PERIODICITY IN THE PERIODIC LAMBDA ALGEBRA
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Abstract. The periodic lambda algebra is a co-Koszul complex of the Steenrod algebra whose homology gives the E_2 term for the Adams spectral sequence. Its elements are closely related to periodic homotopy theory, and exhibit periodic properties. In this paper we discuss an algorithm to compute the homology of the periodic lambda algebra, and investigate the algebraic structure for the E_2 page of the v_n periodic elements.

1. Introduction

Computing the homotopy groups of spheres is of great interest in algebraic topology. One method of achieving this is by the Adams spectral sequence. This is a spectral sequence whose E_2 term is $Ext_{A^*}(\mathbb{Z}/p, \mathbb{Z}/p)$, where A^* is the Steenrod algebra, that converges to the p-component of $\pi_*(S^0)$.

A well studied object that gives an E_1 term for the Adams spectral sequence is the lambda algebra, which is the co-Koszul complex of the Steenrod algebra by taking the dual to the admissible basis. In [1], Gray gives another algebra Λ, which he terms the periodic lambda algebra, whose homology also gives the E_2 term for the Adams spectral sequence. This is a differential graded algebra generated multiplicatively by elements λ_i and v_n subject to certain relations.

Compared to the classical lambda algebra, the periodic lambda algebra is smaller and simpler. Of interest to us is that the v_n generators turn out to correspond to the v_n self maps in periodic homotopy theory.

It is known that $H^*(\Lambda_\lambda)$ decomposes into v_n periodic parts (see §2). This gives us a method to recover $H^*(\Lambda)$ from $H^*(\Lambda_\lambda)$. Our interest here is mainly in determining the algebraic structure of the various periodic parts. We did so by inverting the generator v_n. An understanding of this can give us information about the homotopy groups of spheres. Since when we delete $v_1, \cdots, v_{n-1}$ and invert v_n, we obtain the E_1 term of an Adams spectral sequence that converges to $\pi_*(v_n^{-1}V(n-1))$. If we then run the Bockstein spectral sequence on the E_∞ term, we obtain $\pi_*(S^0)$.

The project was roughly split into two parts. The first part was writing a computer program to compute the homology of the periodic lambda algebra and

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determine the v_n (for $n \leq 3$) periodic parts. We did this for the two cases $p = 3$ and $p = 5$. The second part was attempting to understand the algebraic structure of the periodic parts. This was mainly done through comparing the elements with known results and finding patterns between the cases $p = 3$ and $p = 5$. Our main result is determining the generators for the periodic parts, and proving certain elements to be v_n periodic. Our results also provide some insight into the "telescope conjecture".

In §2 we give a brief overview of the periodic lambda algebra, which will be the main object of study in our paper. In §3 we discuss our implementation of the leading term algorithm to generate Curtis tables for the periodic lambda algebra, and give a proof of its correctness. In §4, §5 and §7 we present some results we have obtained through investigating the Curtis tables of the respective periodic parts. We also include §6 in which we discuss how our work relates to the telescope conjecture. In §8, we give a theorem for proving certain elements to be v_n periodic. Finally, in §9 we mention some problems that we wish to continue to work on.

2. The Periodic Lambda Algebra

The classical lambda algebra is derived from the Steenrod algebra by the taking the Koszul dual of the admissible basis. In [1], Gray derives another co-Koszul complex of the Steenrod algebra by taking the Milnor basis instead, which results in the periodic lambda algebra.

The Milnor basis for the Steenrod algebra (see [3]) is given by the set of generators $\{Q_k\} \cup \{P^i\}$, with $deg(Q_k) = 2p^k - 1$ and $deg(P^n) = 2n(p-1)$. These satisfy the relations

(1) If $a<pb$, then
$$P^a P^b = \sum (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} P^{a+b-t} P^t$$
(2) $Q_k Q_l = -Q_l Q_k$
(3) $P^n Q_k = \begin{cases} Q_k P^n + Q_{k+1} P^{n-p^k} & n>p^k \\ Q_k P^n + Q_{k+1} & n = p^k \\ Q_k P^n & n<p^k \end{cases}$

According to these relations, we may express all elements in the form
$$Q_{k_1} Q_{k_2} \cdots P^{i_1} P^{i_2} \cdots ,$$
where

(1) all Q_k's occur before P^i's
(2) the Q_k's are listed in decreasing order
(3) the portion of the product consisting of P^i's are in admissible form.

By the result given in [4], we get a differential graded algebra, Λ, generated by λ_i and v_n with $deg(\lambda_k) = 2k(p-1)$ and $deg(v_n) = 2p^n - 1$ subject to the relations

(1) $\lambda_i \lambda_{pi+k} = \sum (-1)^{j+1} \binom{(p-1)(k-j)-1}{j} \lambda_{k+i-j} \lambda_{pi+j}$
(2) $v_n v_k = v_k v_n$
(3) $\lambda_k v_n = v_n \lambda_k + v_{n-1} \lambda_{k+p^{n-1}}$ if $n>0$
(4) $\lambda_k v_0 = v_0 \lambda_k$

Homological degree is given by length of monomials. The differentials are given by

(1) $d(v_n) = v_{n-1} \lambda_{p^n-1}$ if $n>0$
(2) $d(v_0) = 0$

(3) $d(\lambda_k) = \sum(-1)^{j+1}\binom{(p-1)(k-j)-1}{j}\lambda_{k-j}\lambda_j$

and satisfy the equality

$$d(xy) = d(x)y + (-1)^{deg(x)}xd(y).$$

Since this is a subcomplex of the cobar complex of the Steenrod algebra, its homology gives $E_{A^*}(\mathbb{Z}/p, \mathbb{Z}/p)$.

Now, if we consider the space Λ/v_0 arising from the short exact sequence $0 \to \Lambda \xrightarrow{v_0} \Lambda \to \Lambda/v_0 \to 0$, we get a Bockstein spectral sequence $H^*(\Lambda/v_0) \otimes P(v_0) \Rightarrow H^*(\Lambda)$. We can continue the same process on Λ/v_0, taking the quotient by v_1, to get $H^*(\Lambda/(v_1, v_0)) \otimes P(v_0, v_1) \Rightarrow H^*(\Lambda)$. If we let Λ_λ denote the subalgebra of Λ generated by the λ_i terms only, then inductively we can obtain $H^*(\Lambda_\lambda) \otimes P(v_0, v_1, \cdots) \Rightarrow H^*(\Lambda)$. This shows that $H^*(\Lambda_\lambda)$ is decomposed into periodic parts.

To find the v_n torsion free parts of Λ, we first compute $\Lambda/(v_0, \cdots, v_{n-1})$ and then invert v_n. Computing this and investigating the structure of the periodic parts will be the topic of the remaining of our paper. To simplify notation, we will abbreviate $v_n^{-1}\Lambda/(v_0, \cdots, v_{n-1})$ by $v_n^{-1}\Lambda$.

3. Leading Term Algorithm

In [8] Tangora describes an algorithm to compute the homology of the classical lambda algebra. In our project, we modified the algorithm to compute the homology of the periodic lambda algebra. We describe the algorithm and give a proof of correctness.

3.1. **The Algorithm.** We first define an ordering for the terms in Λ. We define all v_n terms to be greater than λ_i, while v_n and λ_i are ordered between themselves by topological degree. Monomials in admissible form of the same bidegree are then ordered lexicographically. We say a polynomial is expressed in "proper form" if all monomials are admissible and in decreasing order. Then the order on monomials induces a lexicographic order on polynomials. We will call the greatest monomial in a polynomial the *leading term*.

The algorithm will proceed inductively on total degree and on topological degree for each fixed total degree. At each step, we consider a source and target pair, from a box of bidegree (r, s) to $(r-1, s+1)$. We will say that a proper polynomial x *tags* a cycle basis y if it is the smallest polynomial such that $d(x) = y + (\text{terms} < y)$.

Suppose a basis at target has already been found. The base case is trivial. For each admissible monomial x from small to large at source, run the following procedure.

If $d(x) = 0$, then add the leading term of x to the basis at source. Otherwise, let m be the leading term of $d(x)$. If m has not been tagged, then m is tagged by the leading term of x. Otherwise, suppose m has been tagged by y. Then replace x by $x - y$ and run the above again.

At the end of this process, each basis cycle at target will either remain a basis cycle for the homology group or will be tagged by some element at source in which case both do not belong to the basis. However, the basis elements and tags thus expressed may not necessarily be cycles themselves. They *complete* to *complete cycles* or *full tags*, and are the leading terms of such completions.

The claim is that at each box, the basis is given by the minimal monomial in the homology class it represents.

3.2. **Proof of correctness.**

Theorem 3.1. *The algorithm given in §3.1 is finite and correct.*

Proof. We first consider finiteness and correctness in the case that we record full cycles and tags. In this case, we are working with complete basis cycles, so we know that at each iteration the leading term of the differential is driven strictly downwards, so the algorithm is finite. It must also return the correct result, as all monomials have either been completed to a full tag or complete cycle. Minimality follows from the fact that the procedure is run in increasing order on the elements at source.

To prove the leading term algorithm correct, we first note two things. First, we may assume that in each box the leading terms of the complete basis cycles are different. For if $x + y$ and $x + z$ ($x>y>z$) were both in the basis, we may replace $x + y$ with $y - z$ and still obtain the same space. Next, suppose that x tags y, but $d(x)$ does not contain y. Then the leading term of $d(x)$ must be larger than y. This follows from how we arrive at the full tag of a target cycle.

Now, at each iteration in the former case mentioned, when we add in full tags to complete x, complete basis cycles in $d(x)$ are reduced in whole. The leading term case of the algorithm can be considered a term-by-term version of this algorithm.

First suppose $d(x)$ is a single complete basis cycle with leading term m. Suppose m is tagged by some element t. If $d(t)$ contains m, then since leading terms of basis cycles are unique, its leading basis cycle must be the complete basis cycle $d(x)$. In this case the differential $d(x - t)$ will be driven downwards and the algorithm will be finite. If $d(t)$ does not contain m, then its leading term will be greater than m. When the procedure is iterated now with $x - t$ and $d(x - t)$, it will behave just as if completing t. Thus the whole process is just adding in each term of the full tag of the basis cycle one at a time. Since the completion is finite, by induction on the monomials at each source, the process must terminate.

Now for the general case, $d(x)$ is a combination of complete basis cycles. The algorithm proceeds similarly, adding in terms of full tags. However, in this case, it is possible that the terms of the full tags will overlap each other. This does not have any effect on finiteness, as the worst that can happen is that each term of every full tag in the completion of x is added. So this reduces to the case where complete cycles and full tags are recorded. Therefore finiteness is proved.

It remains to show that the leading term of the completion of x is still x. This is straightforward to check. At any step of the iteration, suppose we determine that m is already tagged by y. Then y must be less than x, or else the procedure would not have been run on y yet. This completes our proof.

□

An important property the ordering we defined gives rise to is that the initial term of $d(\mu)\mu_1 \cdots \mu_n$, where μ denotes any generator and $\mu_1 \cdots \mu_n$ is in admissible form, will be less than μ. This is because by the definition of differentials, the leading monomial of the differential of the generators λ_i and v_n are less than themselves. Since all other terms μ_j will be less than or equal to μ, the property follows. This property is crucial in the proof of following theorem.

Theorem 3.2. *Let μ denote a generator. If the monomial μx tags μy, then x tags y. Furthermore, if x tags y, then $k\mu x$ tags μy (provided μx and μy are admissible), for some coefficient k.*

Proof. First suppose that μx completes to the the tag T that tags μy. This means that

$$d(T) = \mu y + (\text{terms} < \mu y).$$

On the other hand, we have

$$d(T) = d(\mu)x + \mu d(x) + (\text{smaller terms}).$$

By our previous discussion, $\mu d(x)$ is the leading term of this polynomial. This implies that

$$d(x) = y + (\text{terms} < y).$$

To prove that x actually tags y, we need to show that no term smaller than x also satisfies this. In order to prove this, we first prove the second part of the theorem.

Suppose x completes to X, y completes to Y, and that μX is admissible. Then

$$\mu Y = \mu y + (\text{terms} < \mu Y).$$

$$d(\mu X) = k\mu d(X) + (\text{smaller terms}),$$

where k is dependent on the degree of μ.

Since X tags Y, we have as largest term

$$\text{leading term of } k\mu d(X) = k\mu d(x) = \mu y.$$

Now suppose μS is any smaller full tag such that $d(\mu S) = \mu y + (\text{smaller terms})$, then by our previous weaker proof of the first statement, we would have that some term less than or equal to S that tags Y. But this contradicts our hypothesis that X tags Y. Thus the second statement is proved.

Back to the proof of the first statement, suppose that some term $q<x$ tags y. Then by the above, μq would tag μy, a contradiction again. This proves the first statment and we are done.

□

Due to the parity of degree, in the case of the periodic lambda algebra, $k = -1$ if μ is a λ generator and $k = 1$ if μ is a v_n generator.

This theorem allows us to suppress terms whose tails have already been tagged, and played an important role in our implementation.

3.3. **Implementation.** We implemented the algorithm using Java. This provided much flexibility in implementation, as Java is an object oriented and relatively low level programming language.

Our first approach was to list out all admissible monomials up to a certain total degree, and run the algorithm on these elements at once. This gave us a Curtis table up to total degree 60. However, after that the RAM space was not enough to store all admissible monomials. This was somewhat expected as the size of the algebra grows exponentially and can be very large.

To circumvent the memory space issue, we re-structured our computer program to generate admissible monomials inductively after the algorithm had been run for the previous total degrees. This allowed us to omit admissible terms whose tails have already been tagged by theorem 3.2. The only resulting issue is that the leading term of a differential may not be in the list of monomials at target. However, if this is the case, it would imply that the leading term is already tagged, so we only need to trace back using theorem 3.2 and find its tag. Under this scheme, the program did not slow down until total degree 130, after which the program naturally became slow due to the length of each completion, with many terms reaching above 100, 000 iterations.

Two days of computing gave us the pretty looking Curtis table for the periodic lambda algebra with highest total degree 146. The terms coincide with the Curtis table for the classical lambda algebra as should. We include the Curtis table up to total degree 50 in the case $p = 3$, along with the code of the main method (with certain ommisions to increase readability), in the appendix.

We note that computing the homology up to a certain total degree, say a, does not give all terms for topological degrees up to a. However, we know there are vanishing lines in the E_2 terms. This is a line of fixed slope, say k, such that all terms lie below it. Thus, the homology is fully computed at least up to topological degree $\frac{a}{k+1}$. Nonetheless, the terms above $\frac{a}{k+1}$ are still useful in helping to determine generators and structure.

Since we are primarily concerned with the v_n periodic parts, we also ran a variation of the above program. Rather than first computing the full Curtis table then quotienting out the elements $v_1, \cdots, v_{n-1}$, it became clear that it would be more efficient to directly quotient the terms during the algorithm. This saved time as the number of elements in each degree decreased and some terms whose completions were very long were omitted. There seemed to be a balance between the number of admissible monomials we omitted and how high our total degree needed to get for us to see periodicity in the table.

We used the computer to find periodic parts in these generated tables. We did so by implementing a method that would take in a confidence factor c, and search the list for v_n towers of each cycle. This meant that an element x would be deemed v_n periodic if and only if $v_n^q x$ is a cycle for all q such that $v_n^q x$ is in the table, and $\max\{q\} \geq c$. For lower degree terms, where $\max\{q\}$ are relatively large, we can be fairly certain that they are indeed periodic. In fact, many can be proved periodic using theorem 8.1 in §8.

4. v_1 PERIODICITY

We computed $H^*(\Lambda/v_0)$ for $p = 3$ up to total degree 164 and found the v_1 periodic parts with confidence 5. This gave us a table with highest total degree 129. We also computed the same object up to total degree 303 for $p = 5$, which gave us v_1 periodic terms up to degree 247. These gave us enough information to see what happens in lower degrees and to conjecture its general structure for arbitrary p.

We compared our results to the following theorem, given in [2] by Miller.

Theorem 4.1. *The E_2 term for the Adams spectral sequence of $V(0)$ localizes at v_1 to*

$$P[v_1, v_1^{-1}] \otimes E[h_i : i \geq 1] \otimes P[b_i : i \geq 1]$$

for p>2, where the generators are explicit in the cobar complex, with

$$h_i = \{[\bar{\xi}_i]\}\ ,\ |h_i| = \big(1, 2(p^i - 1)\big)$$

$$b_i = \{\sum_{j=1}^{p-1} \binom{p}{j} [\bar{\xi}_i^{\,j}][\bar{\xi}_i^{\,p-j}]\}\ ,\ |b_i| = \big(2, 2p(p^i - 1)\big)\,.$$

For $p = 3$, λ_1 and its Massey product $\lambda_2\lambda_1$ are expected generators. The topological degree of h_2 corresponds to the term λ_4. However, since λ_4 is not a cycle, we inverted $v_1$to obtain $v_1^{-1}v_2\lambda_1$ as a candidate for h_2. Going further down the list, $v_1^{-3}v_2^3\lambda_2\lambda_1$ at bidegree $(46, 2)$ gives an element with correct degree to correspond to b_2. This gave us some clue that the terms h_i and b_i correspond to the v_2 towers of λ_1 and $\lambda_2\lambda_1$. The same pattern was apparent in the $p = 5$ case, in which case b_1 corresponds to $\lambda_4\lambda_1$.

We thus give the following conjecture.

Conjecture 4.2. *The generators in theorem 4.1 correspond to the following elements in* $H^*(v_1^{-1}\Lambda)$

$$h_i \leftrightarrow v_1^{-r}v_2^r\lambda_1$$
$$b_i \leftrightarrow v_1^{-s}v_2^s b_1\ ,\ b_1 \leftrightarrow \lambda_{p-1}\lambda_1$$

where $r = \frac{p^{i-1}-1}{p-1}$ *and* $s = \frac{p^n-p}{p-1}$. *Thus these terms give the polynomial and exterior generators for the* v_1 *periodic part in* Λ.

It is straightforward to verify that the degrees match, and that b_1 is the Massey product for λ_1 as expected.

Each element in the v_1 periodic part can be given a name using these generators. The non-trivial relations for lower degree terms in the case $p = 3$ are

$$h_1h_2 \leftrightarrow \lambda_2\lambda_3$$
$$h_1h_3 \leftrightarrow v_1^{-3}v_2^3\lambda_2\lambda_3$$
$$h_1h_3b_2 \leftrightarrow v_1^{-1}v_2\lambda_2\lambda_3\lambda_6\lambda_3$$
$$h_2h_3 \leftrightarrow v_1^{-1}v_3\lambda_1\lambda_2\lambda_3.$$

These relations completely define the structure of $H^*(v_1^{-1}\Lambda)$ up to toplogical degree 116. Similar relations can be given in the case $p = 5$. A diagram for $p = 3$ in this range is given in figure 1. We omitted the lines indicating multiplication by b_2 to simplify the diagram.

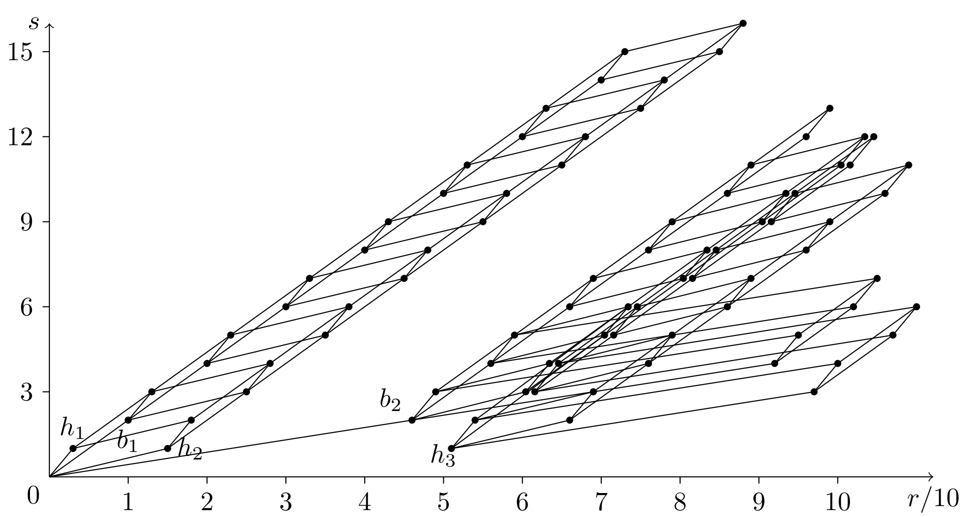

FIGURE 1

As can be seen from the figure, the E_2 page consists of b_1 and b_2 towers of elements as b_i are polynomial generators.

Since it is known that $L_{K(1)}V(0) = v_1^{-1}V(0)$, and that $E(h_1)$ gives the algebraic structure for the left hand side, we would expect all generators except h_1 to kill each other by d_2 or higher differentials. By an analysis of degrees, we give a conjecture for these differentials. The claim is that there are d_2 differentials that send h_{i+1} to $v_1 b_i$. These are indicated roughly in the diagram 2.

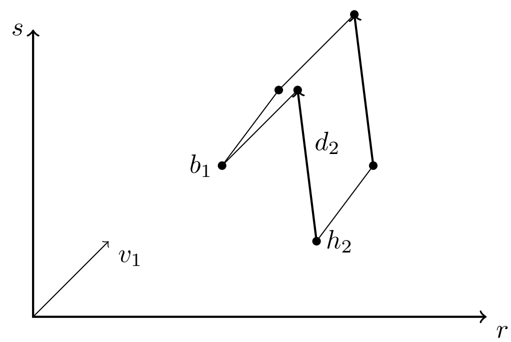

FIGURE 2

5. v_2 PERIODICITY

From the tables we generated, we observed that v_1 periodicity implies v_2 periodicity. However, the structure for higher v_n periodic parts become more and more difficult to identify, as the degrees of v_n increase, and there are more possibilities in choice of terms leading to the same degree.

For v_2 periodicity, we mainly compared our results with results giving the algebraic structures of $H^*(S(2))$ in [6]. $H^*(S(2))$ gives the E_2 term for the Adams spectral sequence that converges to $L_{K(2)}V(1)$. This sepectral sequence collapses, so the E_2 term gives the E_∞ term. On the other hand, we know that $H^*(v_2^{-1}\Lambda)$ gives the E_2 term for the Adams spectral sequence that converges to $v_2^{-1}V(1)$. It is known that there is an injective map $v_2^{-1}V(1) \to L_{K(2)}V(1)$ that respects multiplication. We use this map to compare our v_2 periodic parts.

It is unknown whether a map in the other direction holds, which would thus imply that $L_{K(2)}V(1) = v_2^{-1}V(1)$. This is the *telescope conjecture* given by Ravenel, which claims the affirmative. We discuss this in §6.

5.1. $p = 3$. We first consider the $p = 3$ case. In [6], Ravenel gives a result for the algebraic structure of $H^*(S(2))$ when $p = 3$.

Theorem 5.1. *For $p = 3$, $H^*(S(2)) \simeq E[\zeta_2, \xi] \otimes (E[h_{10}, h_{11}) \otimes P(b_{10}, b_{11})/I)$, where*

$$I = (h_{10}h_{11}, b_{10}^2 + b_{11}^2, h_{10}b_{10} - h_{11}b_{11}, h_{11}b_{10} + h_{10}b_{11}),$$

and

$$\zeta_2 = h_{2,0} + h_{2,1},$$

$$\xi = \left\langle (h_{10}, h_{11}), \begin{pmatrix} h_{10} & -h_{11} \\ h_{11} & h_{10} \end{pmatrix}, \begin{pmatrix} h_{11} & -h_{10} \\ h_{10} & h_{11} \end{pmatrix}, \begin{pmatrix} h_{10} \\ h_{11} \end{pmatrix} \right\rangle.$$

The elements h_{ij} *have bidegree* $\big(i, 2p^j(p^i-1)\big)$*, and* b_{ij} $\big(2, 2p^{j+1}(p-1)\big)$*.*

For this case, we computed $\Lambda/(v_0, v_1)$ up to total degree 149, and found the v_2 periodic parts with confidence 3. This gave us v_2 periodic parts up to topological degree 60. Using this information, we were able to easily identify $E[h_{10}, h_{11}) \otimes P(b_{10}, b_{11})$.

Conjecture 5.2. *The generators in theorem 5.1 correspond to the generators in* $v_2^{-1}H^*(\Lambda/(v_0, v_1))$ *by*

$$h_{10} \leftrightarrow \lambda_1$$

$$h_{11} \leftrightarrow \lambda_3$$

$$b_{10} \leftrightarrow \lambda_2\lambda_1$$

$$b_{11} \leftrightarrow \lambda_6\lambda_3.$$

Apart from these permanent cycles, we also have the generators $\lambda_2\lambda_3$, $\lambda_4\lambda_3$ and $v_2^{-1}v_3\lambda_3$. However, we are uncertain about the relations these generators satisfy. Although we do know that the relation $h_{10}h_{11} = 0$ in theorem 5.1 is apparent in the E_2 term.

We have been unable to determine ξ and ζ_2 for degree reasons up to our computed range. Also, we have not been able to identify any possible element that can kill $\lambda_2\lambda_3$ through d_2 or higher differentials. This suggests the possibility that $\lambda_2\lambda_3$ may possibly be a permanent cycle. These two observations shed some light on the telescope conjecture which we discuss in greater detail in §6.

5.2. $p{>}3$**.** Next we consider the $p = 5$ case. We computed $H^*(\Lambda/(v_0, v_1))$ up to total degree 304, and found the v_2 periodic part with confidence 3. The element of largest toplogical degree was at 131.

In [6] Ravenel proves the following theorem.

Theorem 5.3. *For* $p{>}3$*,* $H^*(S(2)) \simeq \mathbb{F}_p[v_2, v_2^-1]\{1, h_0, h_1, g_0, g_1, h_0g_1\} \otimes E[\zeta]$*, where the bidegrees* (s,t) *of the generators are*

$$|h_0| = (1, 2(p-1))$$
$$|h_1| = (1, -2(p-1))$$
$$|g_0| = (2, 2(p-1))$$
$$|g_1| = (2, -2(p-1))$$
$$|\zeta| = (1, 0)\,.$$

As with the previous discussions, we have conjectured an identification as follows.

Conjecture 5.4. *The elements in theorem 5.3 correspond to elements in the periodic lambda algebra by*

$$h_0 \leftrightarrow \lambda_1$$
$$h_1 \leftrightarrow v_2^{-1}\lambda_5$$
$$g_0 \leftrightarrow v_2^{-1}\lambda_2\lambda_5$$
$$g_1 \leftrightarrow v_2^{-2}\lambda_6\lambda_5$$

With this correspondence we have $h_0 g_1 \leftrightarrow \lambda_1\lambda_6\lambda_5 = \lambda_2\lambda_5\lambda_5$, which completes the "diamond" (see figure 3).

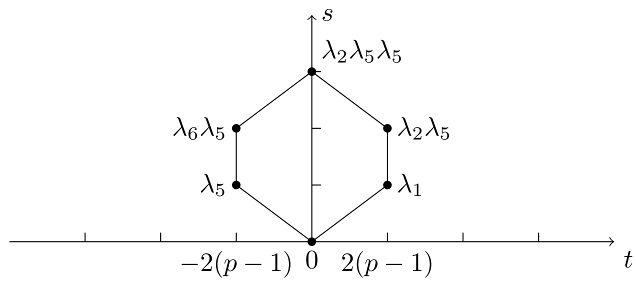

FIGURE 3

In the E_2 page, we also have $v_3\lambda_5$ and the Massey product $\lambda_4\lambda_1$ as generators. There are more conjectured generators which we describe in generality in §7.

As with the case $p = 3$, we have also been unable to identify the element ζ existing in the given algebraic structure. Also, since the terms in our above conjecture gives the permanent cycles, we would expect all other terms not belonging to the diamonds to be killed at some point in the spectral sequence. However, we have been unable to identify these either. We checked the generators $\lambda_4\lambda_1$ and $v_3\lambda_5$, both of which we would expect to be killed. This suggests that $\lambda_4\lambda_1$ and $v_3\lambda_5$ may possibly be permanent cycles, in which case the telescope conjecture would also be disproved. We come back to this issue in §6.

6. The Telescope Conjecture

As discussed in §5, the telescope conjecture conjectures the equality of $L_{K(2)}V(1)$ and $v_2^{-1}V(1)$. Ravenel conjectured it in [5], then gave a disproof of if in [7]. However, the proof was later deemed incorrect. It remains a conjecture, though most people believe it to be wrong. The main approach in the incorrect proof was to prove the nonexistence of ζ in $v_2^{-1}V(1)$, which would imply that equality cannot hold, as their E_∞ terms would not agree. In this aspect, our computations somewhat supports this assertion, despite the proof being incorrect.

For $p = 3$, we have been unable to determine ζ_2 up to total degree 149, and for $p = 5$ up to 304. Although this gives us evidence of its non-existence up to a certain range, we know the degree of ζ may be aritrarily large. Thus our computations are not sufficient to give a disproof.

Apart from determining the ζ terms, we have also run into trouble identifying suitable differentials to kill the non-permanent cycles. This cut-in point seems more promising, as it may be possible to prove the non-existence of differentials, which would also show that the E_∞ terms do not match.

We consider the generator $\lambda_2\lambda_3$ in the case $p = 3$. First, we know that it is v_2 periodic, by theorem 8.1 which we later prove. On the other hand, we conjecture that differentials should increase the number of λ terms. A verification of differentials that are known shows it is likely that differentials increase λ terms by 1 in general. If this does happen to be the case, then for any term to hit $\lambda_2\lambda_3$, it can only contain one λ term. This would narrow down our search, and could provide a possibility of proving the non-existence by degree means. This is still in work.

7. v_n Periodicity for Higher n

We first give a general conjecture showing sufficiency in v_n periodicity.

Conjecture 7.1. *If an element is v_{n-1} periodic and is non-trivial in $v_n^{-1}\Lambda$, then it is also v_{n+1} periodic.*

This holds for v_1, v_2 and v_3 periodicity up to the ranges we have computed.

For higher v_n, we also observed a neat pattern in the $p = 3$ case which we also believe to hold for general p.

From the tables we obtained, for v_1, we have the generators

$$\lambda_1, v_2\lambda_1, v_2^{13}\lambda_1, \cdots$$

$$\lambda_2\lambda_1, v_2^3\lambda_2\lambda_1, v_2^{12}\lambda_2, \cdots;$$

for v_2, there are the extra generators

$$\lambda_3, v_3\lambda_3, \cdots$$

$$\lambda_6\lambda_3, \cdots$$

$$\lambda_2\lambda_3 = \langle 1, 1, 3\rangle, \lambda_4\lambda_3 = \langle 3, 1, 3\rangle$$

and for v_3, there is the extra generator λ_9 and the conjectured Massey product $\lambda_{18}\lambda_9$.

This pattern motivated us to conjecture that in general we have inductive generators for $v_n^{-1}\Lambda$.

Conjecture 7.2. *The set of generators for $v_n^{-1}\Lambda$ consists of the λ generators for $v_{n-1}^{-1}\Lambda$, in addition to the generators*

$$v_n^{-l}v_{n+1}^{l}\lambda_{p^{n-1}}$$

$$v_n^{l-1}v_{n+1}^{l-1}\lambda_{2*p^{n-1}}\lambda_{p^{n-1}}$$

for $i \geq 1$, where $l = \frac{p^{i-1}-1}{2}$, and the Massey products of the λ terms.

If true, this would be a remarkable result, as it shows much simplicity in the periodic parts of the E_2 term.

8. Proof of Periodicity

In this section, we describe a method to prove that a term is v_n periodic. This method works for low degree terms which we know to be cycles, and even better if we know that it is v_n torsion free up to some power of v_n.

Consider a cycle x in $H^*(v_n^{-1}\Lambda)$ with topological degree r.

We first note that $v_n^l x$ cannot tag any term for any $l>0$. For if it did, it must tag a term of the form $v_n^l y$, as any other term will contain some power of v_{n-1} which is trivial in $v_n^{-1}\Lambda$. But then by theorem 3.2, this would imply that x tags y, a contradiction to x being a cycle.

So if x is not v_n periodic, it must be that some element tags an element in the v_n tower of x. Let l be the smallest power such that $v_n^l x$ is tagged. Then its tag cannot contain any v_n terms, otherwise it would contradict the minimality of l by theorem 3.2. So the smallest possible tag is $v_{n+1}^l y$ to get the leading generators v_n^l. In this case, for the correct degrees to be possible, we have the following inequality

$$2l(p^n - 1) + r > 2l(p^{n+1} - 1) + 1.$$

This gives us

$$r - 1 > 2lp^n(p-1).$$

Suppose we know that x is v_n torsion free up to power l_0. Then the above equality needs to hold for some $l > l_0$. If it is the case that $r - 1 \le 2(l_0+1)p^n(p-1)$, then we know that no such term may exist. So it must be that x is v_n periodic.

This proves the following theorem.

Theorem 8.1. *Suppose x is of topological degree r and is such that $v_n^l x$ is a cycle for all $l \le l_0$. If $r - 1 \le 2(l_0+1)p^n(p-1)$, then x must be v_n periodic.*

Recall the generator $\lambda_2\lambda_3$ mentioned in §6. Here $l_0 = 8$ from our Curtis table for $\Lambda/(v_0, v_1)$ and $r = 18$. So $2(l_0+1)p^n(p-1) = 324$, while $r - 1 = 17$. This proves our claim that $\lambda_2\lambda_3$ is v_2 periodic.

9. Further Work

As time was limited in doing this project, and much time was spent on coding the computer program, there are still many problems that we wish to solve. First and foremost is finding proofs (or disproofs) for our conjectures. This will likely require more sophiscated methods, but should be an interesting topic to work on. Ultimately, it would be nice if we could fully determine the algebraic structures for the v_n periodic parts.

Another promising topic is the telescope conjecture as discussed in §6. If our proposed method does follow through, we would be able to give a disproof of the conjecture for the special cases $p = 3$ and $p = 5$.

We also believe it is worth continue computing the homology of the periodic lambda algebra. Our code could be further optimized and run on faster computers to improve results. This could give us more information to either support or disprove our claims. Another problem that could be considered is determining a general condition for which the leading term algorithm can be applied.

Appendix A. Curtis Table, p=3

We include the Curtis table for Λ when $p = 3$ up to total degree 50. Generators are represented by numbers, with positive ones λ and negative ones v. A term such as $2(-5\ 1)$ corresponds to $2v_5\lambda_1$. Tags are listed by the format (monomial)/(tag). For instance, 1(11)/1(2) indicates that $\lambda_1\lambda_1$ is tagged by λ_2.

3,1 1(1)
3,2 1(0 1)/1(-1)
6,2 1(1 1)/1(2)
7,2 1(-1 1)
7,3 1(0 -1 1)/2(-1 -1)
10,2 1(2 1)
10,3 1(0 2 1)
10,4 1(0 0 2 1)/1(-1 -1 1)
11,1 1(3)
11,2 1(0 3)

11,3 1(0 0 3)
11,4 1(0 0 0 3)/1(-1 -1 -1)
13,3 1(1 2 1)
13,4 1(0 1 2 1)/1(-1 2 1)
14,2 1(3 1)/2(4)
15,2 1(-1 3)/1(-2)
15,4 1(-1 -1 -1 1)
15,5 1(0 -1 -1 -1 1)/1(-1 -1 -1 -1)
16,4 1(1 1 2 1)/1(2 2 1)
17,4 1(-1 1 2 1)
17,5 1(0 -1 1 2 1)/2(-1 -1 2 1)
18,2 1(4 1)/1(5) 1(2 3)
18,3 1(0 2 3)/2(-2 1)
19,5 1(-1 -1 -1 -1 1)
19,6 1(0 -1 -1 -1 -1 1)/2(-1 -1 -1 -1 -1)
20,4 1(2 1 2 1)
20,5 1(0 2 1 2 1)
20,6 1(0 0 2 1 2 1)/1(-1 -1 1 2 1)
21,3 1(3 2 1)/2(5 1) 1(1 2 3)
21,4 1(0 1 2 3)/1(-1 2 3)
22,2 1(3 3)/1(6)
22,5 1(-1 -1 -1 2 1)
22,6 1(0 -1 -1 -1 2 1)
22,7 1(0 0 -1 -1 -1 2 1)/1(-1 -1 -1 -1 -1 1)
23,3 1(-1 -2 1)
23,4 1(0 -1 -2 1)
23,5 1(0 0 -1 -2 1) 1(1 2 1 2 1)
23,6 1(0 0 0 -1 -2 1) 1(0 1 2 1 2 1)/1(-1 2 1 2 1)
23,7 1(0 0 0 0 -1 -2 1)/2(-1 -1 -1 -1 -1 -1)
24,4 1(3 1 2 1)/2(4 2 1) 1(1 1 2 3)/1(2 2 3)
25,4 1(-1 1 2 3)/1(-2 2 1)
25,6 1(-1 -1 -1 1 2 1)
25,7 1(0 -1 -1 -1 1 2 1)/1(-1 -1 -1 -1 2 1)
26,2 1(6 1)/2(7) 1(4 3)
26,3 1(0 4 3)/1(-2 3)
26,4 1(-1 -1 2 3)
26,5 1(0 -1 -1 2 3)/1(-1 -1 -2 1)
26,6 1(1 1 2 1 2 1)/1(2 2 1 2 1)
27,6 1(-1 1 2 1 2 1)
27,7 1(-1 -1 -1 -1 -1 -1 1) 1(0 -1 1 2 1 2 1)/2(-1 -1 2 1 2 1)
27,8 1(0 -1 -1 -1 -1 -1 -1 1)/1(-1 -1 -1 -1 -1 -1 -1)
28,4 1(4 1 2 1)/1(5 2 1) 1(2 1 2 3)
28,5 1(0 2 1 2 3)/2(-2 1 2 1)
29,3 1(3 2 3)/2(5 3) 1(2 3 3)
29,4 1(0 2 3 3)/2(-1 4 3)
29,7 1(-1 -1 -1 -1 1 2 1)
29,8 1(0 -1 -1 -1 -1 1 2 1)/2(-1 -1 -1 -1 -1 2 1)
30,2 1(7 1)/1(8)

30,5 1(-1 -1 -1 2 3)
30,6 1(0 -1 -1 -1 2 3)/2(-1 -1 -1 -2 1) 1(2 1 2 1 2 1)
30,7 1(0 2 1 2 1 2 1)
30,8 1(0 0 2 1 2 1 2 1)/1(-1 -1 1 2 1 2 1)
31,3 1(-1 -2 3)/2(-2 -2)
31,5 1(3 2 1 2 1)/2(5 1 2 1) 1(1 2 1 2 3)
31,6 1(0 1 2 1 2 3)/1(-1 2 1 2 3)
31,8 1(-1 -1 -1 -1 -1 -1 -1 1)
31,9 1(0 -1 -1 -1 -1 -1 -1 -1 1)/2(-1 -1 -1 -1 -1 -1 -1 -1)
32,4 1(3 1 2 3)/2(4 2 3) 1(1 2 3 3)/2(2 4 3)
32,7 1(-1 -1 -1 2 1 2 1)
32,8 1(0 -1 -1 -1 2 1 2 1)
32,9 1(0 0 -1 -1 -1 2 1 2 1)/1(-1 -1 -1 -1 -1 1 2 1)
33,3 1(6 2 1)/2(8 1)
33,4 1(-1 2 3 3)/1(-2 2 3)
33,5 1(-1 -2 1 2 1)
33,6 1(0 -1 -2 1 2 1)
33,7 1(0 0 -1 -2 1 2 1)/1(-1 -1 -1 -1 2 3) 1(1 2 1 2 1 2 1)
33,8 1(0 1 2 1 2 1 2 1)/1(-1 2 1 2 1 2 1)
34,2 1(6 3)
34,3 1(0 6 3)
34,4 1(-1 -1 4 3)/2(-2 -2 1) 1(0 0 6 3)
34,5 1(0 0 0 6 3)
34,6 1(0 0 0 0 6 3) 1(3 1 2 1 2 1)/2(4 2 1 2 1) 1(1 1 2 1 2 3)/1(2 2 1 2 3)
34,7 1(0 0 0 0 0 6 3)/1(-1 -1 -1 -1 -2 1)
34,8 1(-1 -1 -1 -1 -1 -1 2 1)
34,9 1(0 -1 -1 -1 -1 -1 -1 2 1)
34,10 1(0 0 -1 -1 -1 -1 -1 -1 2 1)/1(-1 -1 -1 -1 -1 -1 -1 -1 1)
35,1 1(9)
35,2 1(0 9)
35,3 1(0 0 9)
35,4 1(0 0 0 9)
35,5 1(0 0 0 0 9)
35,6 1(-1 1 2 1 2 3)/1(-2 2 1 2 1) 1(0 0 0 0 0 9)
35,7 1(0 0 0 0 0 0 9)
35,8 1(-1 -1 -1 1 2 1 2 1) 1(0 0 0 0 0 0 0 9)
35,9 1(0 -1 -1 -1 1 2 1 2 1)/1(-1 -1 -1 -1 2 1 2 1) 1(0 0 0 0 0 0 0 0 9)
35,10 1(0 0 0 0 0 0 0 0 0 9)/1(-1 -1 -1 -1 -1 -1 -1 -1 -1)
36,4 1(6 1 2 1)/2(7 2 1) 1(4 1 2 3)/1(5 2 3) 1(2 2 3 3)
36,5 1(0 2 2 3 3)/2(-2 1 2 3)
36,6 1(-1 -1 2 1 2 3)
36,7 1(0 -1 -1 2 1 2 3)/1(-1 -1 -2 1 2 1)
36,8 1(1 1 2 1 2 1 2 1)/1(2 2 1 2 1 2 1)
37,3 1(3 4 3)
37,4 1(0 3 4 3)/2(-1 6 3)
37,8 1(-1 1 2 1 2 1 2 1)
37,9 1(-1 -1 -1 -1 -1 -1 1 2 1) 1(0 -1 1 2 1 2 1 2 1)/2(-1 -1 2 1 2 1 2 1)
37,10 1(0 -1 -1 -1 -1 -1 -1 1 2 1)/1(-1 -1 -1 -1 -1 -1 -1 2 1)

38,2 1(9 1)/2(10) 1(7 3)
38,3 1(0 7 3)/2(-1 9)
38,6 1(4 1 2 1 2 1)/1(5 2 1 2 1) 1(2 1 2 1 2 3)
38,7 1(-1 -1 -1 -1 -1 2 3) 1(0 2 1 2 1 2 3)/2(-2 1 2 1 2 1)
38,8 1(0 -1 -1 -1 -1 -1 2 3)/1(-1 -1 -1 -1 -1 -2 1)
39,5 1(3 2 1 2 3)/2(5 1 2 3) 1(1 2 2 3 3)
39,6 1(0 1 2 2 3 3)/1(-1 2 2 3 3)
39,9 1(-1 -1 -1 -1 1 2 1 2 1)
39,10 1(-1 -1 -1 -1 -1 -1 -1 -1 -1 1) 1(0 -1 -1 -1 -1 1 2 1 2 1)/2(-1 -1 -1 -1 -1 2 1 2 1)
39,11 1(0 -1 -1 -1 -1 -1 -1 -1 -1 -1 1)/1(-1 -1 -1 -1 -1 -1 -1 -1 -1 -1)
40,4 1(7 1 2 1)/1(8 2 1) 1(3 2 3 3)/1(4 4 3)
40,7 1(-1 -1 -1 2 1 2 3)
40,8 1(0 -1 -1 -1 2 1 2 3)/2(-1 -1 -1 -2 1 2 1) 1(2 1 2 1 2 1 2 1)
40,9 1(0 2 1 2 1 2 1 2 1)
40,10 1(0 0 2 1 2 1 2 1 2 1)/1(-1 -1 1 2 1 2 1 2 1)
41,3 1(6 2 3)/2(8 3)
41,4 1(-1 3 4 3)/1(-2 4 3)
41,5 1(-1 -2 1 2 3)/2(-2 -2 2 1)
41,7 1(3 2 1 2 1 2 1)/2(5 1 2 1 2 1) 1(1 2 1 2 1 2 3)
41,8 1(0 1 2 1 2 1 2 3)/1(-1 2 1 2 1 2 3)
42,2 1(10 1)/1(11)
42,3 1(-1 7 3)
42,4 1(-1 -1 6 3)/1(-2 -2 3) 1(0 -1 7 3)/1(-1 -1 9)
42,6 1(3 1 2 1 2 3)/2(4 2 1 2 3) 1(1 1 2 2 3 3)/1(2 2 2 3 3)
42,8 1(-1 -1 -1 -1 -1 -1 2 3)
43,5 1(6 2 1 2 1)/2(8 1 2 1)
43,6 1(-1 1 2 2 3 3)/1(-2 2 1 2 3)
43,7 1(-1 -2 1 2 1 2 1)
44,4 1(6 1 2 3)/2(7 2 3) 1(4 2 3 3)/2(5 4 3) 1(2 3 4 3)
44,5 1(0 2 3 4 3)/1(-2 2 3 3)
44,6 1(-1 -1 2 2 3 3)/1(-2 -2 1 2 1)
45,3 1(9 2 1)/2(11 1) 1(3 6 3)
45,4 1(0 3 6 3)
45,5 1(0 0 3 6 3)/2(-1 -1 7 3)
46,2 1(9 3)/2(12)

Appendix B. Code

The following is the main method of our program to compute the homology of the periodic lambda algebra. We created classes "Monomial" and "Polynomial" that simplified manipulations. We hope the code is self-explanatory enough to read.

```
public static void compute() {

  for(int t = 0; t < MAX_ARRAY_SIZE; t++) {

    //find admissible monomials
    for(int i = 0; t - i > 0; i++) {

      int top = i, hom = t - i;
```

```
if(hom == 1) {
  if((top + 1) % (2 * (p - 1)) == 0) {
    ArrayList<Monomial> temp = new ArrayList<Monomial>(1);
    temp.add(new Monomial((top + 1) / (2 * (p - 1))));
    basis[top][hom] = temp;
  }
  else if((top + 2) % 2 == 0){
    double subscript = Math.log((top + 2) / 2) / Math.log(p
        );
    if(subscript - Math.floor(subscript) == 0.0) {
      ArrayList<Monomial> temp = new ArrayList<Monomial>(1)
          ;
      temp.add(new Monomial((int) - subscript));
      basis[top][hom] = temp;
    }
  }
}

else{
  ArrayList<Monomial> temp_box = new ArrayList<Monomial>();
  for(int lambda = 1; 2 * lambda * (p - 1) - 1 < i; lambda
      ++) {
    int lambda_dim = 2 * lambda * (p - 1) - 1;
    for(int item = 0; item < basis[top - lambda_dim][hom -
        1].size(); item++) {
      if(!basis[top - lambda_dim][hom - 1].get(item).
          is_tagged()) {
        if(!basis[top - lambda_dim][hom - 1].get(item).
            tags_something()) {
          Monomial temp = basis[top - lambda_dim][hom - 1].
              get(item).clone();
          temp.add(1, lambda);
          if(temp.is_admissible())
            temp_box.add(temp);
        }

        else {
          Monomial this_tags = basis[top - lambda_dim][hom
              - 1].get(item).this_tags().clone();
          this_tags.add(1, lambda);
          Monomial temp = basis[top - lambda_dim][hom - 1].
              get(item).clone();
          temp.add(1, lambda);
          if(temp.is_admissible() && !this_tags.
              is_admissible())  temp_box.add(temp);
        }
      }

      else {
        Monomial tagged_by = basis[top - lambda_dim][hom -
            1].get(item).tagged_by().clone();
        tagged_by.add(1, lambda);
        Monomial temp = basis[top - lambda_dim][hom - 1].
            get(item).clone();
        temp.add(1, lambda);
        if(temp.is_admissible() && !tagged_by.is_admissible
            ())  temp_box.add(temp);
```

```
          }
        }
      }

      for(int v = 0; 2 * (Math.pow(p, v) - 1) < top; v++){
        int v_dim = (int) (2 * (Math.pow(p, v) - 1));
        for(int item = 0; item < basis[top - v_dim][hom - 1].
            size(); item++) {
          if(!basis[top - v_dim][hom - 1].get(item).is_tagged()
              ) {
            if(!basis[top - v_dim][hom - 1].get(item).
                tags_something()){
              Monomial temp = basis[top - v_dim][hom - 1].get(
                  item);
              temp.add(1, -v);
              if(temp.is_admissible())  temp_box.add(temp);
            }
            else {
              Monomial this_tags = basis[top - v_dim][hom - 1].
                  get(item).this_tags();
              this_tags.add(1, -v);
              Monomial temp = basis[top - v_dim][hom - 1].get(
                  item).clone();
              temp.add(1, -v);
              if(temp.is_admissible() && !this_tags.
                  is_admissible()) temp_box.add(temp);
            }
          }

          else {
            Monomial tagged_by = basis[top - v_dim][hom - 1].
                get(item).tagged_by();
            tagged_by.add(1, -v);
            Monomial temp = basis[top - v_dim][hom - 1].get(
                item).clone();
            temp.add(1, -v);
            if(temp.is_admissible() && !tagged_by.is_admissible
                ())  temp_box.add(temp);
          }
        }
      }
      temp_box = sort(temp_box);
      basis[top][hom] = temp_box;
    }
  }

  //curtis
  if(t != 4){
    for(int i = 4; t - i > 0; i++) {

      int top = i, hom = t - i;

      for(int item = basis[top][hom].size() - 1; item > -1;
          item--) {

        Polynomial differential = Calculator.
            compute_differential(basis[top][hom].get(item));
```

```
          while(true) {

            if(differential.is_null()) {
              break;
            }

            Monomial leading_term = differential.greatest_term();
            int coefficient = leading_term.get_coefficient();
            int index = find_in_list(top - 1, hom + 1,
                leading_term);

            if(index != -1) {
              if(!basis[top - 1][hom + 1].get(index).is_tagged())
                   {
                basis[top - 1][hom + 1].get(index).tagged_by(
                    basis[top][hom].get(item).divide(coefficient)
                    );
                basis[top][hom].get(item).this_tags(basis[top -
                    1][hom + 1].get(index));
                break;
              }
            }

            Monomial tagged_by = find_tag(leading_term);
            differential.sum(Calculator.compute_differential(
                tagged_by.product(-coefficient)));

          }
        }

      }
    }
  }
}
```

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