Correlation Functions of the Schur Process Through Macdonald Difference Operators

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Abstract

Introduced by Okounkov and Reshetikhin in 2003, the Schur Process has been shown to be a determinantal point process, so that each of its correlation functions are determinants of minors of one correlation kernel matrix. In previous papers, this was derived using determinantal expressions of the skew-Schur functions; in this paper, we obtain this result in a different way, using the fact that the skew-Schur functions are eigenfunctions of the Macdonald difference operators.

1 Introduction

1.1 Background and Results

For positive variables $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ and a partition λ with $|\lambda| \leq n$, let $s_\lambda(x)$ be the Schur function associated to λ in the variables $x_1, x_2, \cdots, x_n$, let $s_\lambda(y)$ be the Schur function associated to λ in the variables $y_1, y_2, \cdots, y_n$, and let $F(x,y) = 1/\prod_{1 \leq i,j \leq n}(1 - x_i y_j)$. Now, let **SM** be a function mapping partitions to symmetric functions such that, for any partition λ,

$$\mathbf{SM}(\{\lambda\}) = \frac{s_\lambda(x) s_\lambda(y)}{F(x,y)}.$$

The sum of the values of this function over all partitions may be shown to be equal to 1 (Proposition 2.1.1 below), so we may extend **SM** to be a probability measure on the set of partitions. This measure, introduced by Okounkov in 2000, is the *Schur measure*.

Okounkov and Reshetikhin later generalized the Schur measure and defined the *Schur process*. Suppose that $\lambda = \{\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(n)}\}$ and $\mu = \{\mu^{(1)}, \mu^{(2)}, \cdots, \mu^{(n-1)}\}$ are sets of partitions such that $\lambda^{(1)} \supset \mu^{(1)} \subset \lambda^{(2)} \supset \mu^{(2)} \subset \cdots \subset \lambda^{(n)}$ and such that the maximum weight of any of the partitions in λ and μ is less than some integer m. Then, let x_{ij} and y_{ij} be positive variables for $1 \leq i \leq n$ and $1 \leq j \leq m$ and let

$$\mathcal{W}(\lambda, \mu) = s_{\lambda^{(1)}}(x_1) s_{\lambda^{(n)}}(y_n) \prod_{i=1}^{n-1} (s_{\lambda^{(i+1)}/\mu^{(i)}}(x_{i+1}) s_{\lambda^{(i)}/\mu^{(i)}}(y_i)),$$

where $s_{\lambda^{(i+1)}/\mu^{(i)}}(x_{i+1})$ is the skew-Schur function associated to the partitions $\lambda^{(i+1)}$ and $\mu^{(i)}$ in the variables $x_{(i+1)j}$ and $s_{\lambda^{(i)}/\mu^{(i)}}(y_i)$ is the skew-Schur function associated to the $\lambda^{(i)}$ and $\mu^{(i)}$ in the variables y_{ij}. Let $\mathbf{S}$ be a function mapping pairs (λ, μ) to symmetric functions such that, for any pair (λ, μ),

$$\mathbf{S}(\{(\lambda, \mu)\}) = \frac{\mathcal{W}(\lambda, \mu)}{\prod_{1 \le i \le j \le n} F(x_i, y_j)}.$$

It may be shown that the sum of the values of $\mathbf{S}$ over all pairs of partitions is 1 (Proposition 2.1.3 below), so $\mathbf{S}$ may be extended to form a probability measure, which is the Schur process. Observe that the Schur measure is a special case of the Schur process when $n = 1$.

We now discuss the correlation functions of the Schur measure and Schur process. Let $\mathcal{S}$ be a function mapping partitions to sets of integers such that, for any partition $\lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_{l(\lambda)}\}$, $\mathcal{S}(\lambda) = \{\lambda_1 - 1, \lambda_2 - 2, \lambda_3 - 3, \cdots, \lambda_{l(\lambda)} - l(\lambda)\}$. Now, use the Schur measure $\mathbf{SM}$ to randomly find a partition, selecting any partition λ with probability $\mathbf{SM}(\lambda)$. For any subset S of integers, we define the *correlation function* $\rho_{\mathbf{SM}}(S)$ to be the probability that $S \subset \mathcal{S}(\lambda)$.

Now suppose that λ is a set of partitions $\{\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(n)}\}$. Then, let $\mathfrak{S}$ be a function that maps a set of partitions to a subset of $\{1, 2, \cdots, n\} \times \mathbb{Z}$, mapping λ to the set of pairs $(i, \lambda_j^{(i)})$, where i ranges from 1 to n, j ranges from 1 to $l(\lambda^{(i)})$, and $\lambda^{(i)} = \{\lambda_1^{(i)}, \lambda_2^{(i)}, \cdots, \lambda_{l(\lambda^{(i)})}^{(i)}\}$. Using the Schur process, randomly select a pair (λ, μ) with probability $\mathbf{S}(\{\lambda, \mu\})$ and randomly select a set of partitions λ with probability $\mathbf{S}(\bigcup_\mu \{(\lambda, \mu)\})$, where μ ranges over all sets $\{\mu^{(1)}, \mu^{(2)}, \cdots, \mu^{(n-1)}\}$. Then, if S is a subset of $\{1, 2, \cdots, n\} \times \mathbb{Z}$, let the *correlation function* $\rho_{\mathbf{S}}(S)$ be the probability that $S \subset \mathfrak{S}(\lambda)$.

In [6], Okounkov and Reshetikhin discussed the applications of the correlation functions of the Schur process to probability theory; in particular, they showed how determining the correlation functions of the Schur process may allow one to evaluate aspects of a random three dimensional Young diagram. In order to explicitly determine the correlation functions of the Schur process and measure, they showed that the Schur process is a *determinantal point process*, so that there exists an infinite dimensional *kernel correlation matrix* K such that for any set S, the correlation function $\rho(S)$ is the determinant of the $|S| \times |S|$ minor matrix of K that takes each row and column whose index is an element of S. The same result was shown by Borodin and Rains in [3]; both of these proofs used ways of expressing the skew-Schur functions as ratios of determinants. In [1], Borodin and Corwin wrote a paper on the more general Macdonald process and discussed its relationship with the difference operators introduced by Macdonald in [5].

Here, we use the results from [1] and [5] to determine the correlation kernel matrices of the Schur measure and Schur process without using deteriminantal identities. Since determinantal identities are not known to apply for symmetric functions that generalize Schur functions (for instance, for Macdonald polynomials), the methods used here might be useful in more general contexts where the relationship with determinants is unknown.

In particular, the results from [5] allow one to express the skew-Schur functions as scalar products of Schur functions and the results from [1] yield identities that express the action of the Macdonald difference operators on multiplicative functions in terms of contour integrals. Using these facts and the Cauchy Determinant Identity, we prove Theorems 1.1.1 and 1.1.2 in Section 2; the results of Theorems 1.1.1 and 1.1.2 are also shown to hold in [5] and [6], respectively.

Theorem 1.1.1. *Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be nonnegative numbers less than 1, and let*

$$K(i,j) = \frac{1}{4\pi^2} \oint \oint \left(\frac{1}{z-w} \right) \prod_{i=1}^{n} \left(\frac{1-\frac{y_i}{z}}{1-\frac{y_i}{w}} \right) \prod_{i=1}^{n} \left(\frac{1-x_i w}{1-x_i z} \right) \frac{w^i dw dz}{z^{j+1}},$$

for each $1 \le i, j \le n$, where the contours are positively oriented about circles of different radii, but both radii being greater than 1 and less than $1/y_k$ for each $1 \le k \le n$. Then, for any subset $S = \{s_1, s_2, \cdots, s_m\}$ of integers, $\rho_{\boldsymbol{SM}}(S)$ is $\det K_S$, where K_S is the $m \times m$ matrix obtained by placing $K(s_i, s_j)$ in the ith row and jth column.

Theorem 1.1.2. *Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be sets of v nonnegative numbers less than 1, and let*

$$K(i,g;j,h) = \frac{1}{4\pi^2} \oint \oint \left(\frac{1}{z-w} \right) \left(\frac{\prod_{k=1}^{i} F(y_k, \frac{1}{w}) \prod_{k=j+1}^{n} F(x_k, z)}{\prod_{k=1}^{j} F(y_k, \frac{1}{z}) \prod_{k=i+1}^{n} F(x_k, w)} \right) \frac{z^g dw dz}{w^{h+1}},$$

where the contour integrals above are taken about positively oriented circles of different radii, but both radii being greater than 1 and less than each $1/y_{ij}$ (where y_{ij} are the elements of y_i), such that $z < w$ if $i < j$ (so that $1/(z-w)$ is expanded as $-1/w - z/w^2 - z^2/w^3 - \cdots$) and $z > w$ otherwise (so that $1/(z-w)$ is expanded as $1/z + w/z^2 + w^2/z^3 + \cdots$). Then, if $\{(a_1, b_1), (a_2, b_2), \cdots, (a_m, b_m)\} = S \subset \{1, 2, \cdots, n\} \times \mathbb{Z}$, we have that $\rho_{\boldsymbol{S}}(S)$ is $\det K_S$, where K_S is the $m \times m$ matrix whose entry in the ith row and jth column is $K(a_i, b_i; a_j, b_j)$.

The results of Theorems 1.1.1 and 1.1.2 may be generalized to obtain a similar correlation kernel matrix if arbitrary nonnegative specializations of the Schur functions and skew-Schur functions are taken instead of evaluations at a finite set of analytic variables. We do not pursue this here, but this result is shown to hold in [3] and [6].

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2 Proofs of Theorems 1.1.1 and 1.1.2

2.1 Schur Functions and Scalar Products

Let $x_1, x_2, \cdots, x_n$ be real numbers; for an integer m, let $p_m(x) = \sum_{i=1}^{n} x_i^m$ and for any partition $\nu = (\nu_1, \nu_2, \cdots)$, let $p_\nu(x) = \prod_{i=1}^{\infty} p_{\nu_i}(x)$. Let $y_1, y_2, \cdots, y_m$ be another set of variables and define the functions

$$F(x,y) = \prod_{1\le i\le n; 1\le j\le m} \left(\frac{1}{1-x_iy_j}\right), \qquad H_q(x,y) = \prod_{1\le i\le n; 1\le j\le m} \left(\frac{1-x_iy_j}{1-qx_iy_j}\right),$$

Then we have the following facts from [4].

Proposition 2.1.1. *We have that*

$$F(x,y) = \exp\left(\sum_{j=1}^{\infty} \frac{p_j(x)p_j(y)}{j}\right); \qquad H_q(x,y) = \exp\left(\sum_{j=1}^{\infty} \frac{p_j(x)p_j(y)(q^j-1)}{j}\right).$$

Moreover, letting λ range over all partitions such that $|\lambda| \le n$, we have that

$$\sum_{\lambda} s_\lambda(x)s_\lambda(y) = F(x,y).$$

Now, letting $x_1, x_2, \cdots, x_n$ be variables, consider the bilinear form on the space of symmetric functions in the x_i such that, for all partitions λ, μ such that $|\lambda|, |\mu| \le n$, we have that $\langle p_\lambda, p_\mu\rangle_x$ is $\prod_{i=1}^{\infty} i^{m_i(\lambda)} m_i(\lambda)!$ if $\lambda = \mu$ (where $m_i(\lambda)$ is the number of times i is in the partition λ) and is 0 otherwise. Then, we have the following result from [4] regarding applying the bilinear form to two Schur functions.

Proposition 2.1.2. *We have that $\langle s_\lambda(x), s_\mu(x)\rangle_x = 0$ if $\lambda \ne \mu$ and $\langle s_\lambda(x), s_\mu(x)\rangle_x = 1$ if $\lambda = \mu$.*

Using the above proposition, we may alternatively define the *skew-Schur functions* as follows. Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be variables, let λ and μ be partitions such that $|\lambda|, |\mu| \le n$, and let $s_\lambda(x,y)$ be the Schur function corresponding to λ in the variables x_i and y_i. Then, the skew-Schur function $s_{\lambda/\mu}(x)$ is equal to $\langle s_\lambda(x,y), s_\mu(x)\rangle_y$.

Now, suppose that $\lambda = \{\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(n)}\}$ and $\mu = \{\mu^{(1)}, \mu^{(2)}, \cdots, \mu^{(n-1)}\}$ are sets of partitions such that $\lambda^{(1)} \supset \mu^{(1)} \subset \lambda^{(2)} \supset \mu^{(2)} \subset \cdots \subset \lambda^{(n)}$. Then, we have the following result on the sum of the $\mathcal{W}(\lambda, \mu)$.

Proposition 2.1.3. *Summing over all possible λ and μ defined above,*

$$\sum_{\lambda,\mu} \mathcal{W}(\lambda,\mu) = \prod_{1\le i\le j\le n} F(x_i, y_j).$$

We have the following identity involving the scalar product of exponentials of power series.

Proposition 2.1.4. *Let $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$ be infinite sets of variables and suppose that $S_1, S_2, \cdots$ and $T_1, T_2, \cdots$ are power series in the y_i. Then,*

$$\left\langle \exp\left(\sum_{i=1}^{\infty} \frac{p_i(x)S_i(y)}{i}\right), \exp\left(\sum_{i=1}^{\infty} \frac{p_i(x)T_i(y)}{i}\right)\right\rangle_x = \exp\left(\sum_{i=1}^{\infty}\left(\frac{S_i(y)T_i(y)}{i}\right)\right)$$

if the series converge as power series in the y_i.

Proof. Observe that if $M_1, M_2, \cdots$ are power series in the x_i and $N_1, N_2, \cdots$ are power series in the y_i, then

$$\left(\sum_{i=1}^{\infty} \frac{M_i(x)N_i(y)}{i}\right)^m = \sum_{|\lambda|=m} \prod_{i=1}^{l(\lambda)} \left(\left(\frac{M_{\lambda_i}(x)N_{\lambda_i}(y)}{i^{m_i(\lambda)}}\right)\left(\frac{m!}{\prod_{i=1}^{\infty}(m_i(\lambda))!}\right)\right),$$

where the sum is ranged over the partitions with weight equal to m, and for any partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, $M_\lambda = \prod_{i=1}^{\infty} M_i^{\lambda_i}$ and $N_\lambda = \prod_{i=1}^{\infty} N_i^{\lambda_i}$ (observe that both expressions converge as formal power series since there are finitely many terms of fixed degree in the x and y). Hence, applying this to (M_i, N_i) equal to (p_i, S_i), (p_i, T_i), and (S_i, T_i) gives the result of the proposition since

$$\exp\left(\sum_{i=1}^{\infty} \frac{M_i(x)N_i(y)}{i}\right) = 1 + \sum_{j=1}^{\infty} \frac{1}{j!} \left(\sum_{i=1}^{\infty} \frac{M_i(x)N_i(y)}{i}\right)^j.$$

□

2.2 Difference Operators

Let Λ_n be the space of symmetric functions over the n variables $x_1, x_2, \cdots, x_n$ and let q be a real greater than 0 and less than 1. Let $T_{q,i}$ be the operator that maps any function $f(x_1, x_2, \cdots, x_n) \in \Lambda_n$ to $f(x_1,\ x_2,\ \cdots,\ x_{i-1},\ qx_i,\ x_{i+1},\ x_{i+2},\ \cdots\ ,\ x_n)$. Now, we define difference operators $D_{n;q}^r$ acting on Λ_n as

$$D_{n;q}^r = q^{\frac{r(r-1)}{2}} \sum_I \left(\prod_{i\in I, j\notin I} \left(\frac{qx_i - x_j}{x_i - x_j}\right)\right)\left(\left(\prod_{i'\in I} T_{q,i'}\right)\right),$$

where I ranges over all subsets of $\{1, 2, 3, \cdots, n\}$ of size r. In [1], a variant of the Macdonald difference operator $\tilde{D}_{n,q}^r = q^{n(n-1)/2} D_{n;q}^{n-r} T_{1/q}$ is defined, where $T_{1/q}(F)(x_1, x_2, \cdots, x_n) = F(x_1/q, x_2/q, \cdots, x_n/q)$. The following result from [1] and [4] relates the Schur functions and Macdonald difference operators.

Proposition 2.2.1. *Let e_r denote the r symmetric sum of a set of variables. Then, $s_\lambda(x)$ is an eigenfunction of $\tilde{D}_{n;q}^r$ with eigenvalue $e_r(q^{1-\lambda_1-n}, q^{2-\lambda_2-n}, \cdots, q^{-\lambda_n})$ for all $0 < q < 1$.*

From Remark 2.2.11 of [1], we also have the following way of expressing the action of $\tilde{D}_{n;q}^1$ on particular types of functions $F \in \Lambda_n$.

Proposition 2.2.2. *Suppose that $q, x_1, x_2, \cdots, x_n$ are numbers that are greater than 0 and less than 1 and that $F \in \Lambda_n$ such there exists a rational function f such that $F(x_1, x_2, \cdots, x_n) = \prod_{i=1}^n f(x_i)$ and such that $f(x_i) \neq 0$ for each $1 \leq i \leq n$. Then,*

$$q^n \tilde{D}_{n;q}^1(F(x)) = \frac{F(x)}{2\pi i} \oint \frac{q}{z - zq} \left(\prod_{k=1}^n \left(\frac{1 - qzx_k}{1 - zx_k}\right)\left(\frac{f(\frac{1}{qz})}{f(\frac{1}{z})}\right)\right) dz,$$

where the integral is over the union of the circles where $|z| = r_1$ and $|z| = r_2$ such that, for all $1 \leq i \leq n$, $qr_2 < r_1 < 1/x_i < r_2$, $f(1/qz)$ contains no poles in the torus between $|z| = r_1$ and $|z| = r_2$, and $f(1/z)$ contains no zeroes in this region.

In [2], it is shown that for particular functions F, r_2 may be increased without changing the integral and that, if r_2 tends to ∞, then the contour integral about the circle $|z| = r_2$ is a multiple of q^n. A similar method yields that the integral about $|z| = r_2$ is a multiple of q^n if $f(1/qz)$ has no poles outside of the region determined by $|z| = r_1$ and $f(1/z)$ has no zeroes inside this region. This gives the following result.

Proposition 2.2.3. *Using the notation above, letting $y_1, y_2, \cdots, y_n$ be nonnegative numbers less than 1, and supposing that no pole of $f(1/qz)$ is outside the region $|z| = r_1$ and that no zero of $f(1/z)$ is in the region $|z| = r_1$, we have that*

$$q^n \tilde{D}^1_{n;q}(F(x)) - \frac{F(x)}{2\pi i} \oint \frac{q}{z - zq} \left(\prod_{k=1}^{n} \left(\frac{1 - qzx_k}{1 - zx_k} \right) \left(\frac{f(\frac{1}{qz})}{f(\frac{1}{z})} \right) \right) dz,$$

where the integral is about the circle $|z| = r_1$, is a multiple of q^n.

The above proposition may be generalized as follows.

Proposition 2.2.4. *Using the notation as used in the previous proposition, and letting $q_1, q_2, \cdots, q_m$ be positive reals that are less than 1 and sufficiently close to 1, we have that*

$$\prod_{i=1}^{m} q_i^n \left(\prod_{k=1}^{m} \tilde{D}^1_{n;q_i} \right) (F(x)) - F(x) \left(\frac{1}{2\pi i} \right)^m \oint \oint \cdots \oint \prod_{k=1}^{m} \left(\frac{q_k}{z_k - q_k z_k} \right) \times$$
$$\prod_{1 \leq j < k \leq m} \left(\frac{(q_k z_k - q_j z_j)(z_k - z_j)}{(z_k - q_j z_j)(q_k z_k - z_j)} \right) \prod_{\substack{1 \leq j \leq m \\ 1 \leq k \leq n}} \left(\left(\frac{1 - q_j z_j x_k}{1 - z_j x_k} \right) \left(\frac{f(\frac{1}{q_j z_j})}{f(\frac{1}{z_j})} \right) \right) \prod_{k=1}^{m} dz_k,$$

is a multiple of $\prod_{i=1}^{m} q_i^n$, where each of the contour integrals is a circle centered at 0, such that no contour integral contains one of the $1/x_i$, such that all poles of $f(1/q_j z_j)$ is contained in each of the contours, such that no zero of $f(1/z_j)$ is contained in any of the contours, and such that the contour integrals satisfy $z_{i-1} < z_i \min_{1 \leq j \leq m} q_j$ for all $2 \leq i \leq m$ (we will assume that the q_i are sufficiently close to 1 so that this is possible).

Proof. We proceed by induction on m; if $m = 1$, then this result is the previous proposition. Now suppose that the statement holds for $r = r' - 1$ and we shall show it holds for $r = r'$. Due to the previous proposition, we have that

$$q^n_{r'} \left(\prod_{k=1}^{r'} \tilde{D}^1_{n;q_i} \right) (F(y)) = q^n_{r'} \left(\prod_{k=1}^{r'-1} \tilde{D}^1_{n;q_i} \right) \left(\tilde{D}^1_{n;q_{r'}}(F(y)) \right)$$
$$= \frac{1}{2\pi i} \oint \left(\frac{1}{z_r - z_r q_r} \right) \left(\prod_{k=1}^{r'-1} \tilde{D}^1_{n;q_i} \right) \left(F(y) \prod_{k=1}^{n} \left(\frac{1 - q_r z_r x_k}{1 - z_r x_k} \right) \left(\frac{f(\frac{1}{q_r z_r})}{f(\frac{1}{z_r})} \right) \right) dz_r$$

added to a multiple of $q_{r'}^n$, so using the inductive hypothesis on $f_1(x) = f(x)(1 - q_r z_r y)/(1 - z_r y)$ gives the result of the proposition. $\square$

In order to find the correlation functions for the Schur process, we use ideas from [1] and [2]. In [1], it was suggested to apply the difference operators to the weights of the Schur measure **SM** in order to obtain a formal power series with coefficients equal to the correlation functions of the Schur measure. In [2], Proposition 2.1.2 is used in order to put the skew-Schur functions in terms of a scalar product of Schur functions, which allows one to determine averages of expressions under the Schur process.

We first consider the case when the Schur process is the Schur measure.

2.3 Correlation Functions of the Schur Measure

We now prove Theorem 1.1.1.

Proof. Let $\mathbf{1}_X$ for a statement X be 1 if X holds and 0 otherwise, and for any set $S' \subset \mathbb{Z}$ and $a \in \mathbb{Z}$, let $S' + a$ be the set formed by adding each element of S' to a and let $-S'$ be the set containing all elements $-x$ for $x \in S$ Moreover, let $q_1, q_2, \cdots, q_m$ be formal variables and for any ordered set $T = \{t_1, t_2, \cdots, t_m\}$ let $q^T = \prod_{i=1}^m q_i^{t_i}$. Then,

$$\sum_{T \in \mathbb{Z}^m} \rho_{\mathbf{SM}}(T) q^{-T} = \sum_\lambda \left(\mathbf{SM}(\lambda) \sum_{T \in \mathbb{Z}^m} \mathbf{1}_{T \subset \mathcal{S}(\lambda)} q^{-T} \right)$$
$$= \sum_\lambda \left(\mathbf{SM}(\lambda) \prod_{i=1}^m \left(\sum_{j=1}^\infty q_i^{j-\lambda_j} \right) \right) = \sum_\lambda \left(\left(\frac{s_\lambda(x) s_\lambda(y)}{F(x,y)} \right) \prod_{i=1}^m \left(\sum_{j=1}^\infty q_i^{j-\lambda_j} \right) \right),$$

where λ is summed over partitions of length at most m. Hence, $\rho_{\mathbf{SM}}(T)$ is equal to the coefficient of q^{-T} in

$$\sum_\lambda \left(\left(\frac{s_\lambda(x) s_\lambda(y)}{F(x,y)} \right) \prod_{i=1}^m \left(\sum_{j=1}^n q_i^{j-\lambda_j} \right) \right)$$

if $v \geq \max\{m, m - \min_{t_i \in T} t_i\}$ (if this does not hold, then we may add variables equal to 0 to x and y). Now, by Proposition 2.2.1, the above expression is equal to

$$\sum_\lambda \left(\prod_{i=1}^m q_i^n \left(\prod_{i=1}^m \tilde{D}^1_{n;q_i} \right) (s_\lambda(x)) \right) s_\lambda(y) \prod_{1 \leq i,j \leq n} (1 - x_i y_j)$$
$$= \prod_{1 \leq i,j \leq n} (1 - x_i y_j) \prod_{i=1}^m q_i^n \left(\prod_{i=1}^m \tilde{D}^1_{n;q_i} \right)_x \left(\sum_\lambda s_\lambda(x) s_\lambda(y) \right)$$
$$= \left(\prod_{1 \leq i,j \leq n} (1 - x_i y_j) \right) \left(\prod_{i=1}^m q_i^n \right) \left(\prod_{i=1}^m \tilde{D}^1_{n;q_i} \right)_x \left(\prod_{1 \leq i,j \leq n} \left(\frac{1}{1 - x_i y_j} \right) \right),$$

where, for any operator D on the space of symmetric functions in the x_i and y_i, D_x denotes the action of D on the x_i (note that we may let the q_i be vary since the Schur measure is independent of the q_i). Now, we may suppose that $v > \max_{t \in T} t$, so Proposition 2.2.4 gives that the expression above is

$$\left(\frac{1}{2\pi i}\right)^m \oint\oint\cdots\oint \prod_{\substack{1\le j\le m\\1\le k\le n}} \left(\left(\frac{1-\frac{y_k}{z_j}}{1-\frac{y_k}{q_jz_j}}\right)\left(\frac{1-q_jz_jx_k}{1-z_jx_k}\right)\right)\times$$
$$\prod_{k=1}^m \left(\frac{q_k}{z_k-q_kz_k}\right)\prod_{1\le j<k\le m}\left(\frac{(q_kz_k-q_jz_j)(z_k-z_j)}{(z_k-q_jz_j)(q_kz_k-z_j)}\right)\prod_{k=1}^m dz_k,$$

where $z_1, z_2, \cdots, z_m$ are formal variables and the contours represent the sum of residues in a region that contains that contains the x_i and the y_i, but not the $1/q_jx_i$ and $1/q_jy_i$ (such a region exists since $x_i, y_i < 1$), and where the rational functions have power series functions expansions as if $0 < q_i < 1$ for all i and as if $z_k > q_jz_i > 0$ for all $i < k$ and all j. Substituting $w_i = q_iz_i$ for each i, letting $\mathbf{M}$ be the matrix that has the entry $1/(z_i - w_j)$ in its ith row and jth column, using the Residue Theorem, and applying the Cauchy Determinant Identity gives that $\rho_{\mathbf{SM}}(S)$ is

$$\left(\frac{1}{4\pi^2}\right)^m \oint\oint\cdots\oint \prod_{\substack{1\le j\le m\\1\le k\le n}} \left(\left(\frac{1-\frac{x_k}{z_j}}{1-\frac{x_k}{w_j}}\right)\left(\frac{1-y_kw_j}{1-y_kz_j}\right)\right)\det\mathbf{M}\prod_{k=1}^m\left(\frac{w_k^{s_k}dz_kdw_k}{z_k^{s_k+1}}\right),$$

where the contours represent the sum of the residues of the integrand in a region in a region containing the x_i and y_i but not containing the $1/x_i$ and $1/y_i$, and where the power series expansions of the rational functions in the integrand are such that $w_i < z_i$ and $z_i < w_j$ for $i < j$. Let S_n be the symmetric group on n elements; expanding the above expression as a sum yields

$$\left(\frac{1}{4\pi^2}\right)^m \sum_{h\in S_n}\oint\oint\cdots\oint \operatorname{sgn}(h)\prod_{\substack{1\le j\le m\\1\le k\le n}} \left(\left(\frac{1-\frac{y_k}{z_j}}{1-\frac{y_k}{w_j}}\right)\left(\frac{1-x_kw_j}{1-x_kz_j}\right)\right)\prod_{j=1}^m\left(\frac{w_j^{s_j}dz_jdw_{h(j)}}{(z_j-w_{h(j)})z_j^{s_j+1}}\right).$$

Using the change of variables by letting $h(j)$ become j gives that the above expression is equal to

$$\left(\frac{1}{4\pi^2}\right)^m \sum_{h\in S_n}\oint\oint\cdots\oint \operatorname{sgn}(h)\prod_{\substack{1\le j\le m\\1\le k\le n}} \left(\left(\frac{1-\frac{y_k}{z_j}}{1-\frac{y_k}{w_j}}\right)\left(\frac{1-x_kw_j}{1-x_kz_j}\right)\right)\prod_{j=1}^m\left(\frac{w_j^{s_j}dz_jdw_j}{(z_j-w_j)z_{h(j)}^{s_j+1}}\right)$$
$$=\left(\frac{1}{4\pi^2}\right)^m \sum_{h\in S_n}\operatorname{sgn}(h)\prod_{j=1}^m\left(\oint\oint\left(\frac{1}{z-w}\right)\left(\prod_{k=1}^n\left(\frac{1-\frac{y_k}{z}}{1-\frac{y_k}{w}}\right)\left(\frac{1-x_kw}{1-x_kz}\right)\right)\left(\frac{w^{s_j}dzdw}{z^{s_{h(j)}+1}}\right)\right),$$

which is $\det K_S$, so $\rho_{\mathbf{SM}}(S) = \det K_S$. □

2.4 The Schur Process

We generalize Theorem 1.1.1 and prove Theorem 1.1.2 below.

Proof. Let the integer $i \in \{1, 2, \cdots, n\}$ appear as the left coordinate of $r(i)$ distinct elements of s, say $(i, s_{i1}), (i, s_{i2}), \cdots, (i, s_{ir(i)})$. Then, let q_{ij} for $1 \leq i \leq n$, $1 \leq j \leq r(i)$ (if $r(i) = 0$, then there will be no q_{ij}), be formal variables. Let $\sum_{i=1}^{n} r(i) = m$. For any set $\{t_{ij}\} = T \in \mathbb{Z}^m$, where $1 \leq i \leq n$ and $1 \leq j \leq r(i)$, let $q^T = \prod_{i=1}^{n} \prod_{j=1}^{r(i)} q_{ij}^{t_{ij}}$. As previously, we have that $\rho_{\mathbf{S}}(T)$ is the coefficient of q^{-T} in

$$\sum_{\lambda,\mu} \left(\mathbf{S}(\{(\lambda, \mu)\}) \prod_{i=1}^{n} \prod_{j=1}^{r(i)} \left(\sum_{k=1}^{v+u} q_{ij}^{k-\lambda_k^{(i)}} \right) \right),$$

where the sum is over all pairs of sets of partitions (λ, μ) such that λ and μ each has n elements, and $n \geq \max\{m, m - \min_{t \in T} t, \max_{t \in T} t\}$ (if this is not true, we may add variables equal to 0 to x and y). Now, let $A_1, A_2, \cdots, A_{n-1}$ be infinite sets of formal variables. Now, we have that

$$\mathbf{S}(\lambda, \mu) \prod_{1 \leq i \leq j \leq n} F(x_i, y_j) = s_{\lambda^{(1)}}(x_1) s_{\lambda^{(n)}}(y_n) \times$$

$$\prod_{i=1}^{n-1} \langle s_{\lambda^{(i+1)}}(x_{i+1}, a_{i,u}), s_{\mu^{(i)}}(a_{i,u}) \rangle_{a_{i,u}} \langle s_{\lambda^{(i)}}(y_i, b_i), s_{\mu^{(i)}}(b_i) \rangle_{b_i},$$

where $a_{1,u}, a_{2,u}, \cdots, a_{n-1,u}$ are the first u elements of $A_1, A_2, \cdots, A_{n-1}$, respectively and $b_1, b_2, \cdots, b_{n-1}$ are infinite sets of formal variables. Now, using Proposition 2.2.1, we have that

$$\prod_{1 \leq i \leq j \leq n} F(x_i, y_j) \sum_{\lambda,\mu} \mathbf{S}(\lambda, \mu) \prod_{j=1}^{r(1)} \left(\sum_{k=1}^{v} q_{1j}^{k-\lambda_k^{(1)}} \right) \prod_{i=1}^{n} \prod_{j=1}^{r(i)} \left(\sum_{k=1}^{v+u} q_{ij}^{k-\lambda_k^{(i)}} \right)$$

$$= \prod_{k=1}^{n} \prod_{i=1}^{r(k)} q_i^v \sum_{\lambda,\mu} s_{\lambda^{(n)}}(y_n) \left(\prod_{j=1}^{r(1)} \tilde{D}^1_{v;q_{1j}} \right)_{x_1} (s_{\lambda^{(1)}}(x_1)) \prod_{i=1}^{n-1} \Bigg(\langle s_{\lambda^{(i)}}(y_i, b_i), s_{\mu^{(i)}}(b_i) \rangle_{b_i} \times$$

$$\left\langle \left(\prod_{j=1}^{r(i+1)} \tilde{D}^1_{v+u;q_{ij}} \right)_{x_{i+1}, a_{i,u}} (s_{\lambda^{(i+1)}}(x_{i+1}, a_{i,u})), s_{\mu^{(i)}}(a_{i,u}) \right\rangle_{a_{i,u}} \Bigg).$$

First, we sum over $\lambda^{(1)}$, fixing the other partitions of λ and the partitions of μ. Since the scalar product is bilinear, the difference operators are linear, and the above expression converges, we have that the above term is equal to

$$\prod_{i=1}^{r(1)} q_i^v \prod_{k=2}^{n} \prod_{i=1}^{r(k)} q_i^{v+u} \sum_{\lambda - \{\lambda^{(1)}\}} \sum_{\mu} s_{\lambda^{(n)}}(y_n) \left\langle \left(\prod_{j=1}^{r(1)} \tilde{D}^1_{v;q_{1j}} \right)_{x_1} \left(\sum_{\lambda^{(1)}} s_{\lambda^{(1)}}(x_1) s_{\lambda^{(1)}}(y_1, b_1) \right), s_{\mu^{(1)}}(b_1) \right\rangle_{b_1} \times$$

$$\left\langle \left(\prod_{j=1}^{r(2)} \tilde{D}^1_{v+u;q_{2j}} \right)_{x_2, a_{1,u}} (s_{\lambda^{(2)}}(x_2, a_{1,u})), s_{\mu^{(1)}}(a_{1,u}) \right\rangle_{a_{1,u}} \times$$

$$\prod_{i=2}^{n-1}\left(\langle s_{\lambda^{(i)}}(y_i,b_i),s_{\mu^{(i)}}(b_i)\rangle_{b_i}\left\langle\left(\prod_{j=1}^{r(i+1)}\tilde{D}^1_{v+u;q_{ij}}\right)_{x_{i+1},a_{i,u}}(s_{\lambda^{(i+1)}}(x_{i+1},a_{i,u})),s_{\mu^{(i)}}(a_{i,u})\right\rangle_{a_{i,u}}\right),$$

Summing over $\mu^{(1)}$ and using Proposition 2.1.1 yields that the above term equals

$$\prod_{i=1}^{r(1)}q_i^v\prod_{k=2}^{n}\prod_{i=1}^{r(k)}q_i^{v+u}\sum_{\lambda'}\sum_{\mu'}\left\langle\left\langle\left(\prod_{j=1}^{r(1)}\tilde{D}^1_{v;q_{1j}}\right)_{x_1}(F(x_1;y_1,b_1)),F(b_1,a_{1,u})\right\rangle_{b_1},\right.$$
$$\left.\left(\prod_{j=1}^{r(2)}\tilde{D}^1_{v+u;q_{2j}}\right)_{x_2,a_{1,u}}(s_{\lambda^{(2)}}(x_2,a_{1,u}))\right\rangle_{a_{1,u}}s_{\lambda^{(n)}}(y_n)\prod_{i=2}^{n-1}\left(\langle s_{\lambda^{(i)}}(y_i,b_i),s_{\mu^{(i)}}(b_i)\rangle_{b_i}\times\right.$$
$$\left.\left\langle\left(\prod_{j=1}^{r(i+1)}\tilde{D}^1_{v+u;q_{ij}}\right)_{x_{i+1},a_{i,u}}(s_{\lambda^{(i+1)}}(x_{i+1},a_{i,u})),s_{\mu^{(i)}}(a_{i,u})\right\rangle_{a_{i,u}}\right),$$

where $\lambda'=\lambda-\{\lambda^{(1)}\}$ and $\mu'=\mu-\{\mu^{(1)}\}$. Repeating this process yields that the above is equal to

$$\prod_{i=1}^{r(1)}q_i^v\prod_{k=2}^{n}\prod_{i=1}^{r(k)}q_i^{v+u}\left\langle\cdots\left\langle\left\langle\left(\prod_{j=1}^{r(1)}\tilde{D}^1_{v;q_{1j}}\right)_{x_1}(F(x_1;y_1,b_1)),F(b_1,a_{1,u})\right\rangle_{b_1},\left(\prod_{j=1}^{r(2)}\tilde{D}^1_{v+u;q_{2j}}\right)_{x_2,a_{1,u}}\right.\right.$$
$$\left.(F(x_2,a_{1,u};y_2,b_2))\right\rangle_{a_{1,u}},\cdots,\left(\prod_{j=n-1}^{r(n)}\tilde{D}^1_{v+u;q_{nj}}\right)_{x_n,a_{n-1,u}}(F(x_n,a_{n-1,u};y_n))\Bigg\rangle_{a_{n-1,u}}.$$

Now, due to Proposition 2.2.4, we have that

$$\left\langle\prod_{i=1}^{r(1)}q_i^v\left(\prod_{j=1}^{r(1)}\tilde{D}^1_{v;q_{1j}}\right)_{x_1}(F(x_1;y_1,b_1)),F(b_1,a_{1,u})\right\rangle_{b_1}$$
$$=\left\langle\left(\frac{1}{2\pi i}\right)^{r(1)}\oint\oint\cdots\oint\prod_{j=1}^{r(1)}\left(\frac{q_{1j}}{z_{1j}-q_{1j}z_{1j}}\right)\prod_{1\le j<k\le r(1)}\left(\frac{(q_{1k}z_{1k}-q_{1j}z_{1j})(z_{1k}-z_{1j})}{(z_{1k}-q_{1j}z_{1j})(q_{1k}z_{1k}-z_{1j})}\right)\times\right.$$
$$\left.F(x_1;b_1,y_1)\prod_{j=1}^{r(1)}\left(H_{q_{1j}}(x_1;z_{1j})H_{q_{1j}}\left(b_1,y_1;\frac{1}{q_{1j}z_{1j}}\right)\right)\prod_{j=1}^{r(1)}dz_{1j},F(b_1,a_{1,u})\right\rangle_{b_1}$$

added to a multiple of $\prod_{i=1}^{r(1)}q_{1i}^v$, where the contour represents the sum of the residues of the integrated function in a region containing the x_i and y_i and not containing the $1/q_ix_j$ and $1/q_iy_j$ and where the power series are expanded as if $q_{1j}>1$ and $q_{1j}z_{1j}>z_{1k}$ for all $j>k$. Furthermore, the rational functions H and F are expanded as exponentials as in Proposition 2.1.1 so that

$$F(x,y) = \exp\left(\sum_{i=1}^{\infty} \frac{p_i(x)p_i(y)}{i}\right); \qquad H_q(x,y) = \exp\left(\sum_{i=1}^{\infty} \frac{p_i(x)p_i(y)(q^i-1)}{i}\right).$$

Since each of the power sums converges as an meromorphic function in the a_{ij}, x_{ij}, and y_{ij}, we have that

$$\left\langle \prod_{i=1}^{r(1)} q_i^v \left(\prod_{j=1}^{r(1)} \tilde{D}^1_{v;q_{1j}}\right)_{x_1} (F(x_1;y_1,b_1)), F(b_1,a_{1,u}) \right\rangle_{b_1}$$
$$= \left(\frac{1}{2\pi i}\right)^{r(1)} \oint\oint\cdots\oint \prod_{j=1}^{r(1)} \left(\frac{q_{1j}}{z_{1j}-q_{1j}z_{1j}}\right) \prod_{1\le j<k\le r(1)} \left(\frac{(q_{1k}z_{1k}-q_{1j}z_{1j})(z_{1k}-z_{1j})}{(z_{1k}-q_{1j}z_{1j})(q_{1k}z_{1k}-z_{1j})}\right) \times$$
$$\left\langle F(x_1;b_1,y_1) \prod_{j=1}^{r(1)} H_{q_{1j}}\left(b_1;\frac{1}{q_{1j}z_{1j}}\right), F(b_1,a_{1,u}) \right\rangle_{b_1} \prod_{j=1}^{r(1)} \left(H_{q_{1j}}(x_1;z_{1j}) H_{q_{1j}}\left(y_1;\frac{1}{z_{1j}}\right) dz_{1j}\right)$$

added to a multiple of $\prod_{i=1}^{r(1)} q_i^v$. Applying the same reasoning, we obtain that

$$\left\langle \left\langle \prod_{i=1}^{r(1)} q_i^v \left(\prod_{j=1}^{r(1)} \tilde{D}^1_{v;q_{1j}}\right)_{x_1} (F(x_1;y_1,b_1)), F(b_1,a_{1,u}) \right\rangle_{b_1}, \prod_{i=1}^{r(2)} q_i^{v+u} \left(\prod_{j=1}^{r(2)} \tilde{D}^1_{v;q_{2j}}\right)_{x_2,a_{1,u}} F(x_2,a_{1,u};y_2,b_2) \right\rangle_{a_{1,u}}$$
$$= \left(\frac{1}{2\pi i}\right)^{r(1)+r(2)} \oint\oint\cdots\oint \prod_{l=1}^{2} \left(\frac{\prod_{1\le k<j\le r(l)}(q_{lk}z_{lk}-q_{lj}z_{lj})(z_{lj}-z_{lk})}{\prod_{1\le k,j\le r(l)}(q_{lk}z_{lk}-z_{lj})}\right) \left\langle \left\langle F(x_1;b_1,y_1)\times\right.\right.$$
$$\left.\prod_{j=1}^{r(1)} H_{q_{1j}}\left(b_1;\frac{1}{q_{1j}z_{1j}}\right), F(b_1,a_{1,u})\right\rangle_{b_1}, F(x_2,a_{1,u};b_2,y_2) \prod_{j=1}^{r(2)} \left(H_{q_{2j}}(a_{1,u};z_{2j}) H_{q_{2j}}\left(b_2,\frac{1}{q_{2j}z_{2j}}\right)\right) \Bigg\rangle_{a_{1,u}} \times$$
$$\prod_{j=1}^{r(1)} \left(H_{q_{1j}}(x_1;z_{1j}) H_{q_{1j}}\left(y_1;\frac{1}{q_{1j}z_{1j}}\right) q_{1j}dz_{1j}\right) \prod_{j=1}^{r(2)} \left(H_{q_{2j}}(x_2;z_{2j}) H_{q_{2j}}\left(b_2,y_2;\frac{1}{q_{2j}z_{2j}}\right) q_{2j}dz_{2j}\right)$$

added to a multiple of $\prod_{i=1}^{r(1)} q_i^v \prod_{i=1}^{r(2)} q_i^{v+u}$, where the contour passes through each of the $x_{1j}, x_{2j}, y_{1j}, y_{2j}$ and does not pass through the $1/q_{ij}x_{lk}$ or $1/q_{ij}y_{lk}$. Furthermore, the $1/(q_{lk}z_{lk} - z_{lj})$ terms are expanded as if each $q_{lk} < 1$ and $q_{lk}z_{lk} > z_{lj}$ for $k > j$ and $l \in \{1,2\}$. The terms in the integral dependent on u are now the sets of formal variables $a_{i,u}$ and the correlation functions are not dependent on the $a_{i,u}$, so we may let u tend to ∞.

We now determine the values of the scalar products. Using the exponential form of F and H above and Proposition 2.1.4, we obtain that $\langle H_{q_{1j}}(b_1;1/q_{1j}z_{1j})F(x_1;b_1,y_1), F(b_1,a_{1,u})\rangle_{b_1}$ is equal to

$$F(x_1;y_1)\left\langle \exp\left(\sum_{i=1}^{\infty}\frac{p_i(b_1)((q_{1j}^i-1)/q_{1j}^i z_{1j}^i + p_i(x_1))}{i}\right), \exp\left(\sum_{i=1}^{\infty}\frac{p_i(a_{1,u})p_i(b_1)}{i}\right)\right\rangle_{b_1}$$
$$= F(x_1;y_1)\exp\left(\sum_{i=1}^{\infty}\frac{p_i(a_{1,u})(q_{1j}^i-1)/q_{1j}^i z_{1j}^i + p_i(a_{1,u})(p_i(x_1))}{i}\right),$$

which is $H_{q_{1j}}(a_{1,u}; 1/q_{1j}z_{1j})F(x_1; a_{1,u}, y_1)$. We may then use similar reasoning as in the proof of Proposition 2.1.4 to obtain that, for any $1 \le j \le r(1)$ and $1 \le k \le r(2)$, as u tends to ∞,

$$\left\langle H_{q_{1j}}(a_{1,u}; z_{1j}), H_{q_{2k}}\left(a_{1,u}; \tfrac{1}{q_{2k}z_{2k}}\right) \right\rangle_{a_{1,u}}$$

tends to as a formal power series in q and z (so that the coefficient of any term of fixed degree in q and z in the difference of the two expressions is 0 for sufficiently large u)

$$\left\langle \exp\left(\sum_{i=1}^{\infty}\left(\frac{p_i(a_{1,u})z_{1j}^i(q_{1j}^i-1)}{i}\right)\right), \exp\left(\sum_{i=1}^{\infty}\left(\frac{p_i(a_{1,u})(q_{2k}^i-1)}{iz_{2k}^iq_{2k}^i}\right)\right)\right\rangle_{\lim_{u\to\infty} a_{1,u}}$$
$$= \exp\left(\sum_{i=1}^{\infty}\left(\frac{z_{1j}^iq_{1j}^i}{iz_{2k}^i} - \frac{z_{1j}^i}{iz_{2k}^i} - \frac{z_{1j}^iq_{1j}^i}{iz_{2k}^iq_{2k}^i} + \frac{z_{1j}^i}{iq_{2k}^iz_{2k}^i}\right)\right)$$
$$= \frac{\left(1-\frac{z_{1j}}{z_{2k}}\right)\left(1-\frac{q_{1j}z_{1j}}{q_{2k}z_{2k}}\right)}{\left(1-\frac{q_{1j}z_{1j}}{z_{2k}}\right)\left(1-\frac{z_{1j}}{q_{2k}z_{2k}}\right)} = \frac{(z_{1j}-z_{2k})(q_{1j}z_{1j}-q_{2k}z_{2k})}{(q_{1j}z_{1j}-z_{2k})(z_{1j}-q_{2k}z_{2k})},$$

expanded as if $q_{1j}z_{1j} < z_{1j} < q_{2k}z_{2k} < z_{2k}$ for all $1 \le j \le r(1)$ and $1 \le k \le r(2)$. Applying similar reasoning to the other terms and using the Cauchy Determinant Identity yields that the above scalar product with respect to $a_{1,u}$ tends to

$$\left(\frac{1}{2\pi i}\right)^{r(1)+r(2)} \oint\oint\cdots\oint \det \mathbf{M}_2 F(x_1; y_1)F(x_2, y_2)F(x_1; y_2)F(x_1, x_2; b_2)\times$$
$$H_{q_{2j}}\left(b_2; \tfrac{1}{q_1z_1}, \tfrac{1}{q_2z_2}\right)\prod_{l=1}^{2}\prod_{l'=1}^{l}\left(H_{q_{lj}}(x_{l'}; z_l)H_{q_{lj}}\left(y_l; \frac{1}{q_{l'}z_{l'}}\right)\right)\prod_{j=1}^{r(1)} q_{1j}dz_{1j}\prod_{j=1}^{r(2)} q_{2j}z_{2j},$$

where z_k denotes the set of variables $z_{k1}, z_{k2}, \cdots, z_{kr(k)}$; $1/q_kz_k$ denotes the set of variables $1/q_{k1}z_{k1}, 1/q_{k2}z_{k2}, \cdots, 1/q_{kr(k)}z_{kr(k)}$; and $\mathbf{M}_2$ denotes the matrix whose rows and columns are indexed by pairs (j,k) such that $j \in \{1,2\}$ and $1 \le k \le r(j)$ such that the entry in the (j_1,k_1) row and (j_2,k_2) column is $1/(q_{j_1k_1}z_{j_1k_1} - z_{j_2k_2})$.

By induction, if we let $\mathbf{M}_n$ be the matrix whose rows and columns are indexed by pairs of integers (j,k) such that $1 \le j \le n$ and $1 \le k \le r(j)$ such that the element in the (j_1,k_1) row and (j_2,k_2) column is $1/(q_{j_1k_1}z_{j_1k_1} - z_{j_2k_2})$, then, as u tends to ∞, the term with $2n-2$ nested scalar products divided by q^{v_u} tends to

$$\prod_{1\le i\le j\le n} F(x_i; y_j)\left(\frac{1}{2\pi i}\right)^m \oint\oint\cdots\oint \det \mathbf{M}_n \prod_{l=1}^{n}\prod_{l'=1}^{l}\left(\frac{F(x_l, q_{l'}z_{l'})}{F(x_l, z_{l'})}\right)\times$$
$$\prod_{l=1}^{n}\prod_{l'=l}^{n}\left(\frac{F\left(y_l, \frac{1}{z_{l'}}\right)}{F\left(y_l, \frac{1}{q_{l'}z_{l'}}\right)}\right)\prod_{k=1}^{n}\prod_{j=1}^{r(k)} dz_{kj},$$

where the contour represents the sum of residues in a region containing each of the x_{ij}, each of the y_{ij}, not containing any of the $1/q_{ij}x_{lk}$, and not containing any of the $1/q_{ij}y_{lk}$; where

$\det \mathbf{M}_n$ is expanded as if all $q_{ij} < 1$, $q_{ij}z_{ij} > z_{ik}$ for all $j < k$, and $z_{ij} > z_{ij}q_{ij} > z_{kl} > q_{kl}z_{kl}$ for all $k < i$; and where F and H are expanded as exponentials.

Now, the above contour integral is equal to $\prod_{1 \le i \le j \le n} F(x_i, y_j) \sum_{T \in \mathbb{Z}^m} \rho_{\mathbf{S}}(T) q^{-T}$ added to a multiple of $\prod_{i=1}^{r(1)} q_i^v \prod_{k=2}^{n} \prod_{i=1}^{r(k)} q_i^{v+u}$ as a power series in the q_{ij}. Then, $\rho_{\mathbf{S}}(T)$ is the coefficient of q^{-T} in the above contour integral since $v > \max_{t \in T} t$. Letting $w_{ij} = q_{ij}z_{ij}$ for each $1 \le i \le n$ and $1 \le j \le r(i)$, applying the Residue Theorem, and expanding the determinant as a sum, we obtain that $\rho_{\mathbf{S}}(S)$ is equal to

$$\left(\frac{1}{4\pi^2}\right)^m \sum_{h \in S_m} \operatorname{sgn}(h) \oint \oint \cdots \oint \prod_{i=1}^{n} \prod_{j=1}^{r(i)} \left(\frac{1}{z_{ij} - w_{h(ij)}}\right) \times$$
$$\prod_{k=1}^{n} \prod_{j=1}^{r(k)} \prod_{i=k}^{n} \left(\frac{F(x_i, w_{kj})}{F(x_i, z_{kj})}\right) \prod_{k=1}^{n} \prod_{j=1}^{r(k)} \prod_{i=1}^{k} \left(\frac{F\left(y_i, \frac{1}{z_{kj}}\right)}{F\left(y_i, \frac{1}{w_{kj}}\right)}\right) \prod_{k=1}^{n} \prod_{j=1}^{r(k)} \left(\frac{w_{kj}^{s_{kj}} dw_{kj} dz_{kj}}{z_{kj}^{s_{kj}+1}}\right)$$

where h acts by permuting the pairs of integers (i, j), for $1 \le i \le n$ and $1 \le j \le r(i)$, and the contours are positively oriented circles of different radii such that the contours contain each of the x_{ij} and y_{ij} and do not contain the $1/x_{ij}$ and $1/y_{ij}$. The rational functions are expanded so that each $q_{ij} > 1$, $q_{ij}z_{ij} > z_{ik}$ for all $k < j$, and $w_{i_1 i_2} < z_{i_1 i_2} < w_{j_1 j_2} < z_{j_1 j_2}$ if $i_1 < j_1$ for all i_2 and j_2. Then, as in the proof of Theorem 1.1.1, letting $w_{h(ij)}$ become w_{ij} yields the determinantal identity for the correlation functions of the Schur process, which gives the first result claimed by the theorem. □

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