HAUSDORFF DIMENSION OF CLOSURE OF CYCLES IN d-MAPS ON THE CIRCLE

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Abstract. We study the dynamics of the map $x \mapsto dx (\text{mod } 1)$ on the unit circle. We characterize the invariant finite subsets of this map which are called cycles and are graded by their degrees. By looking at the combinatorial properties of the base-d expansion of the elements in the cycles, we prove a conjecture of Curt McMullen that the Hausdorff dimension of the closure of degree-m cycles is equal to $\log m / \log d$. We also study the connection between this map and the degree-d maps on the hyperbolic disk.

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1. Introduction

Degree-d maps on the unit disk have many interesting geometric, topological and analytic properties, which are closely related to hyperbolic geometry. The dynamics of these maps has an important application in classifying dynamical systems generated by polynomials in single complex variable [3] and it provides useful information about Julia sets and Mandelbrot sets [2, 5]. To get an understanding of this family of maps, people studied the behaviour of these maps restricted to the boundary unit circle [3, 4]. In the paper [6], McMullen gave a complete description of simple (i.e. degree 1) cycles of these boundary maps, showing that those simple cycles are an analogue of simple closed geodesics on hyperbolic surfaces in the sense that the closure of union of simple cycles has Hausdorff Dimension 0 and the closure of union of simple closed geodesics has Hausdorff Dimension 1.

Here, we consider degree-d holomorphic maps from the unit disk onto itself. We focus on one such map $z \mapsto z^d$. The restriction of this map to the boundary circle is equivalent to the map $x \mapsto dx \pmod 1$ on $\mathbb{R}/\mathbb{Z}$. In this paper, we study cycles of higher degrees in this map, and in particular, calculate the Hausdorff dimension of their closure.

Definition 1 (d-map)**.** Let $d \in \mathbb{N}$. We define d-map as the map on the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$ given by

$$x \mapsto dx \pmod 1, \forall x \in S^1 \tag{1}$$

In other words, if the base-d expansion of a point is $(0.b_1b_2b_3....)_d$, then d-map takes it to the point whose base-d expansion is $(0.b_2b_3b_4....)_d$

Definition 2 (Cycle)**.** Let d be a positive integer greater than 1. A finite set $C \subset S^1$ is called a cycle for d-map, iff d-map restricted to C is a transitive permutation. In terms of the base-d expansion, C is given by the following collection of points.

$$C = \{(0.\overline{b_ib_{i+1}....b_nb_1b_2....b_{i-1}})_d | 1 \le i \le n\} \tag{2}$$

where n is a fixed positive integer and $b_1, b_2, ..., b_n$ are fixed digits in $\{0, 1, ..., d-1\}$.

Here are some examples of cycles. Consider the case $d = 2$. Here, $\{0\} = \{(0.\overline{0})_2\}$ is the only 1-element cycle and $\{\frac{1}{3}, \frac{2}{3}\} = \{(0.\overline{01})_2, (0.\overline{10})_2\}$ is the only 2-element cycle. For an integer $n > 2$, there are multiple n-element cycles. For example, there are two 3-element cycles: $\{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\}$ and $\{\frac{3}{7}, \frac{5}{7}, \frac{6}{7}\}$.

Now we define an important invariant of a cycle, called degree.

Definition 3 (Degree of a cycle)**.** Let d be a positive integer greater than 1. Let C be a cycle for d-map. The degree of C is the smallest non-negative integer m for which there exists a degree-m map $f : S^1 \to S^1$ such that f and d-map agree on C. It is denoted by $deg(C)$.

Remark 1. Note that $0 \le deg(C) \le d$ because d-map is a degree-d map.

Remark 2. The cycles which contain only one element are fixed points of d-map and have degree 0. All other cycles have positive degree.

Now we give two examples.

Consider $d = 2$ and $C = \{\frac{3}{7}, \frac{5}{7}, \frac{6}{7}\}$. Under 2-map,

$$\frac{3}{7} \mapsto \frac{6}{7}, \ \frac{5}{7} \mapsto \frac{3}{7}, \ \frac{6}{7} \mapsto \frac{5}{7}$$

i.e. the cyclic order of the points in C is preserved. So, C has degree 1. In other words, as we jump on the points in the cycle from $\frac{3}{7} \to \frac{5}{7} \to \frac{6}{7}$ (jump over 0) and back to $\frac{3}{7}$, we complete one full circle (we jump over 0 exactly once). As we jump on the images of these points, $\frac{6}{7} \to$ (jump over 0 to) $\frac{3}{7} \to \frac{5}{7}$ and back to $\frac{6}{7}$, again we complete one circle (we jump over zero exactly once). This means that the degree of C is 1.

Now consider $d = 3$ and $C = \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$. Under 3-map,

$$\frac{1}{5} \mapsto \frac{3}{5}, \ \frac{2}{5} \mapsto \frac{1}{5}, \ \frac{3}{5} \mapsto \frac{4}{5}, \ \frac{4}{5} \mapsto \frac{2}{5}$$

So, as we jump on the points in the cycle from $\frac{1}{5} \to \frac{2}{5} \to \frac{3}{5} \to \frac{4}{5}$ (jump over 0) and back to $\frac{1}{5}$, we complete one full circle (we jump over 0 exactly once). As we jump on the images of these points, $\frac{3}{5} \to$ (jump over 0 to) $\frac{1}{5} \to \frac{4}{5} \to$ (jump over 0 to) $\frac{2}{5}$ and back to $\frac{3}{5}$, we complete two circles (we jump over zero twice). So, the degree of C is 2.

Definition 4 (Closure of cycles)**.** Let $m, d \in \mathbb{N}$ with $1 < d$ and $1 \leq m \leq d$. We define $E_{m,d}$ as the closure of the union of degree-m cycles for d-map.

$$E_{m,d} = \overline{\{c \in S^1 | c \text{ is in a cycle of degree-}m \text{ for } d\text{-map}\}} \tag{3}$$

In the paper [6] Curtis McMullen discussed the simple (i.e. degree 1) cycles for the d-map and and computed the Hausdorff Dimension of their closure $E_{1,d}$. Recall that Hausdorff Dimension is defined as follows:

Definition 5 (Hausdorff Dimension)**.** The Hausdorff Dimension of a set E is defined as:

$$dim_H(E) = \inf \{\delta \geq 0 | \ \inf \{\sum r_i^\delta | E \subset \bigcup B(x_i, r_i)\} = 0\} \tag{4}$$

Theorem 1 (McMullen)**.** *Let d be a positive integer greater than 1. Then $dim_H(E_{1,d}) = 0$.*

In this paper, we characterize the degree-m cycles for d-map and generalize the above result.

Theorem 2. *Let $m, d \in \mathbb{N}$ with $1 < d$ and $1 \leq m \leq d$. Then*

$$dim_H(E_{m,d}) = \frac{\log m}{\log d}.$$

This paper is organized in the following way: In Section 2, we get a lower bound on the Hausdorff Dimension of $E_{m,d}$ by calculating the Hausdorff Dimension of a subset of $E_{m,d}$. In Section 3, we prove the upper bound by imitating the proof of Theorem 1 in paper [6], using combinatorial arguments. In Section 4, we futher investigate the relation with hyperbolic geometry by looking into a conjecture proposed in [6] and give partial results.

2. Lower Bound

In this section, we define and study two useful invariants of a cycle called crossing number and Digit Portrait, which are directly related to the degree of the cycle. Then we use these properties to construct a subset of $E_{m,d}$ which has Hausdorff Dimension $\frac{\log m}{\log d}$.

Definition 6 (Crossing)**.** Let d be a positive integer greater than 1. Let $C = \{c_1, c_2, ..., c_n\}$ be a cycle for d-map such that

$$0 < c_1 < c_2 < .... < c_n < 1$$

Let $c_{n+1} = c_1$. For any $1 \leq i \leq n$, the pair (c_i, c_{i+1}) is called a crossing generated by C (or simply a crossing) iff

$$0 < dc_{i+1} (\text{mod } 1) < dc_i (\text{mod } 1) < 1 \tag{5}$$

The total number of such crossings is called the crossing number of C.

$$\eta(C) = \# \text{ (crossings generated by C)} \tag{6}$$

Remark 3. As we jump on S^1 in the counterclockwise direction and trace the points of the cycle from $c_1 \to c_2 \to .... \to c_n$ and (jump over 0) back to c_1, we complete one full circle. When we trace the images of these points, $dc_1(\text{mod } 1) \to dc_2(\text{mod } 1) \to ...... \to dc_n(\text{mod } 1)$ and back to $dc_1(\text{mod } 1)$, we may jump over 0 multiple times. Each time we jump over 0, we have a crossing. The crossing number of the cycle is equal to its degree.

Remark 4. If $n = 1$, then there are no crossings. The crossing number is 0 which is the degree of 1-element cycles.

Lemma 1. *Let d be a positive integer greater than 1. Let $C = \{c_1, c_2, ..., c_n\}$ be a cycle for d-map with $0 < c_1 < c_2 < .... < c_n < 1$. Then, the crossing number of C is equal to its degree or*

$$\eta(C) = deg(C)$$

Proof. The case $n = 1$ is obvious. Assume $n > 1$. Let $c_{n+k} = c_k, \forall\ k \in \mathbb{N}$.

Let $1 \le i_1 < i_2 < ... < i_{\eta(C)} \le n$ be such that $\forall\ 1 \le t \le \eta(C), (c_{i_t}, c_{i_t+1})$ is a crossing generated by C.

First, we prove $deg(C) \ge \eta(C)$ by contradiction. Suppose $\eta(C) > deg(C)$. We can divide S^1 or $[0, 1)$ in intervals $I_1, I_2, ..., I_{deg(C)}$ such that on each I_r, there exists a continuous non-decreasing map g_r to $[0, 1)$ which agrees with d-map on $I_r \cap C$.

Note that there are exactly $\eta(C)$ sets of the type $\{c_{i_t}, c_{i_t+1}, ...c_{i_{t+1}}\}$. Since $\eta(C) > deg(C)$, we can find t such that $\{c_{i_t}, c_{i_t+1}, ...c_{i_{t+1}}\} \subset I_r$ for some r. So,

$$0 < g_r(c_{i_t+1}) = dc_{i_t+1}(\text{mod } 1) < dc_{i_t}(\text{mod } 1) = g_r(c_{i_t}) < 1$$

which contradicts the non-decresing nature of g_r. So, $deg(C) \ge \eta(C)$.

Now we prove that $deg(C) \le \eta(C)$ by constructing an appropriate map of degree $\eta(C)$. Define $f : S^1 \to S^1$ as:

$$f(x) = \begin{cases} \frac{dc_{i_t+1}(\text{mod } 1)}{\frac{c_{i_t+1}-c_{i_t}}{2}} \left(x - \frac{c_{i_t}+c_{i_t+1}}{2}\right) & \text{if } x \in [\frac{c_{i_t}+c_{i_t+1}}{2}, c_{i_t+1}] \\ \\ dc_i(\text{mod } 1) + \frac{dc_{i+1}(\text{mod } 1) - dc_i(\text{mod } 1)}{c_{i+1}-c_i} (x - c_i) & \text{if } x \in [c_i, c_{i+1}] \subset [c_{i_t+1}, c_{i_{t+1}}] \\ \\ dc_{i_{t+1}}(\text{mod } 1) + \frac{1-dc_{i_{t+1}}(\text{mod } 1)}{\frac{c_{i_{t+1}+1}-c_{i_{t+1}}}{2}} (x - c_{i_{t+1}}) & \text{if} x \in [c_{i_{t+1}}, \frac{c_{i_{t+1}}+c_{i_{t+1}+1}}{2}] \end{cases}$$

where intervals are taken in counter-clockwise direction.

Here, we have divided S^1 into $\eta(C)$ intervals of the type $[\frac{c_{i_t}+c_{i_t+1}}{2}, \frac{c_{i_{t+1}}+c_{i_{t+1}+1}}{2}]$, each of which is mapped onto S^1.f maps the endpoints of the interval to 0, points of C to their images under d-map. In between, f is linear.

So, f is a continuous function and $\forall\ t$, and restriction of f gives a bijection between $[\frac{c_{i_t}+c_{i_t+1}}{2}, \frac{c_{i_{t+1}}+c_{i_{t+1}+1}}{2})$ and $[0, 1)$ or S^1. Thus, f has degree $\eta(C)$. Also, $f|C$ agrees with d-map. This completes the proof. □

Now that we have established the relation between the degree and the crossing number of a cycle, we need a tool to estimate the crossing number. We observe that the crossing number of a cycle is related to the order of points in the cycle, and hence the digits in the base-d expansion of points of the cycle. Now we define an invariant of the cycle called Digit Portrait which characterizes these digits.

Definition 7 (Digit Portrait)**.** Let d be a positive integer greater than 1. Let C be a cycle for d-map. The Digit Portrait of C is the non-decreasing map $F : \{0, 1, 2, ..., (d-1)\} \to \{0, 1, 2, ..., |C|\}$

which satisfies

$$F(j) = |\ C \ \cap \ [0, (j+1)/d)\ | \ \ \forall\ 0 \leq j \leq (d-1)$$

or

$$F(j) = \#\ (\text{elements of } C \text{ whose base-}d \text{ expansion starts with a digit less than } j+1)$$

Let $dig(C)$ be the number of distinct positive values taken by F. Note that if a digit j is absent in the base-d expansion of a point in C, then $F(j) = 0$ or $F(j) = F(j-1)$. So, $dig(C)$ is also the number of distinct digits which appear in the base-d expansion of any point in C. To estimate the crossing number of C, we need the second interpretation of $dig(C)$.

Here is an example. Consider $d = 4$ and the cycle $C = \{(0.\overline{0012})_4, (0.\overline{0120})_4, (0.\overline{1200})_4, (0.\overline{2001})_4\}$. Note that

$$0 = (0.0)_4 < (0.\overline{0012})_4 < (0.\overline{0120})_4 < \frac{1}{4} = (0.1)_4 < (0.\overline{1200})_4 < \frac{2}{4} = (0.2)_4 < (0.\overline{2001})_3 < \frac{3}{4} = (0.3)_4$$

The Digit Portrait of C is the map $F : \{0, 1, 2, 3\} \to \{0, 1, 2, 3, 4\}$ given by:

$$F(0) = 2, F(1) = 3, F(2) = 4, F(3) = 4$$

F takes the values 2, 3 and 4. So, dig(C) is 3. There are exactly 3 digits (0, 1 and 2) which appear in the base-4 expansion of the points in C.

Now we establish the relation between $dig(C)$ and the crossing number of C:

Lemma 2. *Let d be a positive integer greater than 1. Let $C = \{c_1, c_2, ..., c_n\}$ be a cycle for d-map with $0 < c_1 < c_2 < .... < c_n < 1$ and $n > 1$. Then, the crossing number of C is at most the number of distinct digits which appear in the base-d expansion of a point in C or*

$$\eta(C) \leq dig(C)$$

Proof. Let $1 \leq i < n$. Let $c_i \in [j_1/d, (j_1+1)/d), c_{i+1} \in [j_2/d, (j_2+1)/d)$. Note that $j_1 \leq j_2$ because $c_i < c_{i+1}$.

If $j_1 = j_2 = j$, then

$$0 < dc_i (\text{mod } 1) = dc_i - j < dc_{i+1} - j = dc_{i+1} (\text{mod } 1) < 1$$

In this case, (c_i, c_{i+1}) cannot be a crossing. So, (c_i, c_{i+1}) is a crossing only if $j_1 < j_2$ i.e. c_i is the largest element of $C \cap\ [j_1/d, (j_1+1)/d)$ and c_{i+1} is the smallest element of $C\ \cap\ [j_2/d, (j_2+1)/d)$.

Thus, there are at most $(dig(C) - 1)$ i's for which $1 \leq i < n$ and (c_i, c_{i+1}) is a crossing. For some cycles, (c_n, c_1) is an additional crossing. So, there are at most $dig(C)$ i's for which $1 \leq i \leq n$ and (c_i, c_{i+1}) is a crossing. □

Lemma 1 and Lemma 2 together give a way of estimating the degree of a cycle by looking at the digits in the base-d expansion of a point in the cycle. Now we use this to get a sufficient condition for a point to be in the closure of the cycles of fixed degree.

Lemma 3. *Let $m, d \in \mathbb{N}$ with $1 \leq m \leq d$ and $1 < d$. Then any point in S^1 whose base-d expansion contains atmost m distinct digits lies in $E_{m,d}$.*

Proof. Let $\alpha \in S^1$. Let $\alpha = (0.\alpha_1\alpha_2\alpha_3.....)_d$ such that $\forall\ r, \alpha_r \in \{b_1, b_2, ..., b_m\} \subset \{0, 1, ..., (d-1)\}$. Here, $b_1, b_2, ..., b_m$ are fixed digits in base-d such that $b_1 < b_2 < ... < b_m$.

To prove that α lies in $E_{m,d}$, we will show that $\forall\ q \in \mathbb{N}$, there exists a degree-$m$ cycle C for d-map which intersects d^{-q} neighborhood of α. Any periodic point whose base-d expansion contains exactly m digits is in a cycle of degree at most m. To get the maximum possible degree, we need maximum possible crossings. This can be achieved with the following construction:

Let $N \in \mathbb{N}$ such that $\forall\ t, N$ is greater than the number of times b_t appears in the first q digits of α. Let $< b_t >$ denote $b_t b_t b_t ... b_t$ (N times). Consider the following point:

$$c = (0.\overline{\alpha_1\alpha_2\alpha_3...\alpha_q < b_m >< b_1 >< b_{m-1} >< b_1 > .... < b_2 >< b_1 > b_m})_d \tag{7}$$

Clearly, c is in d^{-q} neighborhood of α. It is a point of cycle $C = \{c_1, c_2, ...c_{q+2N(m-1)+1}\}$ for d-map with $0 < c_1 < c_2 < ... < c_{q+2N(m-1)+1}$.

Now we need to prove that $deg(C) = m$. For each $1 \leq t \leq m$, let i_t be such that the largest element of C whose base-d expansion starts with the digit b_t is c_{i_t}. Note that c_{i_1} is at least $(0.b_1 b_m)_d$. For $t > 1$, c_{i_t} is at least $(0. < b_t >)_d$. For each $1 \leq t < m$, c_{i_t+1} is the smallest element of C whose base-d expansion starts with the digit b_{t+1}. Note that the base-d expansion of c_{i_t+1} starts with $0.b_t < b_1 >$.

So, $\forall\ 1 \leq t < m$,

$$0 < dc_{i_t+1}(\text{mod } 1) < dc_{i_t}(\text{mod } 1) < 1$$

i.e. (c_{i_t}, c_{i_t+1}) is a crossing.

Note that $c_{q+2N(m-1)+1} = c_{i_m}$ and $c_{i_m+1} = c_1 = c_{i_1}$. So,

$$0 < dc_{i_m+1}(\text{mod } 1) < (0.b_2)_d < dc_{i_m}(\text{mod } 1) < 1$$

i.e. (c_{i_m}, c_{i_m+1}) is a crossing.

Thus, the crossing number of C is at least m.

$$\eta(C) \geq m$$

From Lemma 2, we know that the crossing number of C is at most the number of distinct digits in the base-d expansion of a point C, which in this case, is m.

$$\eta(C) \leq dig(C) = m$$

So, $\deg(C) = \eta(C) = m$.

□

This immediately gives the following result:

Proposition 1. *$E_{d,d} = S^1$. Thus, $dim_H(E_{d,d}) = 1 = \frac{\log d}{\log d}$.*

Proof. Note that since base-d has only d digits, the set of all points of S^1 whose base-d expansion contains at most d digits is S^1 itself. So, we have $S^1 \subset E_{d,d} \subset S^1$ or $E_{d,d} = S^1$.

□

Let m and d be positive integers satisfying $d > 1$ and $1 \leq m \leq d$. Now we use Lemma 3 to construct a subset of $E_{m,d}$. Let $A_{m,d}$ be the set of points in S^1 whose base-d expansion contains the digits from 0 to $(m-1)$ only. Clearly, $A_{m,d} \subset E_{m,d}$.

The structure of $A_{m,d}$ is similar to the structure of Cantor's set. Here, we start with $X_0 = [0, 1]$. Write X_0 in d closed intervals of length $1/d$. Let X_1 be union of first m of these intervals.

$$X_1 = \bigcup_{i=0}^{m-1} [i/d, (i+1)/d]$$

X_1 is the union of m intervals of length $1/d$. Divide each such interval in d equal parts and take the first m in X_2. X_2 is the union of m^2 intervals of length $1/d^2$.

Repeat the process. If X_i is the union of m^i intervals of length d^{-i}, then divide each such interval in d equal parts and take the first m in X_{i+1}.

$$A_{m,d} = \bigcap_{i=0}^{\infty} X_i$$

Now we calculate the Hausdorff Dimension of $A_{m,d}$.

Lemma 4. *$dim_H(A_{m,d}) = \frac{\log m}{\log d}$.*

Proof. Let $\beta > 0$.

Note that $\forall\ i, A_{m,d} \subset X_i$. So, for each i, we have m^i intervals of length d^{-i} that form a covering of $A_{m,d}$. If $\beta = \frac{\log m}{\log d} + \varepsilon$ for some $\varepsilon > 0$,

$$\lim_{i\to\infty} m^i (d^{-i})^\beta = \lim_{i\to\infty} m^i (d^{-i\varepsilon})((d^{\frac{\log m}{\log d}})^{-i}) = \lim_{i\to\infty} d^{-i\varepsilon} = 0$$

This means that, for any $\beta > \frac{\log m}{\log d}$, we can cover $A_{m,d}$ such that the summation of β powers of the lengths of the intervals in the cover is as small as we like. So, $dim_H(A_{m,d}) \leq \frac{\log m}{\log d}$.

Now we need to prove that the Hausdorff dimension of $A_{m,d}$ is at least $\frac{\log m}{\log d}$. Note that we can consider $A_{m,d}$ as a subset of $[0, m/d]$ which is a compact set. So, for any countable cover (U_r) of $A_{m,d}$, we can find finitely many open sets $V_1, V_2, ..., V_p$ such that

$$\bigcup_{r=0}^{\infty} U_r \subset \bigcup_{r=0}^{p} V_r \quad \text{and}$$

$$\sum_{r=0}^{p} |V_r|^\beta \leq \sum_{r=0}^{\infty} |U_r|^\beta$$

Now we will get a lower bound on $\sum_{r=0}^{p} |V_r|^\beta$. Let $b \in \mathbb{N}$ such that $\forall\ r, d^{-b} \leq |V_r|$. $\forall\ 1 \leq i \leq b$,let N_i be the number of V_r's which satisfy $d^{-i} \leq |V_r| < d^{-i+1}$. Note that if $|V_r| < d^{-i+1}$, then V_r can intersect at most two intervals in X_{i-1}. Hence, V_r can contain at most $2m^{b-i+1}$ intervals in X_b. X_b has m^b intervals. So,

$$\sum_{i=0}^{b} 2m^{b-i+1} N_i \geq m^b$$

or

$$\sum_{i=0}^{b} m^{-i} N_i \geq \frac{1}{2m}$$

For $\beta = \frac{\log m}{\log d}$,

$$\sum_{r=0}^{p} |V_r|^\beta \geq \sum_{i=0}^{b} (d^{-i})^\beta N_i = \sum_{i=0}^{b} m^{-i} N_i \geq \frac{1}{2m}$$

i.e. the summation of $\frac{\log m}{\log d}$ powers of the lengths of the intervals which form a cover of $A_{m,d}$ has a positive lower bound. Thus, $dim_H(A_{m,d}) \geq \frac{\log m}{\log d}$.

□

Now we have a lower bound on the Hausdorff Dimension of $E_{m,d}$.

Theorem 3. *Let $m, d \in \mathbb{N}$ with $1 < d$ and $1 \leq m \leq d$. Then,*

$$dim_H(E_{m,d}) \geq \frac{\log m}{\log d}$$

Proof. $A_{m,d} \subset E_{m,d}$. So, $dim_H(E_{m,d}) \geq dim_H(A_{m,d}) = \frac{\log m}{\log d}$.

□

3. Upper Bound

In this section, we first find an upper bound on the number of degree-m cycles for d-map which have n elements. We extend this result to precycles. Then, we find an appropriate covering of $E_{m,d}$ to prove that its Hausdorff Dimension is at most $\frac{\log m}{\log d}$.

Definition 8 (Partition generated by a cycle)**.** Let $C = \{c_1, c_2, ..., c_n\}$ be a degree-m cycle for d-map such that $0 < c_1 < c_2 < .... < c_n < 1$. Let σ be the map on $\{1, 2, ..., n\}$ which satisfies:

$$dc_r (\text{mod } 1) = c_{\sigma(r)}, \forall\ 1 \leq r \leq n$$

Note that σ is a permutation of $\{1, 2, ..., n\}$ because C is a cycle.

Let $1 \leq i_1 < i_2 < ... < i_m \leq n$ be such that $\forall\ 1 \leq t \leq m, (c_{i_t}, c_{i_t+1})$ is a crossing generated by C. From the ordering of the elements of C and the definition of crossing, we conclude that:

$$\sigma(i_t) > \sigma(i_t + 1) \text{ and } \sigma(i_t + 1) < \sigma(i_t + 2) < ... < \sigma(i_{t+1}),\ \forall\ 1 \leq t < m$$

and

$$\sigma(i_m) > \sigma(i_m + 1) \text{ and } \sigma(i_m + 1) < ... < \sigma(n) < \sigma(1) < ... < \sigma(i_1)$$

Now we construct a partition of $\{1, 2, .., n\}$ using the above property of σ.

$$P_t = \begin{cases} \{\sigma(r) | i_t < r \leq i_{t+1}\} & \text{if } 1 \leq t < m \\ \{\sigma(r) | i_m < r \leq n\} \cup \{\sigma(r) | 1 \leq r \leq i_1\} & \text{if } t = m \end{cases}$$

$\{P_t | 1 \leq t \leq m\}$ is a partition of $\{1, 2, .., n\}$, called as the partition generated by C and it is denoted by $P(C)$.

Both $P(C)$ and the map σ are useful counting n-element degree-m cycles.

Let us consider $d = 3$ and $C = \{(0.\overline{00102})_3, (0.\overline{01020})_3, (0.\overline{02001})_3, (0.\overline{10200})_3, (0.\overline{20010})_3\}$ as an example. Here, $n = 5$. Under 3-map,

$$c_1 = (0.\overline{00102})_3 \mapsto (0.\overline{01020})_3 = c_2$$
$$c_2 = (0.\overline{01020})_3 \mapsto (0.\overline{10200})_3 = c_4$$
$$c_3 = (0.\overline{02001})_3 \mapsto (0.\overline{20010})_3 = c_5$$
$$c_4 = (0.\overline{10200})_3 \mapsto (0.\overline{02001})_3 = c_3$$
$$c_5 = (0.\overline{20010})_3 \mapsto (0.\overline{00102})_3 = c_1$$

So, σ is the permutation of $\{1, 2, 3, 4, 5\}$ which takes 1, 2, 3, 4, 5 to 2, 4, 5, 3, 1 respectively. (c_3, c_4) and (c_4, c_5) are the crossings generated by the cycle. So, $i_1 = 3, i_2 = 4$ and the partition generated by C is given by:

$$P(C) = \{P_1, P_2\} \text{ where } P_1 = \{\sigma(r) | 3 < r \leq 4\}, P_2 = \{\sigma(r) | 4 < r \leq 5\} \cup \{\sigma(r) | 1 \leq r \leq 3\}$$

or

$$P(C) = \{\{3\}, \{1, 2, 4, 5\}\}$$

Now we will show that if some properties of a cycle such as degree, the partition it generates and its digit portrait are given, then we can construct the cycle (find its points). Later, we will use this result to count the number of cycles of fixed degree and cardinality.

Lemma 5. *Given positive integers d, m, n, $P = \{P_1, P_2, ..., P_m\}$ which is a partition of $\{1, 2, ..., n\}$, positive integer $i_1 < |P_m|$ and a non-decreasing map $F : \{0, 1, 2, ..., (d-1)\} \to \{0, 1, 2, ..., n\}$ with $F(d-1) = n$, there exists at most one cycle C for d-map such that:*

(1) $C = \{c_1, c_2, ..., c_n\}$ *with* $0 < c_1 < c_2 < .... < c_n < 1$
(2) $deg(C) = m$
(3) $P(C) = P$ *or* P *is the partition generated by* C
(4) *If* $i_t = i_{t-1} + |P_t| \; \forall \; 1 < t \leq m$, *then* (c_i, c_{i+1}) *is a crossing* $\forall \; i \in \{i_1, i_2, ..., i_m\}$.
(5) F *is the Digit Portrait of* C.

Proof. Suppose C is a cycle for d-map which satisfies (1), (2), (3), (4), (5). Let σ be the map on $\{1, 2, ..., n\}$ given by:

$$\sigma(r) = \begin{cases} (r - i_t)^{\text{th}} \text{ element of } P_t & \text{if } i_t < r \leq i_{t+1} \text{ and } 1 \leq t < m \\ (r - i_m)^{\text{th}} \text{ element of } P_m & \text{if } i_m < r \\ (r + n - i_m)^{\text{th}} \text{ element of } P_m & \text{if } r < i_1 \end{cases}$$

where $\forall \; 1 \leq t \leq m$, the elements of P_t are in increasing order. Note that σ is uniquely determined by m, n, P and i_1. Comparing this with the definition of partition generated by a cycle, we conclude that

$$dc_r (\text{mod } 1) = c_{\sigma(r)}, \forall \; 1 \leq r \leq n$$

Let $b : \{1, 2, ..., n\} \to \{0, 1, 2, ..., (d-1)\}$ given by:

$$b(r) = \begin{cases} 0 & \text{if } r < F(0) \\ j & \text{if } F(j-1) < r \leq F(j) \text{ and } 1 \leq j \leq (d-1) \end{cases}$$

Note that b is uniquely determined by n, d and k. Comparing this with the definition of Digit Portrait, we conclude that

$$c_r \in [\; \frac{b(r)}{d}, \frac{b(r)+1}{d} \;), \; \forall \; 1 \leq r \leq n$$

or

$$b(r) = \text{the first digit in the base-}d \text{ expansion of } c_r, \forall \; 1 \leq r \leq n$$

Also,

$$\begin{aligned} b(\sigma(r)) &= \text{the first digit in the base-}d \text{ expansion of } c_{\sigma(r)} \\ &= \text{ the second digit in the base-}d \text{ expansion of } c_r \end{aligned}$$

Thus, we conculde that

$$c_r = (0.\overline{b(r) b(\sigma(r)) b(\sigma^2(r)) ..... b(\sigma^{d-1}(r))}\;)_d$$

i.e. the cycle C is uniquely determined by σ and b. □

Remark 5. (c_i, c_{i+1}) can be a crossing only if the first digits of the base-d expansions of c_i and c_{i+1} differ. So, cycle C satisfying all conditions in the above lemma can exist only if

$$\{F(0), F(1), ..., F(d-1)\} \subset \{i_1, i_2, ... i_m\} \cup \{n\}$$

We will use this in the proof of the following lemma.

Lemma 6. *Let* $d, m, n \in \mathbb{N}$ *with* $1 \leq m \leq d$ *and* $n, d > 1$. *Then the number of cycles* C *for* d*-map satisfying* $|C| = n$ *and* $deg(C) = m$ *is at most* $O(n^{d-m+1} m^{n-1})$.

Proof. d, m, n are given. Now we need $P = \{P_1, P_2, ..., P_m\}$ which is a partition of $\{1, 2, ..., n\}$, positive integer $i_1 < |P_m|$ and a non-decreasing map $F : \{0, 1, 2, ..., (d-1)\} \to \{0, 1, 2, ..., n\}$ with $F(d-1) = n$ to fix a degree-m, n-element cycle.

Let T be the set of ordered $(m+1)$-tuples $(P_1, P_2, ..., P_m, i_1)$ such that $P = \{P_1, P_2, ..., P_m\}$ is a partition of $\{1, 2, ..., n\}$, $i_1 \in \mathbb{N}$ and $i_1 \leq |P_m|$. We want an upper bound on $\#(T)$.

$$\begin{aligned}
\#(T) &= \sum_{\substack{a_1+a_2+...+a_m=n \\ a_i\in\mathbb{N}}} a_m \binom{n}{a_1}\binom{n-a_1}{a_2}...\binom{n-a_1-a_2-...-a_{m-1}}{a_m} \\
&= \sum_{\substack{a_1+a_2+...+a_m=n \\ a_i\in\mathbb{N}}} a_m \cdot \frac{n!}{a_1!\ a_2!\ ....a_m!} \\
&= \sum_{\substack{a_1+a_2+...+a_m=n \\ a_i\in\mathbb{N}}} \frac{n!}{a_1!\ a_2!\ ....a_{m-1}!\ (a_m-1)!} \\
&\leq \sum_{\substack{a_1+a_2+...+a_{m-1}+(a_m-1)=n-1 \\ a_i\in\mathbb{N}\cup\{0\}}} n \cdot \frac{(n-1)!}{a_1!\ a_2!\ ....a_{m-1}!\ (a_m-1)!} \\
&= n \cdot m^{n-1}
\end{aligned}$$

After fixing an element of T, we need to fix a non-decreasing map F from $\{0, 1, 2, ..., (d-1)\}$ to $\{0, 1, 2, ..., n\}$ with $F(d-1) = n$ and $\{F(0), F(1), ..., F(d-1)\} \subset \{i_1, i_2, ...i_m\} \cup \{n\}$. We have $(n+1)^{(d-m-1)}$ or $(n+1)^{(d-m)}$ choices for F, depending on if n is in $\{i_1, i_2, ...i_m\}$ or not.

Now, using Lemma 5, we conclude that the number of degree-m n-element cycles for d-map is at most $O(n^{d-m+1}m^{n-1})$.

□

Definition 9 (Precycle)**.** Let d be a positive integer greater than 1. A finite set $C_P \subset S^1$ is called a precycle for d-map, iff $C_P = \{c \cdot d^i \pmod 1 | i \in \mathbb{N} \cup \{0\}\}$ for some $c \in S^1$. So, a precycle is the forward orbit of a rational point in S^1. It can be written in terms of base-d expansion of its points as follows:

$$C_P = \{(0.b_r b_{r+1}...b_{n_1}\overline{b'_1 b'_2....b'_{n_2}})_d | 1 \leq r \leq n_1\} \cup \{(0.\overline{b'_k b'_{k+1}...b_{n_2} b'_1....b'_{k-1}})_d | 1 \leq k \leq n_2\}$$

where n_1 is a fixed non-negative integer, n_2 is a fixed positive integer and all b_r, b'_k are fixed digits in $\{0, 1, 2, ..., d-1\}$.

Remark 6. Every precycle C_P includes a cycle. Note that if $n_1 = 0$, then C_P itself is a cycle.

Here is an example of a precycle. Consider $d = 2$ and $C_P = \{\frac{1}{3}, \frac{2}{3}, \frac{5}{6}\}$. C_P can be written as

$$C_P = \{\frac{5}{6} \cdot d^i (\text{mod } 1) | i \in \mathbb{N} \cup \{0\}\}$$

or

$$C_P = \{\frac{5}{6}\} \cup \{\frac{1}{3}, \frac{2}{3}\} = \{(0.1\overline{10})_2\} \cup \{(0.\overline{01})_2, (0.\overline{10})_2\}$$

Remark 7. We can define degree, crossing, crossing number, digit portrait and partition for a precycle simply by replacing C by C_P in the respective definitions.
Note that if C_P is a precycle which is not a cycle, then the map σ in the definition of partition (Definition 8) is not a permutation. Under σ, one element of $\{1, 2, ..., n\}$ has no preimages, one has two preimages and all other elements have exactly one preimage.

Now by using the logic which proves Lemma 6, we can prove the following result:

Lemma 7. *Let $d, m, n \in \mathbb{N}$ with $1 \leq m \leq d$. Then the number of precycles C_P for d-map satisfying $|C_P| = n$ and $deg(C_P) = m$ is at most $O(n^{d-m+3}m^{n-1})$.*

Remark 8. The increase in the exponent of n is due to the small change in the nature of the map σ in the definition of partition (Definition 8).

Now we find a suitable covering of $E_{m,d}$ and get an upper bound on its Hausdorff Dimension.

Theorem 4. *Let $m, d \in \mathbb{N}$ with $1 < d$ and $1 \leq m \leq d$. Then*

$$dim_H(E_{m,d}) \leq \frac{\log m}{\log d}$$

Proof. Fix a positive integer $n > 1$.

$$\mathscr{C}_P(n) = \{C_P | C_P \text{ is a precycle for } d\text{-map}, deg(C_P) \leq m, |C_P| \leq n\}$$

Lemma 7 implies that $|\mathscr{C}_P|$ is at most $O(n^{d-m+4}m^n)$.

Let C be a degree-m cycle for d-map. Let c be a point in C. There exists a degree-m map $f : S^1 \to S^1$ which agrees with d-map on C. Note that $|(f^n)'(c)| = d^n$. This means that there exists a point x in $O(d^{-n})$ neighborhood of c for which two of $x, f(x), f^2(x), ..., f^n(x)$ coincide. So, c is in $O(d^{-n})$ neighborhood of an element in $\mathscr{C}_P$.

Each point of any degree-m cycle for d-map lies in $O(d^{-n})$ neighborhood of an element in $\mathscr{C}_P$. So, $E_{m,d}$ lies in $O(d^{-n})$ neighborhood of $\mathscr{C}_P$ which has at most $O(n^{d-m+4}m^n)$ elements.

Let $\beta = \frac{\log m}{\log d} + \varepsilon$ for some $\varepsilon > 0$.

$$\begin{aligned}\lim_{n\to\infty} (d^{-n})^{\beta} n^{d-m+4} m^n &= \lim_{n\to\infty} (d^{\frac{\log m}{\log d}})^{-n} d^{-n\varepsilon} n^{d-m+4} m^n \\ &= \lim_{n\to\infty} d^{-n\varepsilon} n^{d-m+4} \\ &= 0\end{aligned}$$

For any $\beta > \frac{\log m}{\log d}$, we can cover $E_{m,d}$ such that the summation of β powers of the lengths of the intervals in the cover is as small as we like. So, $dim_H(E_{m,d}) \leq \frac{\log m}{\log d}$.

□

4. Maps on the disk

Now that we have studied the boundary behaviour, we would like to study some properties of the maps on hyperbolic space. As in paper [6] , we study the following functions:

Let d be a positive integer greater than 1. Let $\mathscr{B}_d$ be the space of maps f on the unit disk which are of the form

$$f(z) = \prod_{r=1}^{d-1} z\left(\frac{z - a_r}{1 - \overline{a_r} z}\right)$$

where $|a_r| < 1, \ \forall\ 1 \leq r \leq d-1$. Note that if $a_r = 0, \ \forall\ 1 \leq r \leq d-1$, we have a map on the disk which takes z to z^d. We denote this map by p_d.

As mentioned in [6], for each map $f \in \mathscr{B}_d$, there is a unique homeomorphism $\phi_f : S^1 \to S^1$ such that -

1. $f(z) = {\phi_f}^{-1} \circ p_d \circ \phi_f(z), \ \forall\ z \in S^1$
2. ϕ_f varies continuously with f
3. ϕ_{p_d} is the identity map on S^1.

Remark 9. Here S^1 is the set of complex numbers z satisfying $|z| = 1$.

In [6] , McMullen defined the following Length function:

Definition 10 (Length of a cycle)**.** The length on f of a cycle C for p_d is given by:

$$L(C,\ f) = \log |(f^q)'(z)| \tag{8}$$

where $q = |C|$ and $z = {\phi_f}^{-1}(c)$ for some $c \in C$.

Note that $f({\phi_f}^{-1}(c)) = {\phi_f}^{-1} \circ p_d \circ \phi_f(\phi^{-1}(c)) = {\phi_f}^{-1}(c^d)$. In general,

$$f^i({\phi_f}^{-1}(c)) = {\phi_f}^{-1}(c^{d^i}),\ \forall\ c \in S^1$$

Now by applying chain rule to calculate $(f^q)'$, we get that

$$L(C,\ f) = \sum_{c \in C} \log |f'({\phi_f}^{-1}(c))| \tag{9}$$

The Length function is an interesting invariant of f, which suggests some properties of the cycles of p_d. For example:

Theorem 5 (McMullen)**.** *Let C be a cycle for p_d. If $L(C, f) < \log 2$, then C is a simple cycle, i.e. $deg(C) = 1$.*

There is a conjecture suggested by McMullen:

Conjecture 1. *Let C be a degree-1 cycle for p_d. Then, $L(C, f)$ has no critical points.*

Now we focus on the case $d = 2$. Since there is only one parameter a which determines f, we write f_a instead of f and ϕ_a instead of ϕ_f. We have

$$f_a(z) = z(\frac{z-a}{1-\overline{a}z}),\ \forall\ z \tag{10}$$

and

$$|f_a'(z)| = 1 + \frac{1-|a|^2}{|z-a|^2},\ \forall\ z \in S^1 \tag{11}$$

When we choose C with less than 3 elements, we can calculate ${\phi_a}^{-1}(c)$ for each c in C and compute $L(C, f_a)$. We get the following result:

Proposition 2. *Let C be a cycle for p_2. If $|C| < 3$, then $L(C,\ f_a)$ has no critical points.*

Proof. For detailed calculations, see Appendix A.
<u>**Case 1:**</u> $C = \{1\}$
We know that

$$f_a({\phi_a}^{-1}(1)) = {\phi_a}^{-1}(1^2) = {\phi_a}^{-1}(1) \text{ and } {\phi_a}^{-1}(1) \in S^1$$

This leads to

$${\phi_a}^{-1}(1) = \frac{1+a}{1+\overline{a}}$$

and

$$L(C, f_a) = \log |f'({\phi_f}^{-1}(1))| = \log(\frac{2+a+\overline{a}}{1-|a|^2})$$

Write a as $x + iy$.

$$L(x, y) = L(C, f_{x+iy}) = \log(\frac{2+2x}{1-x^2-y^2}) \tag{12}$$

Now we compute partial derivatives of L w.r.t. x and y and observe that $\frac{\partial L}{\partial x}$ is 0 iff $(1+x) = \pm y$ and $\frac{\partial L}{\partial y}$ is 0 iff $y = 0$. Thus, both partial derivatives of L are zero iff $a = -1$ which is impossible since $|a| < 1$. Thus, $L(C, f_a)$ does not have any critical points.

Case 2: $C = \{\omega, \omega^2\}$ where $\omega = e^{\frac{2\pi}{3}i}$
We know that

$$f_a({\phi_a}^{-1}(\omega)) = {\phi_a}^{-1}(\omega^2)$$

and

$$f_a({\phi_a}^{-1}(\omega^2)) = {\phi_a}^{-1}(\omega^4) = {\phi_a}^{-1}(\omega)$$

This means that ${\phi_a}^{-1}(\omega), {\phi_a}^{-1}(\omega^2), {\phi_a}^{-1}(1)$ and 0 satisfy the degree-4 equation:

$$f_a(f_a(z)) = z$$

By simplifying this equation and factoring out $[z(z - \phi_a^{-1}(1))]$, we obtain a quadratic equation which has ${\phi_a}^{-1}(\omega)$ and ${\phi_a}^{-1}(\omega^2)$ as its roots. From the coefficients of the quadratic, we get:

$${\phi_a}^{-1}(\omega) + {\phi_a}^{-1}(\omega^2) = a - 1 \text{ and } {\phi_a}^{-1}(\omega){\phi_a}^{-1}(\omega^2) = \frac{1-a}{1-\overline{a}}$$

We use this to show that

$$L(C, f_a) = \log|f_a'({\phi_a}^{-1}(\omega))||f_a'({\phi_a}^{-1}(\omega^2))| = \log(4 + |a|^2 - 2(a + \overline{a}) - \frac{(a-\overline{a})^2}{1-|a|^2})$$

Write a as $x + iy$. Now we have:

$$L(C, f_a) = L(x, y) = \log(4 - 4x + x^2 + y^2 + \frac{y^2}{1 - x^2 - y^2}) \tag{13}$$

Now we compute partial derivatives of L w.r.t. x and y. $\frac{\partial L}{\partial y}$ is 0 iff $y = 0$. Thus, $\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = 0$ iff $x = 2$ and $y = 0$ which is impossible since $|a| < 1$. Thus, $L(C, f_a)$ does not have any critical points. $\square$

We are still studying cycles with more elements. Here are some difficulties that we need to overcome. Let $q \in \mathbb{N}$. Note that if C is a cycle and $|C|$ divides q, then for any point $c \in C, \phi_f^{-1}(c)$ satisfies the degree-d^q equation

$$f^q(z) = z$$

For example, for $d = 2$ and $q = 3, f_a^q(z) = z$ has 8 roots two of which are 0 and $\phi_a{-}1(1)$. We can reduce it to a degree-6 equation whose roots are preimages of two *different* cycles under ϕ_a. In this case, we cannot apply the method used above to calculate L and check its critical points because we first need to separate the roots in two groups - one for each cycle. For $q > 3$, the problem is even harder since the roots of the corresponding equation may include preimages (under ϕ_a) of cycles of higher degree.

Due to our calculations in the proof of Proposition 2 , we belive that in the case $d = 2$, $L(C, f_a)$ is an algebraic function of a and $\overline{a}$, and Conjecture 1 is true.

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Appendix A. Calculating $L(C, f_a)$

Case 1:

$$\begin{aligned} L(C, f_a) &= \log|f'({\phi_f}^{-1}(1))| \\ &= \log(1 + \frac{1-|a|^2}{|(\frac{1+a}{1+\overline{a}}) - a|^2}) \\ &= \log(1 + \frac{|1+\overline{a}|^2}{1-|a|^2}) \\ &= \log(\frac{2+a+\overline{a}}{1-|a|^2}) \end{aligned}$$

$$L(x,y) = L(C, f_a) = \log(\frac{2+2x}{1-x^2-y^2})$$

$$\begin{aligned} \frac{\partial L}{\partial x} &= (\frac{1-x^2-y^2}{2+2x})(\frac{2(1-x^2-y^2)+(2+2x)(2x)}{(1-x^2-y^2)^2}) \\ &= 2(\frac{1+2x+x^2-y^2}{(2+2x)(1-x^2-y^2)}) \\ \frac{\partial L}{\partial y} &= (\frac{1-x^2-y^2}{2+2x})(\frac{(2+2x)(2y)}{(1-x^2-y^2)^2}) \\ &= \frac{2y}{1-x^2-y^2} \end{aligned}$$

Case 2: ${\phi_a}^{-1}(\omega), {\phi_a}^{-1}(\omega^2), {\phi_a}^{-1}(1)$ and 0 satisfy the degree-4 equation:

$$f_a(f_a(z)) = z$$

or

$$z(\frac{z-a}{1-\overline{a}z})(\frac{z(\frac{z-a}{1-\overline{a}z}) - a}{1-\overline{a}z(\frac{z-a}{1-\overline{a}z})}) = z$$

or

$$z(\frac{z-a}{1-\overline{a}z})(\frac{z^2-az-a+a\overline{a}z}{1-\overline{a}z-\overline{a}z^2+a\overline{a}z}) = z$$

So, ${\phi_a}^{-1}(\omega), {\phi_a}^{-1}(\omega^2)$ and ${\phi_a}^{-1}(1)$ satisfy the following cubic:

$$(z-a)(z^2+(a\overline{a}-a)z-a) = (1-\overline{a}z)(1+(a\overline{a}-\overline{a})z-\overline{a}z^2)$$

or

$$(1-\overline{a}^2)z^3 + (a\overline{a}-2a+\overline{a}-\overline{a}^2+a\overline{a}^2)z^2 + (a^2-a^2\overline{a}-a+2\overline{a}-a\overline{a})z + (a^2-1) = 0$$

or

$$[(1+\overline{a})z-(1+a)]\ [(1-\overline{a})z^2+(1-a-\overline{a}+a\overline{a})z+(1-a)] = 0$$

We divide both sides by $[(1+\overline{a})z-(1+a)] = (1+\overline{a})(z-\phi_a^{-1}(1))$ to get the desired quadratic equation.

Let $\phi_a{-}1(\omega) = z_1$ and $\phi_a{-}1(\omega^2) = z_2$. We have

$$z_1 + z_2 = a - 1 \text{ and } z_1 z_2 = \frac{1-a}{1-\overline{a}}$$

Now we can compute the modulus of the product of the derivatives of f at $\phi_a{-}1(\omega)$ and $\phi_a{-}1(\omega^2)$.

$$
\begin{aligned}
&|f_a'(\phi_a{-}1\omega)||f_a'(\phi_a{-}1(\omega^2))| \\
&= |f_a'(z_1)||f_a'(z_2)| \\
&= (1+\frac{1-|a|^2}{|z_1-a|^2})(1+\frac{1-|a|^2}{|z_2-a|^2}) \\
&= 1+\frac{1-|a|^2}{|(z_1-a)(z_2-a)|^2}\ (|z_1-a|^2+|z_2-a|^2+1-|a|^2) \\
&= 1+\frac{1-|a|^2}{|\frac{1-a}{1-\overline{a}}+a(1-a)+a^2|^2}\ (1+|a|^2-\overline{a}z_1-a\overline{z_1}+1+|a|^2-\overline{a}z_2-a\overline{z_2}+1-|a|^2) \\
&= 1+\frac{|1-a|^2}{1-|a|^2}\ (3+|a|^2+\overline{a}(1-a)+a(1-\overline{a})) \\
&= 1+\frac{|1-a|^2}{1-|a|^2}\ (3+a+\overline{a}-|a|^2) \\
&= 1+|1-a|^2+\frac{|1-a|^2}{1-|a|^2}\ (2+a+\overline{a}) \\
&= 2+|a|^2-a-\overline{a}+\frac{(1+|a|^2-a-\overline{a})}{1-|a|^2}(2+a+\overline{a}) \\
&= 4+|a|^2+\frac{(2|a|^2-a-\overline{a})(2+a+\overline{a})}{1-|a|^2} \\
&= 4+|a|^2+\frac{4|a|^2-2(a+\overline{a})(1-|a|^2)-(a+\overline{a})^2}{1-|a|^2} \\
&= 4+|a|^2-2(a+\overline{a})-\frac{(a-\overline{a})^2}{1-|a|^2}
\end{aligned}
$$

Now we take log of both sides and replce a by $x+iy$.

$$L(x,y)=L(C,f_a)=\log(4-4x+x^2+y^2+\frac{y^2}{1-x^2-y^2})$$

$$
\begin{aligned}
\frac{\partial L}{\partial x} &= \frac{1}{4-4x+x^2+y^2+\frac{y^2}{1-x^2-y^2}}\ (-4+2x+\frac{8xy^2}{(1-x^2-y^2)^2}) \\
\frac{\partial L}{\partial y} &= \frac{1}{4-4x+x^2+y^2+\frac{y^2}{1-x^2-y^2}}\ (2y)(1+\frac{4y^2}{(1-x^2-y^2)^2})
\end{aligned}
$$

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