}||\mathbf{x}|}\mathbf{y}[\mathbf{x}, \mathbf{z}] = 0 \tag{3.4}$$

where

$$\mathbf{x} = [x_1|\cdots|x_p]$$
$$\mathbf{y} = [y_1|\cdots|y_q]$$
$$\mathbf{z} = [z_1|\cdots|z_r].$$

In the same manner as Figure 2.1, a 3-way shuffle of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ can be thought of as a walk from $(0,0,0)$ to (p,q,r). Terms appearing in (3.4) are walks with a single bracket (see Figure 3.2), which may be one of the two types in Figure 3.8.

Consider a walk with a diagonal of the first type. It appears in $[\mathbf{x}, \mathbf{yz}]$ and $[\mathbf{x}, \mathbf{y}]\mathbf{z}$ with the same sign and thus disappears in (3.4).

Otherwise, the walk has a diagonal of the second type. It appears in $[\mathbf{x}, \mathbf{yz}]$ and $\mathbf{y}[\mathbf{x}, \mathbf{z}]$, but there is a sign difference of $(-1)^{|\mathbf{y}||\mathbf{x}|}$ because of the interchanged shuffle order of $\mathbf{x}$ and $\mathbf{y}$. Hence it also vanishes in (3.4).

The case with n odd is similar, but there is an additional sign to keep track of (see Theorem 3.2). The antisymmetry condition is straightforward to verify; it reduces to the antisymmetry condition for the bracket on $H_*(\Omega^n X)$. The Jacobi identity is a bit more work, but it also reduces to the Jacobi identity for the bracket on $H_*(\Omega^n X)$. We leave the details to the reader. □

Theorem 3.2 has a simpler description on $E^1_{1,*}$.

Corollary 3.1. *If $x, y \in H_*(\Omega^n X)$, then*

$$[[x],[y]] = [[x,y]] \tag{3.5}$$

where $[x],[y] \in E^1_{1,*}$, $[[x],[y]]$ denotes the bracket in the spectral sequence, and $[x,y]$ denotes the bracket in $H_*(\Omega^n X)$.

This corollary implies Theorem 2-1 in [**1**], as the spectral sequence has an edge homomorphism $E^1_{1,*} \to H_*(\Omega^{n-1}X)$ which sends $[x]$ with $x \in H_*(\Omega^n X)$ to the homology suspension $\sigma x \in H_*(\Omega^{n-1}X)$.

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