A Particle System with Interlacing Pattern
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Abstract

Consider infinite many particles on the real line, infinite in both direction. At next step, we uniformly pick one point between every two consecutive particles to form the next generation of particles. In this way, we obtain Markov process with interlacing pattern. In this paper, we study some basic properties of this Markov chain, such as invariant measure, convergence, fluctuation and some possible generalization of this model.

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Contents

1 Introduction

Interlacing pattern is a natural object in mathematics. It appears in representation theory as Gelfand-Tsetlin pattern, eigenvalues of sub-matrices of Hermitian matrices etc. Correspondingly, there are some very important random objects supported on interlacing patterns such as total asymmetric exclusion process, some Schur and Macdonald process which lead to various interacting particle systems, GUE corner process, lozenge tiling, bead model etc. For the probability related to interlacing patterns, we refer to [1], [2], [3], and reference therein. In this paper, we study a very natural Markov chain under the interlacing condition.

Consider Markov chain Y_n whose state space is 2^R such that it is locally finite and infinite in both direction. At time 0, $Y_0 = \{x_i^0\}_{i\in\mathbb{Z}}$ are infinite points on $\mathbb{R}$. If at time n, $Y_n = \{x_i^n\}_{i\in\mathbb{Z}}$, then $Y_{n+1} = \{x_i^{n+1}\}_{i\in\mathbb{Z}}$ will satisfy the two rules

Rule 1: $x_i^n < x_i^{n+1} < x_{i+1}^n$

Rule 2: x_i^{n+1} will be independently distributed according to $U(x_i^n, x_{i+1}^n)$, where $U(x_i^n, x_{i+1}^n)$ is the uniform distribution in the open interval (x_i^n, x_{i+1}^n).

Denote X_i^n be the distribution of the gap between (x_i^n, x_{i+1}^n), Since it is an Markov particle system, we are interested in two types of questions: equilibrium and dynamic. In the former direction, we give a family of ergodic invariant measure and show that all the invariant under some moment assumption are convex combination of these measure. This type of result is an analogue of that for exclusion process(see [4]). In the latter direction, we study the convergence to equilibrium under certain initial condition and the fluctuation of a single particle.

The paper is organized in the following way: in Section 2, we give a family of invariant measure of this Markov dynamic. In Section 3 we employ moment method to show that under certain moment condition, all the invariant measure are convex combination of the measure defined in Section 2. In Section 4 we show that the if the initial gaps form a stationary sequence satisfying certain moment condition, then it converges to equilibrium. In Section 5, we replace the uniform distribution by other distribution on $[0,1]$ and studied similar questions. In Section 6, we first show that if under extremal invariant measure the trace of one particle is actually a Poisson process. Then we study the fluctuation of on particle under the lattices initial condition. In particular we give the order the variance of the position of a particle.

2 Elementary invariant measure

Theorem. *If $X(\rho)$ is a translational invariant measure which has independent and identically distributed gap $X_i(\rho)$ whose density function is $4\rho^2 xe^{-2\rho x}, (x > 0)$, then $X(\rho)$ is a group of elementary invariant measure for the Markov process.*

Proof. We prove that $X(\rho)$ has the following properties:
1.$X_i(\rho)$ are mutually independent under the Markov process
Since $X_i(\rho)$ are independent,we want to prove that $X_i^1(\rho)$ are independent. It suffice to prove that any consecutive k gap of X_i^1 are independent. Assume f(t) is the characteristic function of $X_i(\rho)$. Then

$$f(t) = \int_{-\infty}^{+\infty} e^{itx} 4\rho^2 xe^{-2\rho x}\, dx = \frac{1}{(1+ct)^2}, c = -i/2\rho$$

The characteristic function of $X_i^1(\rho)$ is $(\int_0^1 f(tu)\, du)^2 = f(t) = \frac{1}{(1+ct)^2}, c = -i/2\rho$. The characteristic function of $X_i^1+\ldots+X_{i+k-1}^1$ is $(\int_0^1 f(tu)\, du)^2 f(t)^{k-1} = f(t)^k$,therefore X_i^1 are independent. Thus $X_i^1(\rho)$ are independent.

2.The distribution of the gap $X_i(\rho)$ are invariant under the Markov process.
Assume U_1 and U_2 are any uniform distributions in [0,1] and f(t) is the characteristic function of $X_i(\rho)$, then

$f(t) = \frac{1}{(1+ct)^2}, c = -i/2\rho, \int_0^1 f(tu)\, du = \frac{1}{1+ct} f(t) = (\int_0^1 f(tu)\, du)^2$
Since $X_i(\rho)$ are mutual independent under the Markov process.Hence, $(1 - U_1)X_i(\rho) + U_2 X_{i+1}(\rho) \sim X_i(\rho)$

3. $X(\rho)$ is translational invariant under the Markov process.
Since $X(\rho)$ is translational invariant, by the definition of the model,$X^1(\rho)$ is translational invariant,therefore at any time n the measure is translational invariant.
Thus,$X\rho$ is a group of translational invariant invariant measure of constant density ρ and the density function of the gap is $4\rho^2 xe^{-2\rho x}, (x > 0)$. □

3 Classification of invariant measure with finite moment

Theorem. *Assume μ is an invariant measure. The distribution of the gap is $X_i, i \in \mathbb{Z}$. If $X_i, i \in \mathbb{Z}$ has finite moment $EX_{i_1}^{j_1} X_{i_2}^{j_2} \ldots X_{i_n}^{j_n}$; $i_1, i_2, \ldots, i_n \in \mathbb{Z}, j_1, j_2, \ldots, j_n \in \mathbb{N}$ and $\exists$ constant c,s.t.$\lim(E(X_i)^k/k!)^{\frac{1}{k}} < c$,then μ is a convex combination of $X(\rho)$ of finite moment.*

Lemma: If an invariant measure with gap distribution X_i has finite moment, then for any distinct $i_1, i_2, \ldots, i_n \in \mathbb{Z}$, and $\forall k, j_1, j_2, \ldots, j_n \in \mathbb{N}, j_1 + j_2 + \ldots + j_n = k, \exists A_k \in R$, s.t. $EX_{i_1}^{j_1} X_{i_2}^{j_2} \ldots X_{i_n}^{j_n} = \frac{(j_1+1)!(j_2+1)!\ldots(j_n+1)!}{2^k} A_k$

Proof. Assume $X_0^i = X_0$ is the distribution containing 0. For k=1,Since X_0^1 is either generated by X_0, X_1(means$X_0 = (1 - U_1)X_0 + U_2 X_1$) or X_{-1}, X_0. Denote $A = X_0$ is generated byX_0, X_1,then we have $0 < P(A) < 1$(if P(A)=0 or 1,then it can't be an invariant measure).Since X_0 is independent with X_i when $i \to \infty$.Then as $|i| \to \infty$,since EX_i is finite, we must have

$$\lim(EX_i - \frac{EX_i + EX_{i+1}}{2}) = lim(EX_{i+1} - \frac{EX_i + EX_{i-1}}{2}) = 0$$

$$\text{Hence}, \lim(EX_{i+1} - EX_i) = 0$$

$$\text{Therefore}, \forall \epsilon > 0, \exists M > 0, s.t. |EX_i - EX_{i+1}| < \epsilon, \text{for any} |i| > M.$$

Since X_0 is arbitrarily chosen, if we choose j and k, s.t. $j - i > M, k - i < -M$, then we can conclude that $|EX_i - EX_{i+1}| < \epsilon$ for any i and ϵ,thus we let $A_1 = EX_i, \forall i \in \mathbb{Z}$
For $k > 1$, set $Y_{i_1}^{j_1} Y_{i_2}^{j_2} \ldots Y_{i_n}^{j_n} = 2^k/(j_1+1)!(j_2+1)!\ldots(j_n+1)!EX_{i_1}^{j_1} X_{i_2}^{j_2} \ldots X_{i_n}^{j_n}$
Then we only have to prove that for any distinct $i_1, i_2, \ldots, i_n \in \mathbb{Z}$, and $\forall k, j_1, j_2, \ldots, j_n \in \mathbb{N}, j_1 + j_2 + \ldots + j_n = k, Y_{i_1}^{j_1} Y_{i_2}^{j_2} \ldots Y_{i_n}^{j_n}$ are equal.

Similar to k=1, $\exists M > 0$,s.t. when $|i_1|, |i_2|, \ldots, |i_k| > M$
$\lim(EX_{i_1}X_{i_2}\ldots X_{i_k} - E((1-U_1)X_{i_1} + U_2X_{i_1+1})\ldots((1-U_k)X_{i_k} + U_{k+1}X_{i_k+1}))$
$= \lim(EX_{i_1+1}X_{i_2+1}\ldots X_{i_k+1} - E((1-U_1)X_{i_1} + U_2X_{i_1+1})\ldots((1-U_k)X_{i_k} + U_{k+1}X_{i_k+1})) = 0$
Therefore,similar to k=1, we can conclude that$EX_{i_1}X_{i_2}\ldots X_{i_k} = EX_{i_1+1}X_{i_2+1}\ldots X_{i_k+1}$
$= E((1-U_1)X_{i_1} + U_2X_{i_1+1})\ldots((1-U_k)X_{i_k} + U_{k+1}X_{i_k+1})$
$= E((1-U_1)X_{i_1-1} + U_2X_{i_1})\ldots((1-U_k)X_{i_k-1} + U_{k+1}X_{i_k})$
Therefore,$\exists a_{j_1j_2\ldots j_k}, b_{j_1j_2\ldots j_k} \in \mathbb{R}, s.t.$
$Y_{i_1}Y_{i_2}\ldots Y_{i_k} = Y_{i_1+1}Y_{i_2+1}\ldots Y_{i_k+1}$
$= \sum_{j_t=i_t,i_t+1;t=1,2,\ldots,k} a_{j_1j_2\ldots j_k}Y_{j_1}Y_{j_2}\ldots Y_{j_k}$
$= \sum_{j_t=i_t,i_t-1;t=1,2,\ldots,k} b_{j_1j_2\ldots j_k}Y_{j_1}Y_{j_2}\ldots Y_{j_k}$
$--(1)$
If we let $X_i \sim X_i(\rho)$, then $EX_{i_1}^{j_1}X_{i_2}^{j_2}\ldots X_{i_n}^{j_n} = EX_{i_1}^{j_1}EX_{i_2}^{j_2}\ldots EX_{i_n}^{j_n} = 2^k/(j_1+1)!(j_2+1)!\ldots(j_n+1)!$
Therefore, we have $Y_{i_1}Y_{i_2}\ldots Y_{i_k} = 1$.
Since $a_{j_1j_2\ldots j_k}, b_{j_1j_2\ldots j_k}$ are only related to U,therefore by (1) we have $\sum a_{j_1j_2\ldots j_k} = \sum b_{j_1j_2\ldots j_k} = 1. --(2)$
Since $E((1-U)X)^i(UX)^j = E(1-U)^iU^jEX^{i+j} = \frac{i!j!}{(i+j+1)!} * \frac{(i+j+1)!}{2^{i+j}}Y^{i+j} = \frac{i!j!}{2^{i+j}}Y^{i+j}$
$E((1-U)X)^iE(UX)^j = \frac{1}{i+1}\frac{(i+1)!}{2^i}Y^i\frac{1}{j+1}\frac{(j+1)!}{2^j}Y^j\frac{i!j!}{2^{i+j}}Y^{i+j}$ Therefore,after we substitute $EX_{i_1}^{j_1}X_{i_2}^{j_2}\ldots X_{i_n}^{j_n}$ by $Y_{i_1}^{j_1}Y_{i_2}^{j_2}\ldots Y_{i_n}^{j_n}$, the coefficient $a_{j_1j_2\ldots j_k}, b_{j_1j_2\ldots j_k}$ are all equal,and by (2) they are equal to $\frac{1}{(j_1+1)(j_2+1)\ldots(j_k+1)}$$-(3)$.
Next by(1)(2)(3)we have $Y_{i_1}^{j_1}Y_{i_2}^{j_2}\ldots Y_{i_n}^{j_n} = \frac{1}{(j_1+1)(j_2+1)\ldots(j_n+1)}\sum_{j_t=i_t,i_t+1;t=1,2,\ldots,n} Y_{i_1}^{j_1}Y_{i_2}^{j_2}\ldots Y_{i_n}^{j_n} = \frac{1}{(j_1+1)(j_2+1)\ldots(j_n+1)}\sum_{j_t=i_t,i_t-1;t=1,2,\ldots,n} Y_{i_1}^{j_1}Y_{i_2}^{j_2}\ldots Y_{i_n}^{j_n} --(*)$,it suffice to prove that $Y_{i_1}^{j_1}Y_{i_2}^{j_2}\ldots Y_{i_n}^{j_n}$are equal:
We prove by contradiction:If $Y_{i_1}^{j_1}Y_{i_2}^{j_2}\ldots Y_{i_n}^{j_n}$are not equal,since $Y_{i_1}^{j_1}Y_{i_2}^{j_2}\ldots Y_{i_n}^{j_n}$ are finite.
Set $A(m) = \{Y_{i_1}^{j_1}Y_{i_2}^{j_2}\ldots Y_{i_n}^{j_n}||i_n - i_1| = m\}$,assume $a(m) = max\{a \in A(m)\}, b(m) = min\{b \in A(m)\}$,then$\exists$ minimal $t > 0, s.t., a(t+1) - a(t) > \epsilon$ or $b(t+1) - b(t) < -\epsilon$.
Assume we have $a(t+1) - a(t) > \epsilon$(For $b(t+1) - b(t) < -\epsilon$,the proof is similar), then by (*) either $\exists (j'_n + j'_1) < (j_n + j_1), Y_{i'_1}^{j'_1}Y_{i'_2}^{j'_2}\ldots Y_{i'_n}^{j'_n} > Y_{i_1}^{j_1}Y_{i_2}^{j_2}\ldots Y_{i_n}^{j_n}$ or $a(t+2) - a(t+1) > \epsilon$ (if not we have $0 > (j_2+1)\ldots(j_{n-1}+1)\epsilon - (j_2+1)\ldots(j_{n-1}+1)\epsilon = 0$ which leads to a contradiction.)
Since $j_n + j_1 \leq k$and $(a(t+2) - a(t+1)) \geq \frac{2}{k}(a(t+1) - a(t))$ always holds, therefore $\lim_{m\to\infty} a(m) = Y_{i_1}^{j_1}Y_{i_2}^{j_2}\ldots Y_{i_n}^{j_n} \to \infty$,when $|i_n - i_1| \to \infty$

, which leads to a contradiction with finite moment. Thus for any distinct $i_1, i_2, \ldots, i_n \in \mathbb{Z}$, and $\forall k, j_1, j_2, \ldots, j_n \in \mathbb{N}, j_1+j_2+\ldots+j_n = k, Y_{i_1}^{j_1} Y_{i_2}^{j_2} \ldots Y_{i_n}^{j_n}$ are equal.
Proof of the theorem: By lemma,if we set $Y_i \sim \frac{X_i}{X_0(\rho)}$.Then we have $EY_{i_1}^{j_1} Y_{i_2}^{j_2} \ldots Y_{i_n}^{j_n} = A_k(j_1 + j_2 + \ldots + j_n = k)$.Since $\exists$ constant c,s.t.$\lim(E(X_i)^k/k!)^{\frac{1}{k}} < c$, therefore Y_i equal to the same distribution Y. $X_i \sim YX_0(\rho)$,so the gap X_i is a convex combination of $X_i(\rho)$. Now we only have to prove that the distribution of 0 in X_0 is unique,then μ is a convex combination of $X(\rho)$. Assume X' is the distribution of the gap between 0 and x_1. Then, we have $X' = X' + U_1 X_1, \text{if} X' < U_0 X_0; X' - U_0 X_0, \text{if} X' > U_0 X_0$
Therefore, (X_0, X')forms a Markov chain and (X_0, X') is ergodic on R*R. Thus the invariant joint distribution of the Markov process is unique. □

4 Convergence to invariant measure

Theorem. *Consider initial measure μ which has independent and identical distributed gap $X_i, i \in \mathbb{Z}$. If the initial gap X_i has finite moment $E(X_i)^m, m \in \mathbb{N}$, then μ converge to an invariant measure.*

Proof. We only have to prove that for any $m \in \mathbb{N}, \lim E(X_i)^m \exists$ and $\exists$ constant c,s.t.$\lim(E(X_i)^k/k!)^{\frac{1}{k}} < c$.
We prove by induction on m that$a_n = \lim E(X_i)^n \exists$.
First, $E(X_i^j) = E(X_i)$ exist. Since μ is transitional invariant, we set $\rho = 1/EX_i$.
Assume for $m < n, a_n = \lim E(X_i)^n = \frac{(n+1)!}{(2\rho)^n}$,then for $m = n$, since $E(X_i^j)$ are independent.$(EX_i^j)^n = E((1-U_1)X_1 + U_2 X_2)^n = \frac{1}{n+1}E(X_1)^n + \frac{1}{n+1}E(X_2)^n + \frac{n!(n-1)}{(2\rho)^n}$
Therefore,$\lim E(X_i)^n = \frac{(n+1)!}{(2\rho)^n}$
Thus, μ converge to $X(\rho)$ which is an invariant measure. □

5 Replace uniform distribution by general distribution on $[0, 1]$

Theorem. *If U is a general distribution on $[0, 1]$, $EU^2 + E(1-U)^2 \neq 1$, then $\exists$ invariant measure of the Markov process and all invariant measure are the* *convex combination of a group of particular invariant measure $X(U,\rho)$.*

Proof. In section 3, we have prove that if X_i is the gap of invariant measure $X(U,\rho)$, then for any distinct $i_1, i_2, \ldots, i_n \in \mathbb{Z}$, and $\forall k, j_1, j_2, \ldots, j_n \in \mathbb{N}*, j_1 + j_2 + \ldots + j_n = k$
If$EX_{i_1}^{j_1}X_{i_2}^{j_2}\ldots X_{i_n}^{j_n}\exists$, then $\exists A_k \in R$, s.t. $E(X_{i_1}/X(U,1))^{j_1}(X_{i_2}/X(U,1))^{j_2}\ldots X(i_nX(U,1))^{j_n} = A_k$
Therefore, consider the k-order moment $EX_1^{j_1}X_2^{j_2}\ldots X_n^{j_n}$. Since $EX_1^{j_1}X_2^{j_2}\ldots X_n^{j_n}$can be written as the linear combination of $EX_1^{j_1}X_2^{j_2}\ldots X_n^{j_n}, j_1+j_2+\ldots+j_n = k$. If we assume vector v=$(EX_1^{j_1}X_2^{j_2}\ldots X_n^{j_n}, j_1+j_2+\ldots+j_n = k)$, then we have v=Av, A is a $|v| * |v|$ matrix. We now have to prove that the linear equations have a non-zero solution. In section 4,since $EU^2 + E(1-U)^2 \neq 1 \Rightarrow EU^n + E(1-U)^n \neq 1$. Hence the initial coefficient of $EX_1^{j_1}X_2^{j_2}\ldots X_n^{j_n}, j_1 + j_2 + \ldots + j_n = k = 1 - EU^n - E(1-U)^n > 0$. We have prove that if we start from an initial measure μ which has i.i.d gap distribution X_i, then $\lim EX_1^{j_1}X_2^{j_2}\ldots X_n^{j_n}, j_1 + j_2 + \ldots + j_n = k\exists$. Therefore, the solution of the linear equations exist and must be a distribution of the limit measure which is also an invariant measure. Thus, all invariant measure are the convex combination of a group of particular invariant measure $X(U,\rho)$ satisfying $EX^n = E((1-U_1)X_1+U_2X_2)^n \Leftrightarrow X(U,\rho) = (1-U_1)X(U,\rho)+U_2X(U,\rho)$. □

6 Dynamics of a single particle

6.1 Invariant initial condition

Theorem. *If we start from an invariant measure $X(\rho)$, then the fluctuation of a particular point x_0^n is a Poisson process.*

Proof. Assume $x_0^0 = 0$. Then $x_0^n = U_0X_0^0 + U_1X_0^1 + \ldots + U_nX_0^n$. U_i are independent uniform distribution on $[0,1]$, suppose g(t) is the characteristic function of $U_iX_0^i$.

$$g(t) = \int_0^1 f(tu)\,du = 1/(1 - i/2\rho)$$

Therefore,$U_iX_0^i$ have the same distribution which is an exponential distribution with density function $2\rho e^{-2\rho x}, x > 0$. We want to prove that $U_iX_0^i$ are mutually independent: In section 2,we have found that $f(t) = g(t)^2$ and X_i^n are mutually independent, therefore $U_iX_0^i$ is independent with $(1-U_i)X_0^i$.

Hence, $U_0X_0^0$ is independent with $U_1X_0^1 = U_1((1-U_0)X_0^0 + U_0'X_1^0)$. Assume for m¡n $U_0X_0^0, \dots, U_mX_0^m$ are mutually independent. Then for m=n, since $U_nX_0^n = U_n((1-U_{n-1})X_0^{n-1} + U_{n-1}'X_1^{n-1})$, $U_0X_0^0, \dots, U_{n-1}X_0^{n-1}$ are independent with X_1^{n-1}, U_{n-1}', U_n Similarly we have $U_{n-1}X_0^{n-1}$ and $(1-U_{n-1})X_0^{n-1}$ are independent.Thus, $U_0X_0^0, \dots, U_nX_0^n$ are mutually independent Therefore x_0^n is a Poisson process. Thus we have $\frac{x_i^n}{\sigma_n} \sim$ normal distribution. □

6.2 Lattices initial condition

In this section, we study the dynamic of the interlacing particle system under the lattice initial condition,which means $Y_0 = \{x_i = i, i = 0, 1, \dots, n\}$.

Theorem. *Assume the distribution is U and*$EU = a, EU^2 = b, a^2 < b \leq a$, *We have there exists constant* $c_1, c_2, s.t., 0 < c_1 < \frac{Var(x_0^n)}{n/\log n} < c_2$

Proof. Assume σ_n^2 is the deviation of $x^n = x_i^n, x = x_i^{n-1}, y = x_{i+1}^{n-1}.EU = a, EU^2 = b$ $Ex^n = an$
$x^n = x + U(y-x), x' = x + (y-x)/2$
Then we have $\quad \sigma_n^2 = E(x+U(y-x))^2 - (an)^2$
$= (1-2a+b)Ex^2 + bEy^2 + 2(a-b)Exy - (an)^2$
$= (1-2a+b+2(a-b))[\sigma_{n-1}^2 + (a(n-1))^2] + b[\sigma_{n-1}^2 + (a(n-1)+1)^2] + 2(a-b)Ex(y-x) - (an)^2$

$$= \sigma_{n-1}^2 + (a^2+b-2ab) + 2(a-b)E(x'(y-x)-an) - (a-b)E(y-x)^2 \quad (1)$$

Substitute a by 1-a,b by 1-2a+b,then U is change to 1-U: due to symmetry we also have:

$$\sigma_n^2 = \sigma_{n-1}^2 + ((1-a)^2 + b - 2(1-a)b) + 2(a-b)E(y'(y-x)-an) - (a-b)E(y-x)^2 \quad (2)$$

By(1)(2)$\Rightarrow$ $\sigma_n^2 = \sigma_{n-1}^2 + (a-a^2) - (a-b)E(y-x)^2$
$= \sigma_{n-1}^2 + (a-a^2) - (a-b)E{X_0^{n-1}}^2 - -(3)$

Set $a_i^k = EX_0^k X_i^k$,then $E{X_0^{n-1}}^2 = a_0^{n-1}$ we have:

$$a_0^{k+1} = (1-2(a-b))a_0^k + 2(a-a^2)a_1^k$$
$$a_1^{k+1} = (1-a-b)a_0^k + (1-2a+2a^2)a_1^k + (a-a^2)a_2^k$$
$$a_i^{k+1} = (a-a^2)a_{i-1}^k + (1-2a+2a^2)a_i^k + (a-a^2)a_{i+1}^k, i > 1$$

Since$\forall i, a_i^0 = 1$,Hence,assume$a_0^n = A_n a_0^0 + B_n a_1^0 + \ldots = A_n + B_n + \ldots$ $(A_0 = 1, B_0 = 0)$
Set $S_n = A_n + B_n + \ldots$ Then$a_0^n = S_n$ and $S_n = S_{n-1} + (2b-2a^2)A_{n-1} - (b-a^2)B_{n-1}$ By(*),$A_n = (1-2(a-b))A_{n-1} + (a-b)B_{n-1} \Rightarrow$

$$E{Y_0^{n-1}}^2 = a_0^{n-1} = S_{n-1} = \frac{a-a^2}{a-b} - \frac{b-a^2}{a-b}A_{n-1} \tag{4}$$

By(3)(4)$\Rightarrow \sigma_n^2 = \sigma_{n-1}^2 + (b-a^2)A_{n-1}$
$\Rightarrow \sigma_n^2 = (b-a^2)\sum\limits_{k=1}^{n-1} A_k$

Thus, we only have to find the estimation of $t_{n-1} = \sum\limits_{k=1}^{n-1} A_k$ By(*), A_n is derived by two sequences:

$$A_{n+2} = 2bA_{n+1} + \frac{1-2b}{4}(B_0A_n + B_1A_{n-1} + \ldots + B_nA_0)$$
$$B_{n+2} = 1/2B_{n+1} + 1/16(B_0B_n + B_1B_{n-1} + \ldots + B_nB_0)$$

$A_0 = 1, A_1 = 1-2(a-b), B_0 = 0, B_1 = 2(a-a^2)$ First it is easily to prove that $\exists$constant $c_1, c_2, s.t., c_1 < \frac{4-B_n}{1/n} < c_2$

Therefore,$\exists$constant $c_1, c_2, s.t. c_1 < \frac{\sum\limits_{k=1}^{n-1} A_k}{\text{the sum of all the entries of} A^{-1}} < c_2$
where A is

$$\begin{pmatrix} 1 & 1/2 & \ldots & 1/n \\ 0 & 1 & \ldots & 1/(n-1) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

Set J=

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

Then the sum all the entries of A^{-1} is the coefficient of x on $(nf(x) - f'(x))$, $f(x) = -\frac{x}{\log(1-x)}$,which is $\sim \frac{n}{\log n}$ $\square$

References

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