Some physical space heuristics for Strichartz estimates

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Abstract

This note records an attempt to prove the bilinear Strichartz estimate on hyperbolic space. The main difficulty is the lack of a convolution formula for the Fourier transform of a product of functions. We attempt to develop physical-space methods on Euclidean space to prove the Strichartz estimate, with the hope that such methods can also be effective on hyperbolic space.

In this spur project, we studied the linear Schrödinger equation given in $\mathbb{R}^2$ by

$$i\partial_t u + \Delta u = 0, \tag{1}$$

$$u(0,x) = u_0. \tag{2}$$

In particular, we would like to develop physical-space methods to understand the dispersive properties of solutions. The hope is that such methods might be more easily extended to prove results on hyperbolic space $\mathbb{H}^2$, where Fourier analysis becomes more difficult. The prototype of dispersive estimate that we will consider is the $L^4_{t,x}$ Strichartz estimate

Proposition 1. *Let $u \in L^2(\mathbb{R}^2)$. Then*

$$\|e^{it\Delta} u\|_{L^4_{t,x}} \lesssim \|u\|_{L^2_x}. \tag{3}$$

This inequality follows from interpolation between the dispersive estimate $\|e^{it\Delta}u\|_{L^\infty} \leq |t|^{-1}\|u\|_{L^1}$ and the conservation of mass $\|e^{it\Delta}u\|_{L^2} = \|u\|_{L^2}$, along with a classical TT^* argument. The main ingredient needed to extend the result to hyperbolic space $\mathbb{H}^2$ is the dispersive estimate. This dispersive estimate was proved by Ionescu and Staffilani [8], and independently by Anker and Pierfelice [1]. In addition, a weighted version of the Strichartz estimates is known in the radial case [2]. These results have been used to prove global existence for problems with critical nonlinearities on $\mathbb{H}^d$ [2, 7].

A refined version of the $L^4_{t,x}$ Strichartz estimate was introduced by Bourgain [3]

Proposition 2. *For functions $u, v \in L^2(\mathbb{R}^2)$, we have*

$$\|(e^{it\Delta}P_N u)(e^{it\Delta}P_M v)\|_{L^2_{t,x}} \lesssim \left(\frac{M}{N}\right)^{1/2} \|u\|_{L^2}\|v\|_{L^2}. \tag{4}$$

Here P_N and P_M stand for Littlewood-Paley projections, defined in Section 1. We would like to prove a version of this estimate in hyperboic space.

However, the proofs of Proposition 2 in [3] and [4] rely on Fourier analysis, in particular on the formula

$$\widehat{fg}(\xi) = \widehat{f} * \widehat{g}(\xi),$$

which is not true on $\mathbb{H}^2$. We should mention that there is an alternative proof discovered by Tao [10] which relies on a one-dimensional local smoothing estimate and the tensor-product structure of $\mathbb{R}^n$. However, it is unclear to me how this might extend to hyperbolic space.

To our knowledge, the most robust result available for the bilinear Strichartz estimate in non-flat situations is due to Hani [5], who showed that for closed manifolds a short-time inequality is available. This is, for generic closed manifolds, a sharp result. It enabled Hani to prove a global existence result for the cubic nonlinear Schrödinger equation on closed manifolds [6]. Understanding these methods and how they could extend to a result on hyperbolic space is the topic of a future investigation.

After introducing some basic notation in Section 1, we discuss some basic dispersive properties of solutions to the Schrödinger equation in Section 2. Then, we dicuss a naive strategy for proving the bilinear estimate (4) in Section 3. It is found that this does not strongly exploit cancellations that occur, so in Section 4, we explore some ideas that might have a better chance of capturing cancellation. Also, we should make the comment that although the ultimate goal of this project is to understand dispersion in hyperbolic space, this write-up deals only with Euclidean space. This simplification does not take away any of the difficulty of the problem, so long as we restrict ourselves to geometric arguments.

1 Notation and Preliminaries

First we describe some notation used to discuss the Schrodinger equation. Since the equation (1) is linear and translation invariant, its evolution operator $e^{it\Delta}$ can be given by the Fourier multiplier:

$$\widehat{e^{it\Delta}u} = e^{-it|\xi|^2}\widehat{u}(\xi).$$

From this representation, and Parseval's theorem, it is evident that $e^{it\Delta}$ is unitary. In our computations, we use the convolution formula for $e^{it\Delta}$

$$e^{it\Delta}\, u(x) = \int t^{-1} e^{i|x-y|^2/4t} u(y)\, dy.$$

From this formula the dispersive estimate immediately follows

$$\|e^{it\Delta}\, u\|_{L^\infty} \leq |t|^{-1}\|u\|_{L^1}.$$

This fact, and the unitarity mentioned above, are the key ingredients to the Strichartz estimate (3), which controls the spacetime norm

$$\|e^{it\Delta}\, u\|_{L^4_{t,x}} := \left(\int_{\mathbb{R}}\int_{\mathbb{R}^2} |e^{it\Delta}\, u|^4(x)\, dx\, dt\right)^{1/4}.$$

Now we explain the notation we use to provide heuristic arguments. First, we shall need a notion of localization in frequency. For $f \in L^2(\mathbb{R}^2)$, let $P_N f$ denote a smooth Littlewood-Paley projection of f onto frequencies roughly. That is, suppose $m_N(\xi) \in C_c^\infty(\mathbb{R}^2)$ with support in $\{\xi; N/2 < |\xi| < 2N\}$. Then we define $P_N f$ according to the multiplier

$$\widehat{P_N f}(\xi) = m_N(\xi)\widehat{f}(\xi).$$

For our purposes the most relevant properties of P_N are its continuity on L^p for $1 \leq p \leq \infty$

$$\|P_N f\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^p(\mathbb{R}^2)}$$

and the Bernstein inequality for $q > p$

$$\|P_N f\|_{L^q(\mathbb{R}^2)} \leq N^{2(1/p-1/q)} \|P_N f\|_{L^p(\mathbb{R}^2)}.$$

In addition to localizing in frequency space, we will be discussing localizations in physical space. To this end, let $B_r(x) \subset \mathbb{R}^2$ denote the ball centered at x of radius r. We write B_r for $B_r(0)$.

Finally, we introduce some asymptotic notation. The notation $A \lesssim B$ means that $A \leq CB$ for some universal constant C. If $A \lesssim B$ and $B \lesssim A$, we write $A \sim B$. This is in contrast with the notation $A \approx B$, which is used here in a nonrigorous fashion to state that A and B are morally the same.

2 General properties of Schrödinger Evolution

Before we can understand how solutions of different frequencies interact with each other, we must first come to terms with how frequency localized functions evlove. To this end, let $u_0 \in L^2(\mathbb{R}^2)$ have frequency support in $|\xi| \sim N$, and be concentrated in a ball of radius roughly N^{-1}, and normalized so that $\|u_0\|_{L^2} = 1$. Then

$$e^{it\Delta} u_0(x) = \int_{|\xi|\sim N} e^{i(x\cdot\xi - t|\xi|^2} \widehat{u}(\xi)\, d\xi.$$

Notice that $\nabla_\xi(x \cdot \xi - t|\xi|^2) = 0$ implies $x = 2\xi$. Thus, unless $|x| \sim 2Nt$, the phase oscillates rapidly and the contribution to $|e^{it\Delta} u(x)|$ is negligible ($\widehat{u}$ is smooth because u is localized in physical space). This calculation, in combination with the fact that $e^{it\Delta}$ preserves mass, suggests that $|e^{it\Delta} u|$ can be modeled by

$$|e^{it\Delta} u(x)| \approx \begin{cases} (N^{-1} + Nt)^{-1} & |x| \sim N|t| \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, the method of stationary phase can make this rigorous and provide the estimate (see [9])

Proposition 3. *For any $m \geq 0$,*

$$|(P_N e^{it\Delta})(x)| \lesssim_m \begin{cases} |t|^{-1} & |x| \sim N|t| \geq N^{-1} \\ \frac{N^2}{\langle N^2 t\rangle^m \langle N|x|\rangle^m}, & \textit{otherwise.} \end{cases} \tag{5}$$

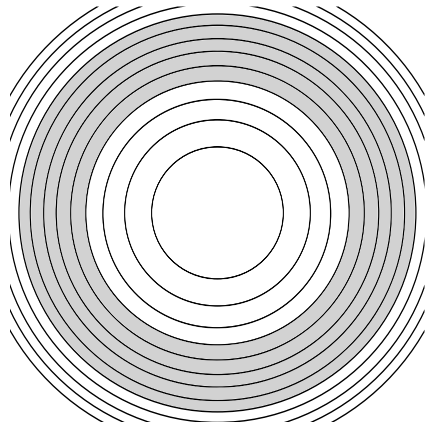

Figure 1: A diagram of the fundamental solution $t^{-1}e^{i|x-y|^2/4t}$. The circles are level sets of the phase. A solution localized in frequency, and concentrated near the origin, would expand on the annulus $|x| \sim 2Nt$ (shaded), where the distance between level sets (and thus the wavelength) is approximately N^{-1}.

Here $\langle A \rangle$ is defined as $(1 + A^2)^{1/2}$. A pictorial representation of this is provided in Figure 1. This estimate is the main ingredient of the approach outlined in Section 3, and we will find it is insufficient. This, it seems we need a more detailed understanding of the way different frequencies behave. Let us return to the example of an initial data concentrated at the origin. Using the fundamental solution, we can write

$$e^{it\Delta} u_0(x) = \int_{y \leq 1/N} t^{-1} e^{i|x-y|^2/4t} u_0(y)\, dy.$$

Observe that

$$|x - y|^2 = |x|^2 + |y|^2 - 2x \cdot y.$$

If we ignore the $|y|^2$ contribution (since $|y| \leq 1/N$), this phase function is linear in y. Thus we can write

$$e^{it\Delta} u_0(x) \approx t^{-1} e^{ix \cdot x/4t} \widehat{u}_0(x/2t),$$

so that near x, $e^{it\Delta} u(x)$ looks like a plane wave with frequency $x/4t$ and amplitude $t^{-1}\widehat{u}(x/2t)$. This estimate can be made precise [9]

Proposition 4. *As* $|t| \to \infty$, *for any* $\psi \in L^2(\mathbb{R}^2)$,

$$\|e^{it\Delta} \psi(x) - t^{-1} e^{i|x|^2/4t} \widehat{\psi}(x/2t)\|_{L^2_x} \to 0.$$

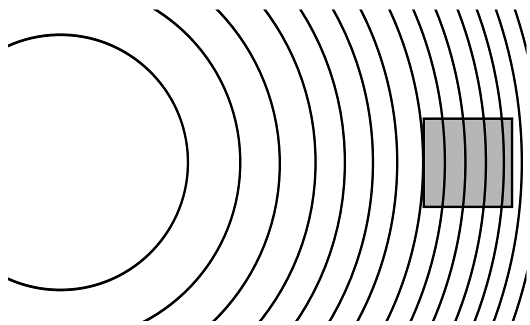

Figure 2: On a small square, the fundamental solution $t^{-1}e^{i|x-y|^2/4t}$ (whose level sets are represented by the cocentric rings) looks like a plane wave. This illustrates how different frequencies travel at different velocities, a statement made precise in Proposition 4.

Figure 2 illustrates this computation. A strategy that uses this property of the Schrödinger evolution is described in Section 4.

3 Concentrated initial data and a naive approach

We are now prepared to understand the exponent of M/N in (4). Let $u \in L^2(\mathbb{R}^2)$ be localized at frequency roughly N, and concentrated in a ball of radius roughly N^{-1}, with $\|u\|_{L^2} = 1$. Also let $v \in L^2(\mathbb{R}^2)$ be localized at frequency M, near the origin in a ball of radius M^{-1}. Because of the kernel estimate (5), we write

$$|e^{it\Delta} u(x)| \approx N^{-1}|t|^{-1}\chi_N(t,x) \quad \text{and} |e^{it\Delta} v(x)| \approx M^{-1}|t|^{-1}\chi_M(t,x),$$

where $\chi_N(t,x)$ (resp. $\chi_M(t,x)$) is a smooth cutoff for the annulus $\{|x| \sim N^{-1} + N|t|\}$ (resp. $\{|x| \sim M^{-1} + M|t|\}$). We use this heuristic to estimate

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} |e^{it\Delta} u|^2 |e^{it\Delta} v|^2 \, dx \, dt.$$

For $t < (MN)^{-1}$, the supports overlap on $\{|x| \sim N^{-1} + N|t|\}$, so

$$\int_{|t| \leq (MN)^{-1}} \int_{\mathbb{R}^2} |e^{it\Delta} u|^2 |e^{it\Delta} v|^2 \, dx \, dt \lesssim \int_{|t| \leq (MN)^{-1}} M^2 \, dt \lesssim \frac{M}{N}.$$

Taking the square roots of both sides we can see that the exponent matches (4). For larger t, the annuli have separate supports, and the contribution is negligible.

While this calculation can be made rigorous using the kernel estimate (5), a more simple argument can be made if we allow for logarithmic losses.

Lemma 1. *Let $u, v \in L^2(\mathbb{R}^2)$ have concentrated supports, in the sense that $|\operatorname{supp} u| \lesssim N^{-2}$, and $|\operatorname{supp} v| \lesssim M^{-2}$. Then*

$$\|(e^{it\Delta} P_N u)(e^{it\Delta} P_M v)\|_{L^2_{t,x}} \lesssim \left(\frac{M}{N}\right)^{1/2} \log(M/N) \|u\|_{L^2} \|v\|_{L^2}.$$

Proof. By conservation of mass, Bernstein's inequality, Hölder's inequality, and a standard dispersive estimate:

$$\|e^{it\Delta} P_N u\|_{L^2_x} \lesssim \|u\|_{L^2_x} \tag{6}$$

$$\|e^{it\Delta} P_N u\|_{L^\infty_x} \lesssim N \|u\|_{L^2_x} \tag{7}$$

$$\|e^{it\Delta} P_N u\|_{L^\infty_x} \lesssim |t|^{-1} \|P_N u\|_{L^1} \lesssim |t|^{-1} \|u\|_{L^1} \lesssim |t|^{-1} N^{-1} \|u\|_{L^2_x}. \tag{8}$$

Analogous estimates hold for $e^{it\Delta} v$. To simplify the clutter suppose $\|u\|_{L^2} = \|v\|_{L^2} = 1$. We wish to estimate the left-hand side:

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}^2} |e^{it\Delta} P_N u|^2 |e^{it\Delta} P_M u|^2 \, dx \, dt &= \left(\int_0^{N^{-2}} + \int_{N^{-2}}^{M^{-2}} + \int_{M^{-2}}^{\infty} \right) \int_{\mathbb{R}^2} |e^{it\Delta} P_N u|^2 |e^{it\Delta} P_N v|^2 \, dx \, dt \\ &= I_1 + I_2 + I_3.\end{aligned}$$

To estimate I_1, notice $\|e^{it\Delta} u\|_{L^4}^2 \lesssim N$, and $\|e^{it\Delta} v\|_{L^4}^2 \lesssim M$ by interpolation of (6) and (7). Now apply Hölder's inequality to obtain

$$I_1 \lesssim \int_0^{N^{-2}} NM \, dt \lesssim \frac{M}{N}.$$

To estimate I_2, use the same bound for $\|e^{it\Delta} v\|_{L^4}^2$, but use the dispersive estimate (8) to obtain $\|e^{it\Delta} u\|_{L^4}^2 \lesssim t^{-1} N^{-1}$. Then

$$I_2 \lesssim \int_{N^{-2}}^{M^{-2}} M (Nt)^{-1} \, dt \lesssim \frac{M}{N} \log(N/M).$$

Finally, for I_3, use $\|e^{it\Delta} v\|_{L^4}^2 \lesssim t^{-1} M^{-1}$. Again calculate

$$I_3 \lesssim \int_{M^{-2}}^{\infty} (MN)^{-1} t^{-2} \, dt \lesssim \frac{M}{N}.$$

The claim follows upon taking square roots. □

The advantage of this argument is that it is more easily extended to the setting of hyperbolic space (with a modified notion of the operator P_N, see for example [7]). The reason is we only use basic properties of the Littlewood-Paley projection operator and kernel estimates for $e^{it\Delta}$, which are well-known.

Now we might hope that, by splitting an initial condition u into small pieces, we can control the interaction between the pieces in a way that sums. Indeed, suppose we took a partition of unity $\{\chi_i\}$ where each χ_i has support in a box of side length N^{-1}, and we write

$$P_N u = \sum_i u_i \quad u_i = P_N \chi_i u.$$

Then we might try to prove the $L^4_{t,x}$ Strichartz estimate in the following way:

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}^2} |e^{it\Delta} P_N u|^4 \, dx \, dt &= \sum_{ijkl} \int_{\mathbb{R}} \int_{\mathbb{R}^2} (e^{it\Delta} u_i)(e^{it\Delta} u_j)\overline{(e^{it\Delta} u_k)(e^{it\Delta} u_l)} \, dx \, dt \\ &\leq \sum_{ijkl} \int_{\mathbb{R}} \int_{\mathbb{R}^2} |(e^{it\Delta} u_i)(e^{it\Delta} u_j)\overline{(e^{it\Delta} u_k)(e^{it\Delta} u_l)}| \, dx \, dt.\end{aligned}$$

The idea is that, by carefully understanding the support properties of each piece, it would be possible to significantly beat Hölder's inequality. This strategy is illustrated in Figure 3. However we can show that this last step of taking the absolute values is already wrong

Lemma 2. *Suppose $u \in L^2(\mathbb{R}^2)$ has support in B_R and can be written as*

$$u = N^{-1} R^{-1} \sum_{i=1}^{N^2 R^2} u_i,$$

where $\|u_i\|_{L^2} = 1$ is concentrated in $B_{1/N}(x_i)$, and $B_{1/N}(x_i)$ is disjoint from $B_{1/N}(x_j)$ for any $i \neq j$. Then

$$\sum_{ijkl} \int_{\mathbb{R}} \int_{\mathbb{R}^2} |(e^{it\Delta} u_i)(e^{it\Delta} u_j)\overline{(e^{it\Delta} u_k)(e^{it\Delta} u_l)}| \, dx \, dt \gtrsim R^3 N^3.$$

Proof. The first step is a simplification. Define the convolution kernel

$$K_{N,t} = \begin{cases} N^2 \chi_{|y| \leq 1/N} & |t| \leq N^{-2} \\ |t|^{-1} \chi_{|y| \sim Nt} & |t| \geq N^{-2}. \end{cases} \tag{9}$$

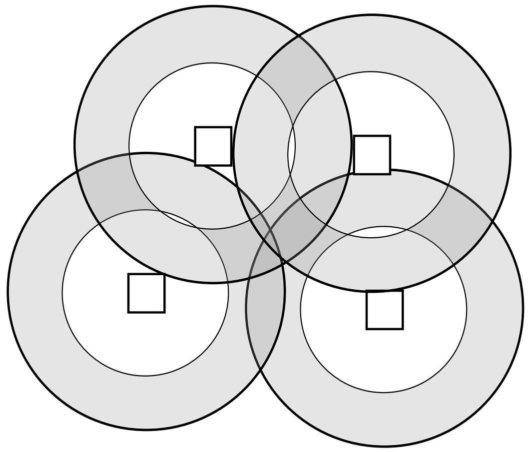

Figure 3: This figure shows the interaction between four localized pieces (represented by squares) after some time. The data originally in each square has expanded to the shaded annulus surrounding it. The four annuli intersect to form a complex geometric shape. One might hope that by estimating the area of this shape, it could be possible to prove the L^4 Strichartz estimate.

By the estimate (5), for $|t| \geq N^{-2}$,

$$|e^{it\Delta}\, u_j(x)| \gtrsim (K_{N,t} * |u_j|)(x).$$

Thus it remains to show that

$$\int_{|t|\geq N^{-2}} \int_{\mathbb{R}^2} |K_{N,t} * |u||^4 \, dx \, dt \gtrsim R^3 N^3.$$

Indeed we can restrict attention to $|t| \geq R/N$. In this case, $K_{N,t} * |u|$ has height $|t|^{-1}R$ on $|x| \sim Nt$. Then

$$\int_{R/N}^{\infty} \int_{\mathbb{R}^2} |K_{N,t} * |u||^4 \, dx \, dt \sim \int_{R/N}^{\infty} \int_{|x|\sim Nt} |t|^{-4} R^4 \, dx \, dt \sim \int_{R/N}^{\infty} R^4 |t|^{-2} N^2 \sim R^3 N^3,$$

□

4 A new approach and final thoughts

As discussed in the proof of Lemma 2, the strategy of dividing initial data into small pieces and estimating their interactions is similar to applying the estimate

$$|e^{it\Delta}P_N u(x)| \leq |e^{it\Delta}P_N| * |u(x)|. \tag{10}$$

The major problem with this approach is that the kernel $|e^{it\Delta}P_N|$ is not uniformly in L^1, since its L^1 norm grows as $N^2|t|$. As we observed earlier, this is not a problem for concentrated initial data, but the problem is severe for diffuse initial data.

A cheap way to try to get around this problem is to simply wait for a solution $u \in L^2(\mathbb{R}^2)$ to concentrate. Thus, we might hope that there is some time T_0 such that $\|e^{iT_0\Delta}P_N u\|_{L^1} \lesssim N^{-1}$, corresponding to concentration at a scale $1/N$. In this case, the estimate

$$|e^{it\Delta}P_N u(x)| \leq |e^{i(t-T_0)\Delta}P_N| * |e^{iT_0\Delta}u|(x)$$

is far more efficient than (10). However, this strategy only works on very special initial data. A slightly more general situation is that a solution u could be a superposition of wavepackets that concentrate at different places and different times. Indeed, one might hope for a decomposition of the form

$$u = \sum_j e^{-iT_j\Delta}\psi_j + \text{error},$$

where each ψ_j is concentrated in a small region, and error never concentrates. Two ingredients would be needed to make this strategy work. First, we need some guarantee that $e^{i(t-T_j)\Delta}\psi_j$ and $e^{i(t-T_k)\Delta}\psi_k$ don't interact much at any time. Second, we would need to know something about the behavior of the error. Somehow the fact that it never concentrates might be useful in proving an L^4 estimate.

My guess is that the solution to this problem will look quite different from this, and will need to use some information about the movement of different frequencies, such as Proposition 4. Indeed, let me now propose a decomposition of free solutions that takes this into account. Suppose we are given an initial condition $u \in L^2$ localized to frequencies $|\xi| \sim N$. As before, we can write

$$P_N u = \sum_i u_i, \quad u_i = P_N \chi_i u,$$

where the partition of unity χ_i is composed of smooth bump functions with supports in balls of radius N^{-1}. Suppose we want to have an accurate estimate for $e^{it\Delta}P_N u$ for times $|t| \lesssim T_0$. Now we can break up frequency space

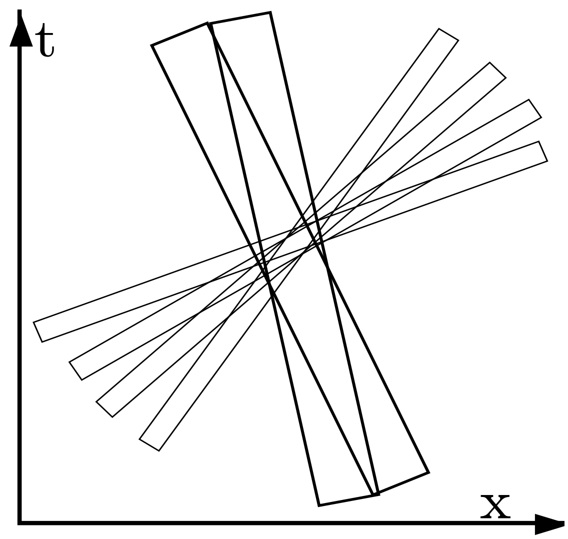

Figure 4: A combinatorial interpretation of the bilinear Strichartz estimate. The wavepackets of the high-frequency wave are more localized and move in a different space-time direction than the wavepackets of the low-frequency wave packets. The transversality and the difference in widths of the space-time tubes are the main features.

into $T_0^2 N^2$ pieces of width $O(T_0^{-1})$. Each piece in frequency space corresponds to a spacetime tube that doesn't spread out by much for $|t| \lesssim T_0$. This allows us to write u_i a a sum of $T_0^2 N^2$ pieces, each of which are spatially localized at a scale N^{-1}, and have unique directions. Figure 4 illustrates a simple example of this decomposition for two functions of different frequencies, each of which are localized at some point in space and time.

With this kind of decomposition, the bilinear Strichartz estimate turns into a problem of estimating the area of intersection of transverse tubes in spacetime with different widths. If this connection can be made precise, then the problem of proving a bilinear Strichartz estimate on hyperbolic space reduces to the problem of understanding the incidence properties of geodesics. This is the global ingredient that is not available, for example, in Hani's result for closed manifolds [5], because closed manifolds have geodesics that can self-intersect in some finite time.

5 Acknowledgements

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