LINEARIZABLE LAURENT PHENOMENON SEQUENCES
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Abstract. Laurent phenomenon sequences are sequences $(x_i)_{i\in\mathbb{N}}$ such that there is a polynomial P with integer coefficients and an integer N such that $x_{n+N}x_n = P(x_{n+1}, ..., x_{n+N-1})$ and all of whose terms are Laurent polynomials in $x_1, ..., x_N$. We study Laurent phenomenon sequences that additionally satisfy a linear recurrence whose coefficients depend on the initial values of the sequence. We study several Laurent phenomenon sequences that were constructed by Alman, Cuenca, Huang, using period one seeds in Laurent phenomenon algebras introduced by Lam and Pylyavskyy. We prove that two such families of such sequences are linearizable and give a conjectural necessary and sufficient condition for linearizability of sequences arising from period one seeds. For a particular sequence $x_{n+p+q}x_n = x_{n+p}x_{n+q} + 1$, the cluster algebra generated by the seed has an associated marked surface. By studying triangulations of this surface, we give a combinatorial formula in terms of almost perfect matchings for the linear recurrence coefficients.

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1. Introduction

For positive integer N and polynomial $P \in \mathbb{Z}[x_1, x_2, ..., x_{N-1}]$ we say that P generates a *Laurent phenomenon sequence* if all the terms of the sequence $(x_i)_{i \geq 1}$ generated by $x_{n+N}x_n = P(x_{n+1}, ..., x_{n+N-1})$ for $n \geq 1$ are in $\mathbb{Z}[x_1^{\pm 1}, ..., x_n^{\pm 1}]$.

For example, the polynomial $P(x_1, ..., x_{N-1}) = \sum_{1 \leq i \leq N/2} x_i x_{N-i}$ generates the well known Somos-N sequence when we set initial values equal to 1. It is know that for $2 \leq N \leq 7$, the polynomial generates a Laurent phenomenon sequence.

We are interested in classifying those polynomials which generate linearizable Laurent phenomenon sequences. An example of such a polynomial is $P(x_1, x_2, x_3) = x_1x_3 + 1$, which generates a sequence satisfying the recurrence relation $x_{n+6} - \mathcal{K}x_{n+3} + x_n = 0$, where

$$\mathcal{K} = \frac{x_1}{x_4} + \frac{x_4}{x_1} + \frac{1}{x_1x_2} + \frac{1}{x_2x_3} + \frac{1}{x_3x_4}. \tag{1}$$

Before we proceed, we describe two methods for generating Laurent phenomenon sequences. The first method comes from cluster algebras and quivers. A quiver is a directed graph with no 1 cycles or 2 cycles. Given a quiver Q and vertex v of Q, we can mutate Q at v, to produce another quiver on the same set of vertices, $\mu_v(Q)$. For now, we disregard the details of mutation. They are covered in Section 3.

Now, given a quiver, Q, with vertices labelled $1, 2, ..., N$, we may assign to vertex i an initial variable x_i. The pair $(Q, \mathbf{x})$ is a labelled seed. Given vertex k of Q, we may mutate the seed at k to obtain the seed $(\mu_k(Q), \mathbf{x})$, where $\mathbf{x}$, defined as follows. $x_i = x'_i$ for $i \neq k$ and

$$x'_k x_k = \prod_{j \to k} x_j + \prod_{k \to j} x_j, \tag{2}$$

where the empty product is taken to be 1.

Thus, each time we mutate, we produce a new variable, related to the old variables via the above *exchange relation.* The polynomial on the right is called the *exchange polynomial* Iterating we can produce several new variables, all of which can be expressed as rational functions in terms of the initial variables $x_1, ..., x_N$. The remarkable property of quivers discovered by Fomin and Zelevensky in [7] is that all the new variables are in fact Laurent polynomials in $x_1, ..., x_N$. Now, for *mutation periodic quivers* it is possible to perform a sequece of mutations, producing variables $x_{N+1}, x_{N+2}, ...$ such that the exchange polynomial at each mutation is equal to some fixed polynomial P. Then, the sequence $x_1, x_2, ...$ consists of Laurent polynomials in $x_1, x_2, ..., x_N$ and it also satisfies $x_{n+N}x_n = P(x_{n+1}, ..., x_{n+N-1})$ for all n. We see that the Laurent property of quivers implies that any exchange polynomial in a mutation-periodic quiver generates a Laurent phenomenon sequence. Such polynomials are studied in [9] and [8].

Note that the exchange polynomial is always a binomial, so mutation-periodic quivers cannot explain all Laurent phenomenon sequences. For example, the Somos-5 sequence is generated by a three term polynomial. Another method for generating Laurent phenomenon sequences is described by Alman, Cuenca, and Huang, in [1], based on *period one seeds* of Laurent phenomenon algebras, a generalization of cluster algebras discussed in [11]. This is the most general method known for generating Laurent phenomenon sequences.

The following theorem lists several polynomials that are obtained using these two methods.

Theorem 1.1. [8],[1] *The following families of polynomials generate Laurent phenomenon sequences.*

(1) $P(x_1, x_2, ..., x_{N-1}) = x_p x_q + A$, *for* $A \in \mathbb{Z}$, $p + q = N$, *and* $\gcd(p, q) = 1$
(2) $P(x_1, x_2, ..., x_{N-1}) = x_p x_q + Ax_m$, *for* $A \in \mathbb{Z}$, $p + q = N = 2m$, *and* $\gcd(p, m) = 1$
(3) $P(x_1, x_2, ..., x_{N-1}) = x_1 x_{N-1} + A \sum_{i=1}^{N-1} x_i + B$, *for* $A, B \in \mathbb{Z}$
(4) $P(x_1, x_2) = x_1 x_2 + Ax_1 - Ax_2$, *for* $A \in \mathbb{Z}$
(5) $P(x_1, x_2, ..., x_{N-1}) = x_{2k} x_{N-2k} + Ax_k + Ax_{N-k}$, *for* $A \in \mathbb{Z}$

Note that the condition $\gcd(p, q) = 1$ in polynomial (1) is because when p and q have a common factor d, the sequence generated by P splits into d simultaneous recurrences $\{(x_{i+dn})_{n\in\mathbb{N}}\}_{i=1}^{d}$, all generated by $P(x_1, ..., x_{N/d-1}) = x_{p/d} x_{q/d} + A$. Similar degeneracy occurs in polynomial (2) when p and m have a common factor.

We will see that the first two families are examples of polynomials corresponding to mutation-periodic quivers in [8] and the last three are examples of polynomials corresponding to period one seeds in Laurent phenomenon algebras. Because period one seeds are generalizations of mutation-periodic quivers, all of these polynomials arise from some period one seed in a Laurent phenomenon algebra.

In fact, more is known about these particular sequences. For polynomials (1) and (2) it is known that the sequence they generate is linearizable and in fact the form of the recurrence is known [8]. In this paper, we further, show that the polynomials (3) and (4) generate linearizable sequences and we show the form of the recurrence. Numerical evidence suggests that polynomial (5) also generates a linearizable sequence. Based on these results we give the following conjectural condition for when a Laurent phenomenon sequence generated by a period one seed is linearizable.

Conjecture 1. Suppose $P(x_1, x_2, ..., x_{N-1})$ is a polynomial with positive coefficients arising from a period one seed, then, P the Laurent phenomenon sequence generated by P is linearizalble if and only if $P = x_q x_q + Q$, where where $p + q = N$ and Q has degree 1. Further, the minimal recurrence has coefficients which are Laurent polynomials in $x_1, ..., x_{N-1}$.

The justification of the necessity of the condition is as follows. We can consider the recurrence polynomial as a polynomial whose coefficients are rational functions in $\mathbf{x} = (x_1, x_2, ..., x_N)$. Then, suppose that there is some open set U in $\mathbb{C}^N$ such for that for $\mathbf{x} \in U$, the recurrence polynomial has unique root, $\lambda(\mathbf{x})$ of maximal absolute value. Then for all such $\mathbf{x}$ the sequence generated by P must asymptotically satisfy $x_n = c\lambda^n$. Then, If we let P_0 be the polynomial obtained by dropping all terms of P not of leading order, then asymptotically, we have $c^2\lambda^{2n+N} = P_0(c\lambda^n, ..., c\lambda^n)$. Then, either λ can only take a finite set of values $\mathbf{x}$ or because P_0 has positive integer coefficients, $P_0 = x_p x_q$ for some $p + q = N$.

Recall that the sequence generated by $x_1x_3 + 1$, satisfies the recurrence $x_{n+6} - \mathcal{K}x_{n+3} + x_n = 0$, with $\mathcal{K}$ given in Equation 1. In general, in [8] it is shown that the recurrence generated by polynomial (1) in Theorem 1.1 satisfies $x_{n+2pq} - \mathcal{K}x_{n+pq} + x_n = 0$, for some $\mathcal{K} \in \mathbb{Q}(x_1, ..., x_N)$. Notice that in the case $(p, q) = (1, 2)$ and $A = 1$ discussed above, we see that $\mathcal{K}$ a Laurent polynomial with positive coefficients. It turns out that this is true for general (p, q) and $A > 0$. We prove this by explicitly giving a formula for $\mathcal{K}$ as a sum of several monomials corresponding to almost perfect matchings of a particular graph. This is similar to the formula given in [13] for terms of the octahedron recurrence and the formula given by Lam in [10].

2. Summary of Results

We study the sequences presented in the Theorem 1.1 and prove some special cases of our conjecture. Recall that in [8], it is shown that the polynomials (1) and (2) in Theorem 1.1 satisfy a linear recurrence and the form of the recurrence is given. We extend this by showing that polynomials (3) and (4) also satisfy a linear recurrence and give its form. We also improve on their result by giving a formula in terms of almost perfect matchings for the recurrence coefficient of the sequence generated by polynomial (1).

Theorem 2.1. [8]
For $\gcd(p,q) = 1$*, polynomial* (2) *from Theorem 1.1 generates a sequence which satisfies a linear recurrence,*

$$x_{n+3pq} - \mathcal{K}x_{n+2pq} + \mathcal{K}x_{n+pq} - x_n = 0,$$

for some $\mathcal{K} \in \mathbb{Q}(x_1, ..., x_N)$*. For* $\gcd(p,q) = 2$*, the polynomial satisfies a linear recurrence*

$$x_{n+3pq} - \mathcal{K}_1 x_{n+5pq/2} + \mathcal{K}_2 x_{n+2pq} - \mathcal{K}_3 x_{n+3pq/2} + \mathcal{K}_2 x_{n+pq} - \mathcal{K}_3 x_{n+pq/2} = 0,$$

for some $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_2 \in \mathbb{Q}(x_1, ..., x_N)$*.*

This result is shown in [8] in the case where $A = 1$. The sequence generated by $x_p x_q + A x_m$ is obtained for the sequence generated by $x_p x_q + x_m$ when we multiply all terms by A, so the general result follows easily.

We prove similar result for polynomial (3) and (4).

Theorem 2.2. *Polynomial* (3) *from Theorem 1.1 generates a sequence which satisfies a linear recurrence,*

$$x_{n+3(N-1)} - \mathcal{K}x_{n+2(N-1)} + \mathcal{K}' x_{n+N-1} - x_n = 0,$$

$\mathcal{K} \in \mathbb{Q}(x_1, ..., x_N)$.

In fact, based on computing the coefficients $\mathcal{K}$ and $\mathcal{K}'$ for small values of p, q it appears that $\mathcal{K} = \mathcal{K}'$ in general, although we do not prove this.

Theorem 2.3. *Polynomial* (4) *from Theorem 1.1 generates a sequence which satisfies the linear recurrence*

$$x_{n+6} - \mathcal{K}x_{n+4} + \mathcal{K}x_{n+2} - x_n = 0$$

for some $\mathcal{K} \in \mathbb{Q}(x_1, x_2, x_3)$*.*

From numerical computation of the recurrence coefficients in Theorems 2.2 and 2.3, and it appears that they are in fact also Laurent polynomials in $x_1, x_2, ..., x_N$, supporting our main conjecture.

Now, before we state the theorem for polynomial (1), we define *almost perfect matchings*. Suppose we are given a planar graph G embedded in a closed disk. Then, an almost perfect matching of G is a matching in which every vertex not on the boundary of the disk is matched. We denote by $M'(G)$ the set of almost perfect matchings of G.

Now, the following theorem holds, where $G_{p,q}$ is a graph defined in Definition 5.2 and $\mathrm{wt}(m)$ is a weight function defined in Definition 5.3 that assigns to each almost perfect matching $m \in M'(G_{p,q})$ a Laurent monomial in $x_1, ..., x_N$. For example, $G_{2,3}$, and all of its 10 perfect matchings with their weights are illustrated in Figure 2.

Theorem 2.4. *Polynomial* (1) *from Theorem 1.1 generates a sequence which satisfies the linear recurrence* $x_{n+2pq} - \mathcal{K} x_{n+pq} + x_n = 0$. $\mathcal{K}$ *is given by the formula*

$$\mathcal{K} = \sum_{m \in M'(G_{p,q})} \mathrm{wt}(m)$$

.

Corollary 2.5. *The coefficient* $\mathcal{K}$ *from Theorem 2.4 is a Laurent polynomial in* $x_1, x_2, ..., x_N, A$ *which is a sum of monomials which have degree* $1, 0,$ *or* -1 *in each variable. The coefficients of this polynomial are positive.*

3. Generating Laurent Phenomenon Sequences

We describe different methods for generate Laurent Phenomenon sequences from cluster algebras and LP algebras.

3.1. **Quivers and Cluster Algebras.** A quiver is a directed graph with no 1 cycles or 2 cycles. Given a quiver Q and vertex v of Q, we construct the mutation of Q at v, denoted $\mu_v(Q)$, as follows.

(1) For every pair of edges $(i \to v, v \to j)$ in Q, add the edge $i \to j$.
(2) Reverse all edges with an endpoint at v.
(3) If there are any vertices i, j, such that there are p vertices from i to j and q vertices from j to i, with $p \geq q > 0$, then remove q vertices from i to j and j vertices from j to i.

The last step is simply to remove any 2 cycles that were created in the first two steps.

Now, given a quiver, Q, with vertices labelled $1, 2, ..., N$, we may assign to vertex i an initial variable x_i. The pair $(Q, \mathbf{x})$ is a labelled seed. Given vertex k of Q, we may mutate the seed at k to obtain the seed $(\mu_k(Q), \mathbf{x})$, where $\mathbf{x}$, defined as follows. $x_i = x'_i$ for $i \neq k$ and

$$x'_k x_k = \prod_{j \to k} x_j + \prod_{k \to j} x_j, \tag{3}$$

where the empty product is taken to be 1.

Thus, each time we mutate, we produce a new variable, related to the old variables via the above *exchange relation*. Iterating we can produce several new variables, all of which can be expressed as rational functions in terms of the initial variables $x_1, ..., x_N$. The remarkable property of quivers, discovered by Fomin and Zelevensky in [7] is that all the new variables are in fact Laurent polynomials in $x_1, ..., x_N$.

Theorem 3.1. *(Laurent Phenomenon, [7])*
Given an initial seed $(Q, \boldsymbol{x})$, any variable that can be produced by mutating the initial seed is can be expressed as an element of $\mathbb{Z}[x_1^{\pm 1}, ..., x_N^{\pm 1}]$ *by the exchange relations.*

Now with this theorem in mind, we describe cluster algebras.

Definition 3.2. (Cluster Algebra) Given a labelled seed $(Q, \mathbf{x})$ we define the cluster algebra associated with Q, $\mathcal{A}$, as the follows. Let S be the set of all cluster variables that can be achieved by mutation from the initial seed $(Q, \mathbf{x})$. Then, $\mathcal{A}$ is the $\mathbb{Z}$-algebra generated by S along with the exchange relations.

Then, by Theorem 3.1, every new cluster variable produced can be expressed as a Laurent polynomial in $x_1, ..., x_N$, so we may view $\mathcal{A}$ as a subalgebra of $\mathbb{Z}[x_1^{\pm 1}, ..., x_N^{\pm 1}]$.

There are many interesting properties related to the sequences discussed in this paper of cluster algebras. We return to them in Section 5.]

Now, we define a method for constructing Laurent phenomenon sequences using mutation periodic quivers.

Definition 3.3. Given quivers Q and Q' on the same vertex set $\{1, 2, ..., N\}$ we say that Q' is a *rotation* of Q if there is a (directed graph) isomorphism from Q to Q' that maps vertex i to $i+1$ for $i = 1, 2..., N-1$ and vertex N to vertex 1.

We say that Q with vertices $1, 2, ..., N$ is *mutation periodic* if $\mu_1(Q)$ is a rotation of Q.

We say that a vertex v of Q is a *sink* if all edges with and endpoint at v have their head at v. Similarly, we say vertex v of Q is a *source* if all edges with and endpoint at v have their tail at v. If vertex 1 is a sink we say Q be is of *sink-type*.

Now, given a mutation periodic quiver Q, then the we can generated a Laurent phenomenon sequence as follows. Start with the seed $(Q, \mathbf{x})$. Then, let $P(x_2, ..., x_N)$ be the exchange polynomial when we mutate at 1. Then, for $i = 1, 2, ...$ at step i if we mutate Q at vertex i (reduced modulo N), and call the cluster variable produced at i x_{i+N}, then the sequence (x_i) satisfies $x_{n+N}x_n = P(x_{n+1}, ..., x_{n+N-1})$ for $n \in \mathbb{N}$. Thus, by Theorem 3.1 the we see that the sequence (x_i) is a Laurent phenomenon sequence.

Now, we describe a class of mutation periodic called primitives.

Definition 3.4. For positive integer N and $p < N$, we define the *primitive quiver* $P_N^{(p)}$ as follows. $P_N^{(p)}$ has vertices $1, 2, ..., N$. For every

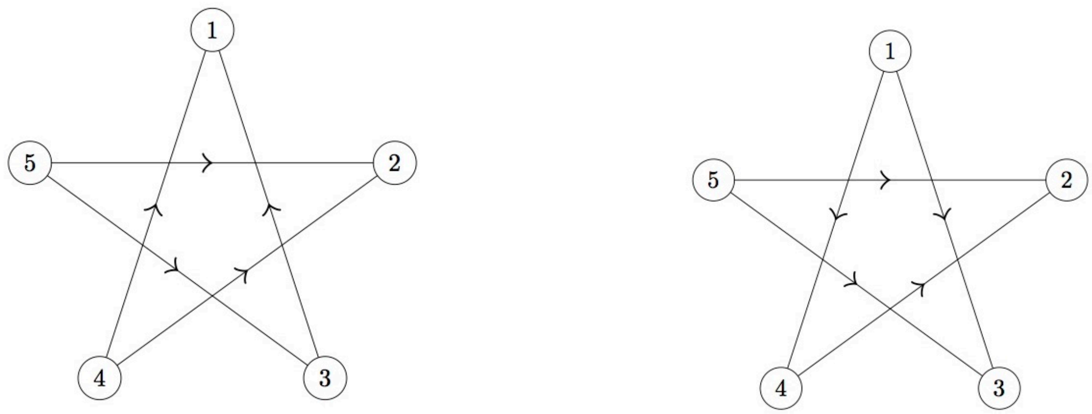

Figure 1. Quiver $P_5^{(2)}$ on left and its mutation at 1 on right; note the mutation is obtained by rotating the entire quiver clockwise

two vertices i, j with $j = p + i$ or $i = p + j - N$, there is an edge from j to i.

An examples is given in Figure 1.

Note that the undirected graph obtained from $P_N^{(p)}$ is a union of disjoint cycles where each vertex i is adjacent to $i + p$ and $i - p$ (vertices taken modulo N) therefore symmetric under rotation. Further all directed edges $i \to j$ in $P_N^{(p)}$ have $j < i$. Note that vertex 1 is a sink, so mutation at 1 simply reverses the arrows incident at 1; we see that this rotates the entire quiver, so $P_N^{(p)}$ is mutation periodic. It gives the exchange polynomial $P(x_1, ..., x_N) = x_p x_q + 1$. We see that for $\gcd(p, q) = 1$ and $A = 1$ this gives polynomial (1) in Theorem 1.1. We return to this sequence and primitives in Section 5.

3.2. **Laurent Phenomenon Algebras and Period One Seeds.** In [11] a new type of seed $(\mathbf{P}, \mathbf{x})$ is presented, where $\mathbf{P}$ is a N dimensional vector of polynomials in $\mathbb{Z}[x]$ that satisfy certain conditions. $\mathbf{P}$ directly generalizes the structure of a quiver; each polynomial somehow represents an exchange relation at that vertex. These seeds $(\mathbf{P}, \mathbf{x})$ can then be mutated by picking an index $k \in \{1, 2, ..., N\}$ to generated a new seed $(\mathbf{P}', \mathbf{x}')$. The polynomials $\mathbf{P}$ all change but only the cluster variable x_k changes and x_k' is related to $\mathbf{x}$ through some relations. There is a theorem analogous to 3.1 for these types of seeds that says that any cluster variable produced through some sequence of mutations is a Laurent polynomial in $x_1, ..., x_N$. The algebra generated by all possible cluster variables that can arise form the initial seed $(\mathbf{P}, \mathbf{x})$ is then called the *Laurent phenomenon algebra*. Then, we can define *period one seeds* as seeds where if we mutate at 1 and rewrite the new polynomials $\mathbf{P}$ in terms of the cluster variables $\mathbf{x}'$, then we get the same polynomials as $\mathbf{x}$. These period one seeds generalize mutation-periodic quivers and several such period one seeds are discussed in [1]. Some of these seeds give rise to the polynomials (3),(4),(5) in Theorem 1.1. We do not give further exposition on these period one seeds because they are not needed for our results, but we refer the reader to [1] section 2.1, 2.3, and Theorem 3.9.

4. Proofs of Linearizability

In this section we give proofs of Theorems 2.2, and 2.3.

4.1. **Proof of Theorem 2.2.**

First, we state a well-known useful theorem from [2].

Theorem 4.1. *Let $M = (a_{i,j})$ be an N dimensional square matrix. Then, let A be the matrix obtained by deleting the last row and last column of M. Let B be the matrix obtained by deleting the last row and first column of M. Let C be the matrix obtained by deleting the first row and last column of M. Let D be the matrix obtained by deleting the first row and first column of M. Finally, let X be the matrix obtained by deleting the first and last rows and columns of M. Then,* $\det M \det X = \det A \det D - \det B \det C$.

Let N be a positive integer and $A, B \in \mathbb{Z}$. Then, let $(x_i)_{i\in\mathbb{N}}$ be defined such that

$$x_{n+N}x_n = x_{n+N-1}x_{n+1} + A\sum_{i=1}^{N-1} x_{n+i} + B, \tag{4}$$

for all positive integer n.

Lemma 4.2. *For all positive integer n,* $\frac{x_{n+2}+x_n+A}{x_{n+1}} = \frac{x_{n+1+N}+x_{n+N-1}+A}{x_{n+N}}$

Proof. If we consider Equation 4 with $n = k$ and $n = k+1$ and subtract, we get that for all positive integer k, $x_{k+N+1}x_{k+1} - x_{k+N}x_k = x_{k+N}x_{k+2} - x_{k+N-1}x_{k+1} + A(x_{k+N} - x_{k+1})$. Rearranging and factoring gives, $x_{k+1}(x_{k+N+1} + x_{k+N-1} + A) = x_{k+N}(x_k + x_{k+2} + A)$. Dividing through by $x_{k+1}x_{k+N}$ gives the desired result. □

Now, for positive integer n, let $M_n = (a_{i,j})_{0\le i,j\le 2}$, where $a_{i,j} = x_{n+(N-1)i+j}$. We will show that $\det M_n$ is constant. By Theorem 4.1 and Equation 4.

(5)

$$\det M_n x_{n+N} = A^2\left(\sum_{i=1}^{N-1} x_{n+i}\right)\left(\sum_{i=N+1}^{2N-1} x_{n+i}\right) - A^2\left(\sum_{i=2}^{N} x_{n+i}\right)\left(\sum_{i=N}^{2N-2} x_{n+i}\right)$$

Lettin $U = \sum_{i=1}^{N-1} x_{n+i}$ and $V = \sum_{i=N+1}^{2N-2} x_{n+i}$, we have

(6)

$$\begin{aligned} A^{-2}\det M_n x_{n+N} &= (x_{n+1}+U)(V+x_{n+2N-1}) - (U+x_{n+N})(x_{n+N}+V) \\ &= x_{n+1}V + x_{n+2N-1}U + x_{n+1}x_{n+2N-1} - x_{n+N}(U+V+x_{n+N}) \end{aligned}$$

Now, from Equation 4, we may substitute $U = A^{-1}(x_n x_{n+N} - x_{n+1}x_{n+N-1} - Ax_{n+1})$ and $V = A^{-1}(x_{n+N}x_{n+2N} - x_{n+N+1}x_{n+2N-1} - Ax_{n+2N-1})$ into the above, and collect terms with x_{n+N}. Dividing through by x_{n+N} gives

(7)

$$\begin{aligned} A^{-2}\det M_n &= -\left(\sum_{i=2}^{2N-2} x_{n+i}\right) + A^{-1}x_{n+1}x_{n+2N-1}x_{n+1}\frac{(x_{n+N-1}+x_{n+N+1}+A)}{x_{n+N}} \\ &= -\left(\sum_{i=2}^{2N-2} x_{n+i}\right) + A^{-1}x_{n+1}x_{n+2N-1}(x_n + x_{n+2} + A). \end{aligned}$$

The last step above follows from lemma 4.2. Now, plugging in $n = k$ and $n = k+1$ into the above and subtracting, then applying Lemma 4.2 and equation 4 gives $\det M_{n+1} - \det M_n = 0$. It follows that $\det M_n$ is constant for positive integer n. Then, applying Dodgson condensation to the matrix $J_n = (a_{i,j})_{0\le i,j\le 3}$, we get that $\det J_n = 0$. Now, because J_n and J_{n+1} share three rows in common, for all n, it follows that $J_1, J_2, ...$ all have a common vector in the null space. Normalize this vector so its first entry is 1 and let it be $v = (-1, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3)^t)$. It follows that $\mathcal{K}_3 x_{n+3(N-1)} + \mathcal{K}_2 x_{n+2(N-1)} + \mathcal{K}_1 x_{n+N-1} - x_n = 0$ for all n. We can solve for $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ by solving $J_n v = 0$, which reduces to a 3 by 3 system of equations and use Cramer's rule. Because all matrices M_k have the same determinant, it follows that $\mathcal{K}_1 = 1$. Also $\mathcal{K}_2, \mathcal{K}_3$ are rational functions in the entries of J_n, so they are rational functions in $x_1, ..., x_N$, as desired.

4.2. **Proof of Theorem 2.3.** We first prove the theorem for $A = 1$.

P generates the sequence $(x_i)_{i\in\mathbb{N}}$, where $x_{n+3}x_n = x_{n+1}x_{n+2} + x_{n+1} - x_{n+2}$. For postive integer $n > 2$, we then have the following

$$x_{n-2} = \frac{-x_n x_{n+1}^2 + x_n x_{n+2} + x_n^2 - x_n + x_n}{x_{n+1}x_{n+2}}$$

$$x_{n-1} = \frac{x_n x_{n+1} + x_n - x_{n+1}}{x_n + 2}$$

$$x_{n+3} = \frac{x_{n+1}x_{n+2} + x_{n+1} - x_{n+2}}{x_n}$$

$$x_{n+4} = \frac{x_{n+1}x_{n+2}^2 + x_n x_{n+2} - x_{n+2}^2 - x_{n+2} + x_{n+2}}{x_n x_{n+1}}$$

$$x_{n+5} = (x_{n+1}^2 x_{n+2}^3 + x_n x_{n+1}^2 x_{n+2} + x_{n+1}^2 x_{n+2}^2 - 2x_{n+1}x_{n+2}^3 + x_n x_{n+1}^2$$
$$-x_n^2 x_{n+2} - x_{n+1}^2 x_{n+2} + x_{n+2}^3 + x_n x_{n+1} - x_{n+1}^2 - x_n x_{n+2} + 2x_{n+1}x_{n+2} - x_{n+2}^2)(x_n^2 x_{n+1} x_{n+2})^{-1}$$

Then, by explicit computation we may verify that

$$\frac{x_{n+5} - x_{n-1}}{x_{n+3} - x_{n+1}} = \frac{x_{n+4} - x_{n-2}}{x_{n+2} - x_n}$$

This implies the theorem because then $\frac{x_{n+4}-x_{n-2}}{x_{n+2}-x_n}$ is constant for all n. We may explicitly compute $\mathcal{K}$ by plugging $n = 3$ into the RHS of the above.

$$\mathcal{K} = \frac{x_1}{x_3} + \frac{x_3}{x_1} + \frac{x_1}{x_2 x_3} - \frac{x_3}{x_1 x_2} + \frac{1}{x_1 x_2} - \frac{1}{x_1 x_3} + \frac{1}{x_2 x_3} + 1. \tag{8}$$

Now, the sequence generated by $x_1x_2 + Ax_1 - Ax_2$ is obtained for the sequence generated by $x_1x_2 + x_1 - x_2$ when we multiply all terms by A, so the general result follows easily. The coefficient $\mathcal{K}$ is obtained by substituting Ax_i for x_i in Equation 8.

5. Matching Formulas for Recurrence Coefficients

In this section we give proof Theorem 2.4. We use results relating surfaces and cluster algebras that follow from the theory of tensor diagrams in [4] The [5], and [12].

But, first we define almost perfect matchings and the graph $G_{p,q}$.

Definition 5.1. (Planar Graphs in Disks and Almost Perfect Matchings) Let G be a planar graph embedded in a disk and let ∂G be the set of vertices of G that lie on the boundary of the disk. An *almost perfect matching* of G is a matching of G such that all the vertices of

G not in ∂G are matched (some of the vertices in ∂G can be matched). We let $M'(G)$ denote the set of almost perfect matchings of G.

Now, we define the graph $G_{p,q}$ which is a planar graph embedded in a disk.

Definition 5.2. If $p \leq q$ then, let $G_{p,q}$ be the following planar graph, G, embedded in a disk. G has $p+q+1$ faces. One of the faces is *central face* and the remaining $p+q$ are *boundary faces.* The central face is a $2p$-gon and has all its vertices in the interior of the disk. The boundary faces are labelled $1, 2, ..., p+q$ and each have one edge on the boundary of the disk; the remaining vertices of each boundary face are shared with the central face. The boundary face labelled i is adjacent to the two boundary faces labelled $i-p$ and $i+p$. The face labelled i is a triangle if $p < a_i < q+1$ and is a quadrilateral otherwise; note there are $2p$ quadrilateral boundary faces and $N-2p$ triangle boundary faces. If $p > q$, let $G_{p,q} = G_{q,p}$.

In particular notice that, two faces i, j are adjacent in G if and only if vertex i and j are adjacent in $P_N^{(p)}$ and face i is a quadrilateral if and only if vertex i is a sink or source in $P_N^{(p)}$.

Now, we define the *weight function* $\text{wt}(m)$ of a matching.

Definition 5.3. Given an almost perfect matching, m of $G_{p,q}$ and a non-central face, i, we define $f_i(m)$ as follows. Let m' be the be the matching of the face i induced by m (m' consists of those edges in m which are an edge of i) then, let α be the number of vertices of i that are matched in m' that don't lie on the boundary of the disk and let β be the number of vertices of i not matched m' that don't lie on the boundary of the disk. Then, $f_i = 1$ if $(\alpha, \beta) = (2, 0)$, $f_i = 0$ if $(\alpha, \beta) = (1, 1)$ or $(\alpha, \beta) = 1, 0)$ and $f_i = -1$ if $(\alpha, \beta) = (0, 2)$ or $(\alpha, \beta) = (0, 1)$. Then, $\text{wt}(m) = \prod_{i=1}^{N} (A^{1/2} x_i)^{f_i(m)}$.

Note that the sequence produced by the polynomial $x_p x_q + A$ is obtained from the sequence produced by $x_p x_q + 1$ by multiplying each term by $A^{1/2}$; hence, it suffices to prove the theorem for $A = 1$. From now on, we assume that $A = 1$.

5.1. **Surfaces and Cluster Algebras.** We begin with some basic definitions.

Definition 5.4. (Bordered Marked Surface)

Let S be a bordered orientable surface and let M be a finite set of *marked points* on the boundary of S. We call (S, M) a *bordered marked surface.* Note that the boundary of S is divided into curves which have

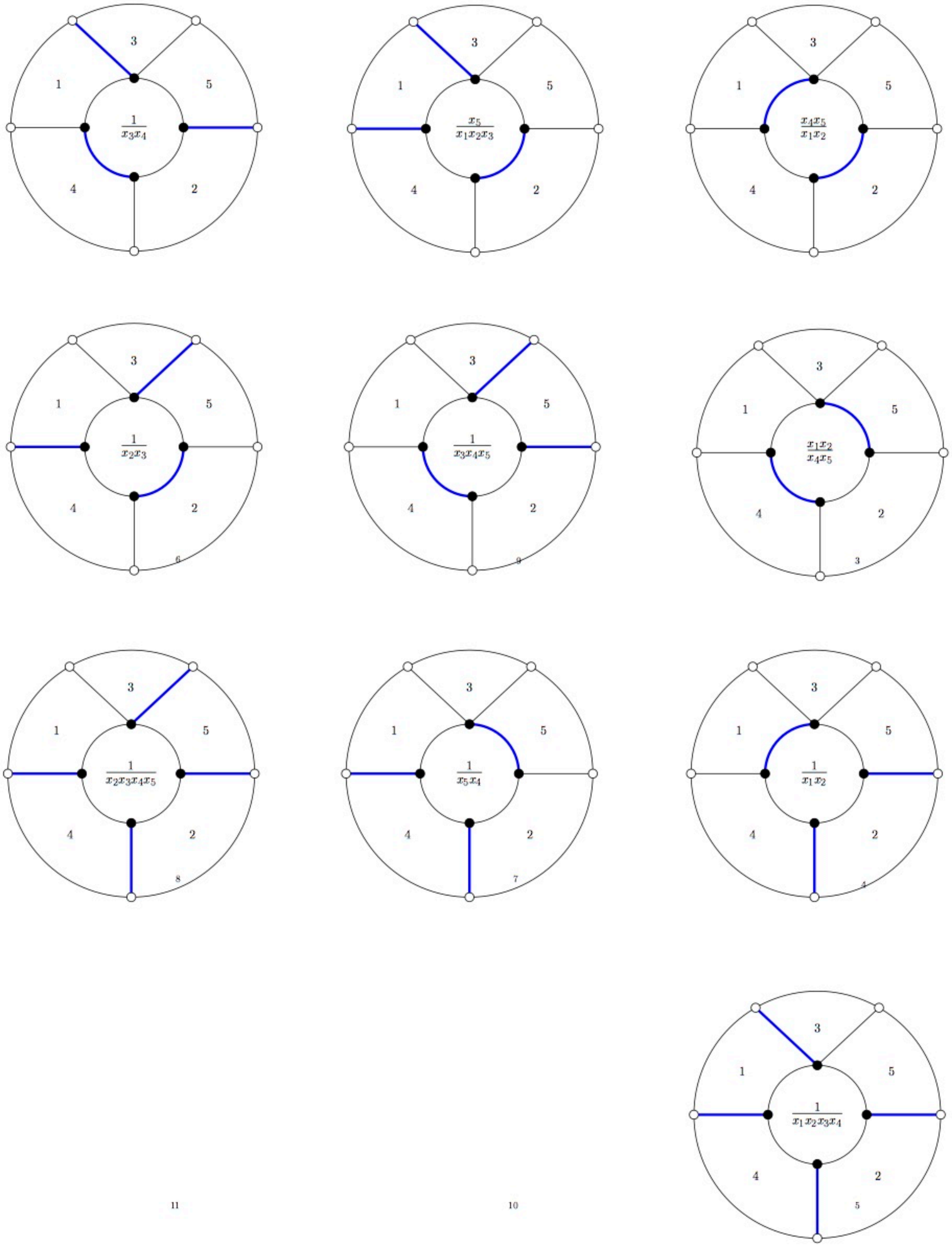

FIGURE 2. $G_{2,3}$ and its 10 almost matchings and corresponding weight monomials for $A = 1$

points of M has endpoints and whose interiors contain no points of M. We call these curves *boundary segments* of (S, M).

Definition 5.5. (Arcs and Loops on Surfaces)
A *ordinary arc* in (S, M) is a non-self-intersecting curve in S whose endpoints lie in M which does not intersect the boundary of S except at the endpoints. A *closed loop* in (S, M) is a closed curve which does not intersect the boundary of S. We say that two ordinary arcs, γ and γ' are isotopic if there is an isotopy of curves from γ to γ' whose image consists of ordinary arcs. In general, we consider ordinary arcs and closed loop to up to isotopy.

Now, given two ordinary arcs, γ and γ', we say that γ and γ' are *compatible* if there are ordinary arcs γ_1 and γ_1' isotopic to γ and γ', respectively, such that γ_1 and γ_1' do not intersect not the interior of S. A *triangulation* of (S, M) is a maximal collection of pairwise nonequivalent and compatible ordinary arcs. The ordinary arcs in a triangulation along with the boundary segments cut S into several triangles.

Now, given a triangulation T of (S, M), we can pick two triangles $\triangle_1$ and $\triangle_2$ of T that share a side γ. Let the other sides of $\triangle_1$ be α_1, β_1 and let the other sides of $\triangle_2$ be α_2, β_2. Then, $\alpha_1, \beta_1, \alpha_2, \beta_2$ bound a quadrilateral in (S, M) one of whose diagonals is γ. We call the other diagonal of this quadrilateral the *flip* of γ in the triangulation T. We call the triangulation obtained be replacing γ with its flip the flip of T at γ.

Now, we see that flips allow us to mutate between different triangulations of a marked bordered surface.

Now, we show how to generate a quiver from a triangulated surface. We will see that mutation of this quiver corresponds to flipping arcs in the triangulation.

Definition 5.6. (Surface Triangulations and Quivers)
Let (S, M) be a bordered marked surface. Given a triangulation T and an enumeration $\tau_1, ..., \tau_N$ of the arcs we construct the quiver Q_T. First, construct directed graph G as follows. Place a vertex on each ordinary arc of T and each boundary segment of (S, M). Label the vertices corresponding to $\tau_1, ..., \tau_N$ as $v_{\tau_1}, v_{\tau_2}, ..., v_{\tau_N}$. Then, for each triangle, $\triangle$, in T, cyclically connect the vertices corresponding to the three sides of T such that when these edges are drawn on S they are in counter-clockwise orientation. Then, let Q_T be the subgraph of G induced on vertices $v_{\tau_1}, v_{\tau_2}, ..., v_{\tau_N}$. Also, let $\mathbf{x}_T = (x_{\tau_1}, x_{\tau_2}, ..., x_{\tau_N})$.

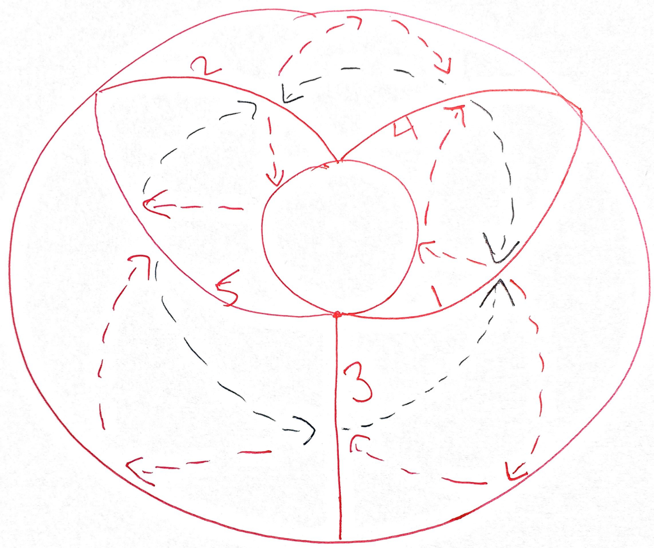

FIGURE 3. Triangulation T of marked annulus; ordinary arcs labelled $1, 2, 3, 4, 5$ correspond to cluster variables at the vertices of the quiver $Q = P_5^{(2)}$ and the dotted black arrows correspond to edges of Q;

Now, we are ready to state the theorem, based on work in [6] and [5] that allows us to relate surfaces and cluster algebras. It is also stated as Theorem 2.11 in [12].

Theorem 5.7. *Given a bordered surface (S, M) and a triangulation T, let $\mathcal{A}$ be the cluster algebra generated from the initial seed $(Q_T, \boldsymbol{x}_T)$. Then, the labelled seeds of $\mathcal{A}$ are in bijection with the labelled triangulations of (S, M) and the cluster variables of $\mathcal{A}$ are in bijection with the ordinary arcs in (S, M) that are not boundary segments. Further, given seed $(Q', \boldsymbol{x}')$ of $\mathcal{A}$ and vertex v of Q', mutating at v corresponds to flip of the arc corresponding to the vertex v in the triangulation corresponding to the quiver Q'.*

To illustrate this theorem, we refer to Figure 3.

Now, because all seeds are related by mutation, the choice of initial seed is arbitrary, so in fact, we can associate with each marked bordered surface (S, M) a cluster algebra $\mathcal{A}_{S,M}$ without picking an initial seed for the algebra. Then, the cluster variables in $\mathcal{A}_{S,M}$ correspond to the

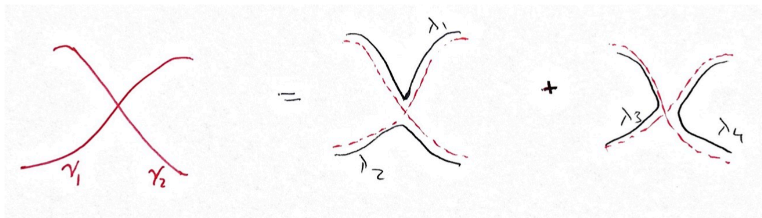

FIGURE 4. : Skein relations for product of intersecting curves

(classes of) ordinary arcs in (S, M) and unlabelled seeds correspond to triangulations.

Now, in general the correspondence in Theorem 5.7 between ordinary arcs and cluster variables can be extended to assign to each collection of transversally intersecting curves (not necessarily ordinary arc or closed loop) an element of $\mathcal{A}$. Given a curve γ we will denote its corresponding element of $\mathcal{A}$ by x_γ. We can then use the following results introduced in [3] to relate these elements to the cluster variables as follows.

Theorem 5.8. *The following are true about curves in S and the corresponding elements of $\mathcal{A}$.*

(1) *Superimposing two curves (ordinary arc or closed loop) corresponds to multiplication of the corresponding of $\mathcal{A}$*
(2) *(Skein Relation) Given two curves γ_1 and γ_2 that intersect transversally, we have*

$$x_{\gamma_1} x_{\gamma_2} = x_{\lambda_1} x_{\lambda_2} + x_{\lambda_3} x_{\lambda_4},$$

where (λ_1, λ_2) and (λ_3, λ_4) are the two ways to resolve the intersection as shown in Figure 4.
(3) *Any curve contractible to a point in M corresponds to the element 0 in $\mathcal{A}$.*

Note that if two curves intersect at t different points, then iterating the skein relation we can write their product as sum of 2^t terms, one for each way to resolve all t intersections.

Now, we are ready to prove Theorem 2.4.

5.2. **Proof of Theorem 2.4.** Let p, q be positive integer such that $\gcd(p, q) = 1$ and $p < q$ (we discard the case $(p, q) = (1, 1)$; it can be explicitly verified in a manner similar to Theorem 2.3). Consider the bounded marked surface (S, M) where S is an annulus and that M consists of p points on the inner boundary of the annulus and q points on the outer boundary. We write $\mathcal{A}$ to denote $\mathcal{A}_{S,M}$. For any curve (ordinary arc or closed loop) γ, we let x_γ be the corresponding element of $\mathcal{A}$. For any triangulation T of S, we let $(Q_t, \mathbf{x}_T)$, denote the corresponding seed of $\mathcal{A}$.

Also, from now on, we will refer to ordinary arcs in S that have one endpoint on each boundary of S as *radial arcs*

Label the marked points on the inner boundary $u_1, ..., u_p$ and the points on the outer boundary $w_1, ..., w_q$ both in clockwise order. From now on, take the indices of the u-s modulo p and the indices of the w-s modulo q. Let $\mathcal{C}$ denote the unique (up to isotopy) non contractible non-self-intersecting closed loop contained in S.

Now, we define operators $\mathcal{T}_1$ and $\mathcal{T}_2$. Given a radial arc γ with one endpoint, u_i, on the inner boundary of S and one endpoint, w_j, on the outer boundary of S, we define $\mathcal{T}_1\gamma$ and $\mathcal{T}_2\gamma$ as follows. We may continuously move the endpoint u_i clockwise along the boundary of S to u_{i+1}. This, can be extended to an isotopy from γ to a curve γ' with endpoints u_{i+1} and w_j. We call this curve $\mathcal{T}_1\gamma$. Similarly, we may continuously move the endpoint w_j clockwise along the boundary of S to w_{j+1}. This, defines an isotopy from γ to a curve γ'' with endpoints u_i and w_{j+1}. We call this curve $\mathcal{T}_2\gamma$.

We see that $\mathcal{T}_1$ and $\mathcal{T}_2$ define bijections on the set of radial arcs. Note that $\mathcal{T}_1$ and $\mathcal{T}_2$ commute and further, $\mathcal{T}_1^p\mathcal{T}_2^q = I$. In particular, we have $\mathcal{T}_1^p = \mathcal{T}_2^{-q} = \mathcal{D}$, where $\mathcal{D}$ denotes the Dehn twist. Further, given any two radial curves we can get from one to the other by application of $\mathcal{T}_1, \mathcal{T}_2$ and their inverses.

Now, we define an operation ϕ, which given a radial arc γ produces a labelled triangulation, T, of (S, M) one of whose arcs is γ. We define the radial arcs $\tau_1, \tau_2, ..., \tau_N$ as follows (take indices of the τ's modulo N).

1. Let $\tau_1 = \gamma$
2. Then, for $k = 1, 2, ..., N-1$ do the following.
   1. (a) Let $i, j \in \{1, 2, ..., N\}$ such that $i \equiv 1 + (k-1)p \pmod N$ and $j \equiv 1 + kp \pmod N$
   2. (b) If $j < i$ then $\tau_j = \mathcal{T}_1\tau_i$ and if $i < j$, then $\tau_i = \mathcal{T}_2\tau_j$

Note that in defining $\tau_1, \tau_{1+p}, ..., \tau_{1+(N-1)p}$, we have started from γ and applied $\mathcal{T}_1$ and $\mathcal{T}_2$ several times. Note that modulo N, $\{1, 1+p, ..., 1+(N-2)p\} = \{1, 2, ..., N\} \setminus \{N+1-p\}$. Therefore, we have applied $\mathcal{T}_2$ a total of $N-p-1 = q-1$ times and $\mathcal{T}_1$ a total of p times. Thus, because there are q points on the outer boundary of S and p points on the inner boundary of S, we have that $\tau_1, ..., \tau_N$ are all distinct. Further, $\mathcal{T}_2\tau_{1+(N-1)p} = \mathcal{T}_1^q\mathcal{T}_2^p\tau_1 = \tau_1$. Thus, for each $k = 1, 2, ..., N$, because $\tau_{1+(k-1)p}$ and τ_{1+kp} are related by either $\mathcal{T}_1$ or $\mathcal{T}_2$, they from a triangle along with one of the boundary segments of S.

Thus, The segments $\tau_1, \tau_2, ..., \tau_N$ form a triangulation of (S, M). Call this triangulation $T = (\tau_1, ..., \tau_N)$.

We see by the construction of T that the following important property holds.

Lemma 5.9. *$T = \phi\gamma$ is the unique triangulation $\tau_1, ..., \tau_N$, consisting of radial arcs, such that $\tau_1 = \gamma$ and the quiver Q_T is isomorphic to $P_N^{(p)}$ if we send v_{τ_i} to vertex i in $P_N^{(p)}$, for $i = 1, 2, ..., N$*

Proof. Suppose $T' = (\tau'_1, ..., \tau'_N)$ with $\tau'_1 = \gamma$ is a triangulation, consisting of radial arcs, such that $Q_{T'}$ is isomorphic to $P_N^{(p)}$ after sending $v_{\tau'_i}$ to vertex i in $P_N^{(p)}$. Because every vertex i of $P_N^{(p)}$ is adjacent to only vertex $i+p$ and $i-p$ (taking vertex labels modulo N), it follows that τ'_i is in a triangle with only τ'_{i+p} and τ'_{i-p} (taking indices modulo N). Now, τ'_i is an edge in two triangles of T', and no triangle in T' can consist of two boundary segments (because T' consists of radial arcs), it follows that every triangle in T' consists of exactly one boundary segment. Hence, τ'_i and τ'_{i+p} are related by either $\mathcal{T}_1$ or $\mathcal{T}_2$ for each, i. Thus, it follows that for each i, we can determine, based on the direction of the edge between i and $i+p$ in $P_N^{(p)}$ whether we have $\tau'_{i+p} = \mathcal{T}_1\tau'_i$ or $\tau'_{i+p} = \mathcal{T}_2^{-1}\tau'_i$ or we have $\tau'_{i+p} = \mathcal{T}_2\tau'_i$ or $\tau'_{i+p} = \mathcal{T}_1^{-1}\tau'_i$. Then, by induction, we see that the choice we make for τ'_{i+p} uniquely determines $\tau'_{i+2p}, ..., \tau'_{i+(N-1)p}$, because no two arcs in T' can cross. Finally, only one of the choices for τ'_{i+p} will give valid triangulation, because the other will apply $\mathcal{T}_1$ a total of $q-1$ times and $\mathcal{T}_2$ a total of p times, so $\tau'_{i+(N-1)p}, \tau'_1$, no longer form a triangle, because $p \neq q$. □

Now, from the definition of ϕ, the following result is clear.

Lemma 5.10. *For radial arc γ, we have $\phi\mathcal{T}_1\gamma = \mathcal{T}_1\phi\gamma$ and $\phi\mathcal{T}_2\gamma = \mathcal{T}_2\phi\gamma$, where $\mathcal{T}_1$ and $\mathcal{T}_2$ act on $\phi\gamma$ by acting on each arc in the triangulation $\phi\gamma$.*

The result follows from the fact that $\mathcal{T}_1$ and $\mathcal{T}_2$ commute.

Now, we prove another important lemma that follows from the fact that $P_N^{(p)}$ is mutatation-periodic.

Lemma 5.11. *There are integers a, b satisfying $pb - qa = 1$, such that following holds: Let γ is an radial arc, let $\phi\gamma = T = (\tau_1, ..., \tau_N)$, and let τ_{N+1} is the flip τ_1 with respect to T; then, $(\tau_2, ..., \tau_{N+1}) = \mathcal{T}_1^a\mathcal{T}_2^b(\tau_1, ..., \tau_N)$.*

Proof. We have defined T above, so that $\tau_2 = \mathcal{T}_1^a\mathcal{T}_2^b\tau_1$, for some positive integer a, b with $a \leq p$ and $b \leq q$. Let a, b be these integers. Now,

$(Q_t, \mathbf{x}_T)$, where $\mathbf{x}_T = x_{\tau_1}, ..., x_{\tau_N}$, be the seed in $\mathcal{A}$ corresponding to T. Recall that Q is isomorphic to the mutation-periodic quiver $P_N^{(p)}$ after sending v_{τ_i} to vertex i. Let $(Q', \mathbf{x}')$ be the mutation of this seed at v_{τ_1}. Because mutation of seeds corresponds to flip of diagonal, we may label the vertices of Q' $(v_{\tau_2}, ..., v_{\tau_{N+1}})$ such that $\mathbf{x}' = (x_{\tau_2}, ..., x_{\tau_{N+1}})$. Then, Q' is isomorphic to $P_N^{(p)}$ after sending v_{τ_i} to vertex $i-1$ (take vertices modulo N). Now, by the uniqueness property in Lemma 5.9, it must follow that $\phi\tau_2 = (\tau_2, ..., \tau_{N+1})$. Thus, $(\tau_2, ..., \tau_{N+1}) = \mathcal{T}_1^a\mathcal{T}_2^b(\tau_1, ..., \tau_N)$ follows from $\tau_2 = \mathcal{T}_1^a\mathcal{T}_2^b\tau_1$ and repeated application of 5.10.

It remains to show that $pb - qa = 1$. Consider the sequence $J = a_0, ..., a_{N-1}$, where $a_k \equiv 1+kp$ and $a_k \in \{1, 2, ..., N\}$ for $k = 0, 1, ..., N-1$. Let $a_t = 2$. Then, the number of $a = \#A$ and $b = \#B$ where $A = \{a_{k-1} > a_k : k \in \{1, 2, ..., t\}\}$ and $B = \#\{a_{k-1} < a_k : k \in \{1, 2, ..., t\}\}$. Thus,

(9)
$$pb-qa = \sum_A(-q)+\sum_B p = \sum_A(a_k-a_{k-1})+\sum_B(a_k-a_{k-1}) = a_t-a_0 = 1$$

□

Let a, b be as in the Lemma.

Now, let γ_1 be an arbitrary radial arc and let $T_0 = \phi\gamma_1 = (\gamma_1, \gamma_2, ..., \gamma_N)$. Then, for $k = N+1, N+2, ...,$ construct γ_k and T_{k-N} inductively as follows

(1) $T_{k-N} = (\gamma_{k-N}, ..., \gamma_{k-1})$ is a triangulation of S, so flip γ_{k-N} with respect to this triangulation to produce γ_k
(2) By Lemma 5.2, it follows that $T_{k-N+1} = (\gamma_{k-N}, ..., \gamma_{k-1})$ is a triangulation and it is obtained by applying $\mathcal{T}_1^a\mathcal{T}_2^b$ to T_{k-N}

From this construction two properties follow. First, we see that $\gamma_k = \mathcal{T}_1^a\mathcal{T}_2^b\gamma_{k-1}$ for positive integer k. Also, we see that because all T_k are isomorphic to $P_{(p)}^N$ and we obtain T_{k+1} by flipping the arc corresponding to vertex 1, because $P_{(p)}^N$ is mutation periodic, by the correspondence in Theorem 5.7, we have that the sequence cluster variables $(x_{\gamma_n})_{n\in\mathbb{N}}$ satisfies the same recurrence as the cluster variables produced by cyclically mutating the quiver $P_{(p)}^N$; namely, $x_{\gamma_{n+N}}x_{\gamma_n} = x_{\gamma_{n+p}}x_{\gamma_{n+q}} + 1$. Thus, because T_1 is a triangulation, the cluster variables $x_{\gamma_1}, ..., x_{\gamma_N}$ are algebraically independent, so in fact it suffices to work with the sequence $(x_{\gamma_n})_{n\in\mathbb{N}}$.

Now let $n > pq$. Suppose we superimpose the curves γ_n and $\mathcal{C}$. From the skein relations we have that $x_{\gamma_n}x_{\mathcal{C}} = \mathcal{D}x_{\gamma_n} + \mathcal{D}^{-1}x_{\gamma_n}$, where, recall, $\mathcal{D} = \mathcal{T}_1^p = \mathcal{T}_2^{-q}$ is the Dehn twist. Now, note that $(\mathcal{T}_1^a\mathcal{T}_2^b)^{pq} =$

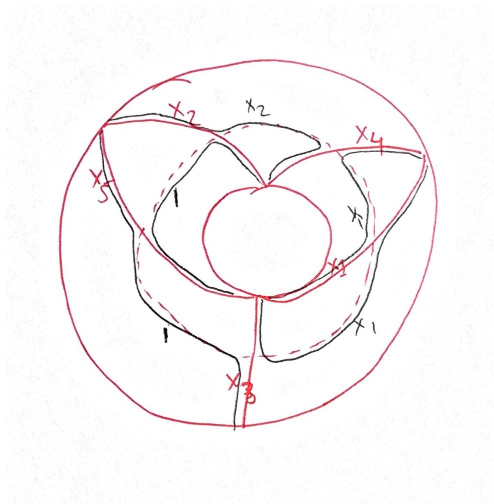

FIGURE 5. One of the ways to resolve the intersections that gives $x_1x_2x_3$

$\mathcal{T}_1^{pqa-pqa}\mathcal{T}_2^{pqb-qab} = \mathcal{T}_2^q = \mathcal{D}^{-1}$ because $\mathcal{T}_1^p\mathcal{T}_2^q = 1$ and $pb - qa = 1$. It follows that

$$x_{\mathcal{C}}x_{\gamma_n} = (x_{\gamma_{n-pq}} + x_{\gamma_{n+pq}}). \tag{10}$$

Thus, shown that proper form of the recurrence hold, it remains to compute $\mathcal{K}$. That is, we expand $x_{\mathcal{C}}$ in terms of $\mathbf{x}_{T_1} = (x_{\gamma_1}, x_{\gamma_2}, ..., x_{\gamma_N})$. Now, to simplify notation, we will, for the remainder of the proof, refer to the arc γ_i as i, so that the cluster variable corresponding to arc i is simply x_i.

Now, superimpose the curve $\mathcal{C}$ on the triangulation T. Because, the element of $\mathcal{A}$ corresponding to T is $x_1...x_N$, we may then expand $x_{\mathcal{C}}x_1...x_N$ using the skein relations, to get a formula for $\mathcal{C}$. For (p,q) this gives us a sum of 10 terms, one of which is shown in This is shown in Figure 5 for $(p,q) = (2,3)$.

Note that for $(p,q) = (2,3)$, there $2^5 = 32$ ways to resolve the intersection, but only 10 contribute to the expansion of $x_{\mathcal{C}}x_1...x_N$. Recall from Theorem 5.8 we know that any curve contractible to a point in M corresponds to 0. There are only 10 ways to revolve the intersections such that no curve contracts to a point.

In general, what we want to show, that the product expands using the skein relations

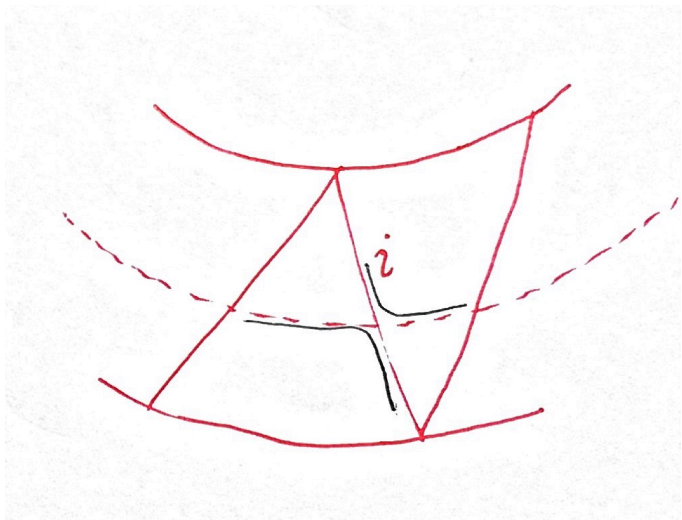

FIGURE 6. *efficient resolution* of the intersection at arc i

$$x_{\mathcal{C}}x_1 \dots x_N = \sum_{m \in M'(G_{p,q})} \prod_{i=1}^{N} x_i^{1+f_i(m)}. \tag{11}$$

We do this by showing a bijection between the almost perfect matchings of $M'(G_{p,q})$ and the ways to resolve the intersection of $\mathcal{C}$ of T. We will leave verifying that the weights $1 + f_i(m)$ are in fact the correct weights to the reader as the details are uninteresting and tedious.

Suppose we are given a resolution, R, of $\mathcal{C}$ and T such that no resulting curves go to 0. We construct an almost perfect matching, m, of $G_{p,q}$ as follows. First, we say that arc i is sink or source if vertex i is a sink or source, respectively, in $P_N^{(p)}$ ($i \leq p$ or $i > q$, respectively). For arc i that is a sink or source, we say that the *efficient resolution* of the intersection at i is as in Figure 6. If arc i is a sink or source, then face i in $G_{p,q}$ is a quadrilateral with an edge on the center face. Call this edge e_i. The edge e_i is in the matching m if and only if arc i is not efficiently resolved in R. Now, it remains to define which of the edges of $G_{p,q}$ from the center face to the boundary are in m.

Suppose we have several adjacent arcs $i, i+p, i+2p, \dots, i+kp$ in S (indices are taken modulo N) such that i and $i+kp$ are each either a sink or a source and the remaining arcs are not (note i and $i+kp$ cannot both be sinks or both be sources). Note that of i and $i+kp$,

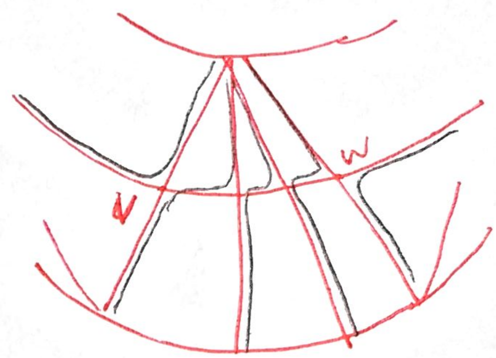

Figure 7. If exactly one of the intersections v and w is resolved efficiently then there is one way to resolve the remaining intersections so that no curve contracts to the boundary

at least one must be resolve efficiently, or the resolution R will have a curve that contracts to 0. This corresponds to the fact that both e_i and e_{i+kp} cannot be in the matching m because they have a common vertex. Now, if exactly one of i and $i+kp$ are resolved efficiently, then the resolutions at $i+p, i+2p, i+(k-1)p$ are determined. This is shown in Figure 7. If both i and $i+kp$ are efficiently resolved, then there are k ways to resolve the intersections at $i+p, ..., i+(k-1)p$. These correspond to the k different ways to match the common vertex of e_i and e_{i+kp} in $G_{p,q}$. This correspondence is also shown in Figure 8. Thus, we have shown how to construct the remainder of the matching m from R.

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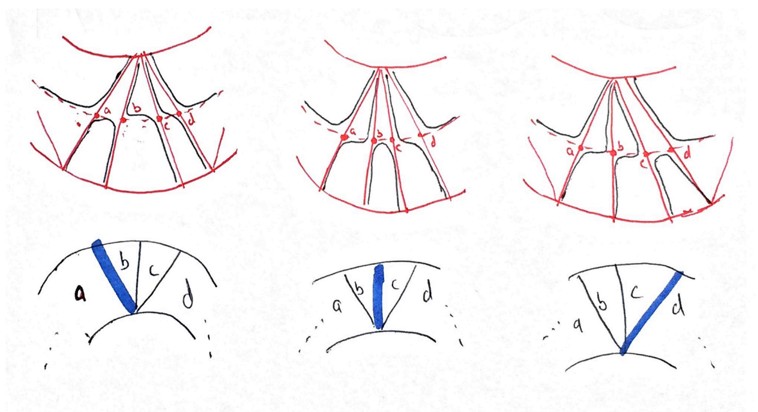

Figure 8. If both of the intersections a and d are resolved efficiently then there are 3 ways to resolve the remaining intersections; they correspond to the the three ways to match the the common interior vertex of face a and d

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