Curve Shortening Flow on n-Loop Curves

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Abstract

In this paper, we explore the properties of n-loop curves under Curve Shortening Flow. In differential geometry, n-loop curves, which are essentially generalized figure-8 curves, are of particular interest because of their dramatically different properties under Curve Shortening Flow from those of simple closed curves. We study specific families of n-loop curves and their parameter spaces. In particular, we divide the parameter spaces of these families into categories based on order of lobe disappearance. Through our work, we expand on recent results regarding the parameter space of a particular family of 3-loop curves. We additionally run numerical studies to approximate the parameter space of a family of 5-loop curves. From the results of our studies, we formulate a conjecture regarding the parameter space of n-loop curves for odd n. Additionally, we identify and prove several properties of the parameter space of 5-loop curves. For both the cases of $n = 3$ and $n = 5$, our results serve to partially generalize the Avoidance Principle beyond simple closed curves. From here, we make progress towards proving the n-loop conjecture for the case of $n = 5$.

Summary

We explore properties of n-loop curves, or generalized figure-8 curves, under Curve Shortening Flow, an operation which shrinks a curve so that each point moves at a speed proportional to curvature. Curve Shortening Flow is fairly well-understood for smooth non-intersecting curves, but little is known about how it acts on curves which intersect themselves. The behavior of Curve Shortening Flow on n-loop curves is thus of particular interest to mathematicians.

1 Introduction

Study of Curve Shortening Flow began in the late 1950s as a tool in materials science [1]. In particular, it was used to model the evolution of grain boundaries in polycrystalline material. Since then, however, Curve Shortening Flow has emerged within the broader field of differential geometry as one of the simplest known geometric flows.

Curve Shortening Flow refers to a form of flow which acts to shrink a curve so that each point moves at a speed proportional to its curvature. For simple closed curves, this definition can be taken to state, equivalently, that Curve Shortening Flow shifts a curve so that its perimeter decreases as quickly as possible with respect to its area. Curve Shortening Flow as it acts on these curves is fairly well-understood. The Gage-Hamilton-Grayson Theorem, shown by Michael Gage, Richard Hamilton [2], and Matthew Grayson [3] in the late 1980s, states that under Curve Shortening Flow, a simple closed curve asymptotically approaches a circle as it collapses to a point. Figure 1 shows how a curve might transform in shape as it shrinks.

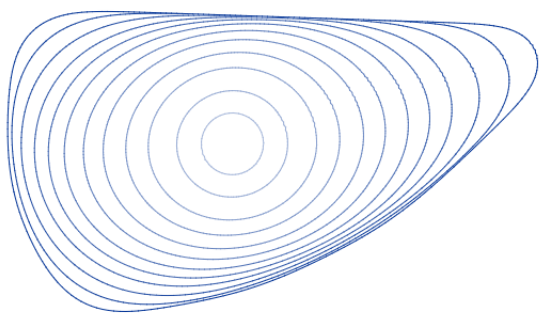

Figure 1: A representation of how CSF shrinks a curve (Eppstein, 2015) [4].

While significantly less is known about self-intersecting curves in general, mathematicians do possess a reasonably well-developed understanding of curves with one self-intersection, known as figure-8 curves: in 1989, Grayson showed that when Curve Shortening Flow acts on figure-8 curves with lobes of unequal area, the smaller lobe collapses to a point [5]. He additionally demonstrated that on the other hand, a figure-8 curve with lobes of equal area would shrink down to a point under Curve Shortening Flow, preserving the intersection. Roughly thirty years later, in 2021, mathematicians Matei Coiculescu and Richard Schwartz developed on this idea for symmetric figure-8 curves, or 2-loop curves [6]. They showed that much like how simple closed curves converge to circles, under Curve Shortening Flow, such curves converge to bowtie shapes. Figure 2 shows a figure-8 curve which, when anisotropically re-scaled, is beginning to approach the shape of a bowtie.

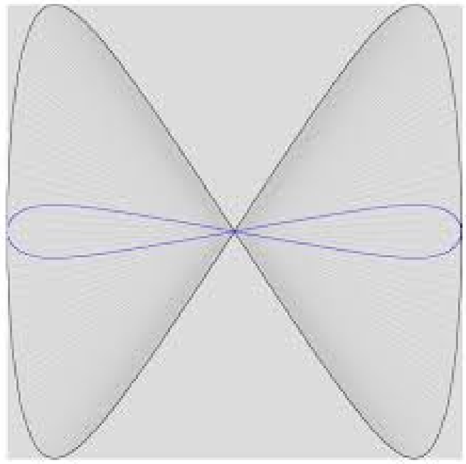

Figure 2: A figure-8 curve approaching a bowtie shape as it shrinks under Curve Shortening Flow (Coiculescu, Schwartz, 2021) [5].

Mathematicians have only recently begun to explore in-depth properties of more complex intersecting curves. In March of 2024, Sigurd Angenent and his team formulated two conjectures about n-loop curves, each of which attempted to generalize results regarding figure-8 curves. Their n-loop conjecture, which they proved for $n = 3$ and $n = 4$, states that for every integer $n \geq 2$, there exists an n-loop such that under Curve Shortening Flow, intersections are preserved. Angenent et al. [7] additionally conjectured that an n-loop curve, re-scaled appropriately, would converge to a squeezed bowtie shape. To support their second conjecture, which would generalize the Coiculescu-Schwartz Theorem, they performed numerical studies for $n = 3$ and $n = 4$.

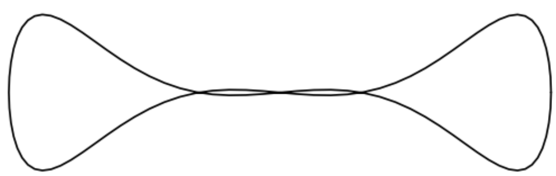

Figure 3: A 4-loop curve [7]

This paper observes and proves the existence of properties of n-loop curves and their behavior under Curve Shortening Flow. We develop on Angenent et al.'s results by showing uniqueness within a particular family for curves satisfying the 3-loop conjecture. We also conduct numerical studies for $n = 5$ in order to determine whether Angenent et al.'s conjectures are most likely true for larger n. Guided by these numericals, we prove additional statements on the behaviors of 5-loop curves. In particular, we show the existence of open and continuous regions corresponding to different orders of lobe disappearance.

2 Preliminaries

We define the curvature κ of a curve at a point to be the constant which satisfies the equation $\frac{dT}{ds} = \kappa N$, where T refers to the unit tangent vector at the point, s refers to the arc length of the curve, and N refers to the normal vector at the point [8]. Curvature is crucial in defining Curve Shortening Flow, which is governed by the equation $\frac{\partial X}{\partial t} = -\kappa N$ for any curve X at time t. In this paper, we assume for the sake of convenience that all curves are smooth.

2.1 Immersed, Embedded, and Simple Closed Curves

We consider a curve parametrized by u to be *immersed* if satisfies the immersion condition. The immersion condition states that $|\frac{dX}{du}| \neq 0$ for every u in I, the image set of X [8].

An immersed curve can self-intersecting or not self-intersecting. In this paper, the term will primarily refer to self-intersecting curves. We instead describe curves which are not self-intersecting as *embedded*. A curve X is *embedded* if it may be represented as an injective map which takes a closed set to a closed set [8].

One special case of an embedded curve is a *simple closed curve*. A closed curve refers to a curve which is periodic with respect to X, or a curve for which there exists some $A > 0$ such that $X(x + A) = X(x)$ for all $x \in I$. Such a curve is embedded if $X(x) = X(y)$ implies that $y - x = \ell A$ for some integer ℓ. We refer to an embedded closed curve as a *simple closed curve* [8].

2.2 Properties of Curve Shortening Flow

A few properties are intrinsic to Curve Shortening Flow and the way in which it acts on curves.

A general equation for Curve Shortening Flow is given by the equation

$$\frac{\partial X}{\partial t} = -\kappa N.$$

However, we also consider a curve $X(x, t)$ to be a solution to Curve Shortening Flow if it satisfies a reparametrized version of the Curve Shortening Flow equation, as below:

$$\frac{\partial X}{\partial t} = -\kappa N + \phi \frac{\partial X}{\partial t}$$

for some function ϕ [9]. For some solution to Curve Shortening Flow $X(x, t) = (x, y(x, t))$, this equation is equivalent to the following:

$$y_t = \frac{y_{xx}}{1 + (y_x)^2}$$

where we use y_x to denote $\frac{\partial y}{\partial x}$.

Curve Shortening Flow follows short-time existence [9]: that is, given an initial curve $X_0 : [a, b] \to \mathbb{R}^2$, there exists a solution to Curve Shortening Flow $X : [a, b] \times [0, \epsilon) \to \mathbb{R}^2$ for some $\epsilon > 0$ such that $X(\cdot, 0) = X_0$. This solution to Curve Shortening Flow is unique.

Curves under Curve Shortening Flow also follow the avoidance principle [9]. Given two solutions to Curve Shortening Flow $X : [a, b] \times [0, T) \to \mathbb{R}^2$ and $Y : [a, b] \times [0, T) \to \mathbb{R}^2$, the avoidance principle states that if the graphs of $X(\cdot, 0)$ and $Y(\cdot, 0)$ do not meet at any point, then $X(\cdot, t)$ and $Y(\cdot, t)$ will also not meet at any point for any $t \in [0, T)$. The avoidance principle provides a proof that a closed immersed curve under Curve Shortening Flow reaches singularity in finite time. It is trivial to show that a circle with finite dimensions reaches singularity by collapsing to a point in finite time. Then, by enclosing a closed immersed curve in a larger circle, we can easily show that such a curve must reach singularity in finite time.

Finally, it is known that when Curve Shortening Flow is applied to initial embedded curves, those curves remain embedded as they shrink.

2.3 Curve Shortening Flow on Simple Closed Curves

Given a simple closed solution to Curve Shortening Flow X, let us denote the area enclosed within the curve as $A(X)$. Then, the following holds:

$$\frac{dA}{dt} = -2\pi.$$

We then see that under Curve Shortening Flow, the area enclosed within a simple closed curve decreases at a constant rate. It is also known that with respect to area, the arc length s of a simple closed curve decreases as quickly as possible.

As a result of the following theorem by Gage and Hamilton [2] and Grayson [3], the properties of simple closed curves acted on by Curve Shortening Flow are understood to be more or less fully known.

Theorem 2.1 (Gage-Hamilton [2] and Grayson [3])**.** *Under Curve Shortening Flow, a simple closed curve becomes asymptotically circular as it collapses to a point.*

2.4 Curve Shortening Flow on Figure-8 Curves

The properties of a figure-8 curve acted on by Curve Shortening Flow are also more or less fully understood.

The following theorem by Grayson [5] characterizes the differences in behavior of figure-8 curves with lobes of unequal area and figure-8 curves with lobes of equal area under Curve Shortening Flow.

Theorem 2.2 (Grayson [5])**.** *Under Curve Shortening Flow, a figure-8 curve with lobes of unequal area reaches singularity when the area enclosed by the smaller lobe shrinks to zero.*

Similarly, the following theorem by Coiculescu and Schwarz [6] characterizes the behavior of figure-8 curves symmetric about the x- and y-axis under Curve Shortening Flow.

Theorem 2.3 (Coiculescu-Schwarz [6])**.** *Under Curve Shortening Flow, a symmetric figure-8 curve, scaled so that the bounding box is equal to a square, approaches a 'bowtie,' given by:*

$$(x, y) : |x| \leq 1, y = \pm x \cup (\pm 1, y) : |y| \leq 1.$$

2.5 Generalizations of Theorems on Figure-8 Curves

We define an *n-loop* as a closed, immersed curve symmetric about the x- and y-axes which can be expressed as the union of a function and its negative $f(x) \cup -f(x)$. A symmetric figure-8 curve is then considered a 2-loop.

The rate of area decrease of an n-loop curve is not constant. However, it is known that if we denote by $\alpha(X, p, t)$ the angle between the tangent vectors to point p on the curve X at time t, then the change in area of one outermost lobe on a curve X at time t is $-2\pi + \alpha(X, p_1, t)$, where p_1 is the self-intersection contained in the lobe. The change in area of a lobe which is not outermost is similarly $-2\pi + \alpha(X, p_2, t) + \alpha(X, p_3, t)$ where p_2 and p_3 are the self-intersections contained in the lobe.

We also define the *bounding box* of an n-loop to be the smallest rectangle with sides parallel to the axes which the loop may be contained inside.

Angenent et al. [7] formulated the following conjecture and proved it for the cases of $n = 3$ and $n = 4$.

Conjecture 2.4 (Angenent et al. [7])**.** *For every integer $n \geq 2$, there exists an n-loop curve such that intersections are preserved under Curve Shortening Flow.*

Angenent et al. also generalized the Coiculescu-Schwarz Theorem on symmetric figure-8 curves with the following conjecture, which they ran numerical programs to support for the cases of $n = 3$ and $n = 4$ [7].

Conjecture 2.5 (Angenent et al., 2024)**.** *Under Curve Shortening Flow, an n-loop curve, scaled so that the bounding box is equal to a square, approaches a 'squeezed bowtie,' given by:*

$$\{(x, y) : |x| \leq 1, y = \pm x^{n-1}\} \cup \{(\pm 1, y) : |y| \leq 1\}.$$

3 Curve Shortening Flow on 3-Loop Curves

In their paper on the singularity shapes of n-loop curves, Angenent et al. [7] proved the n-loop conjecture (2.4) for the cases of $n = 3$ and $n = 4$. In order to do so, they limited their study to the family of 3-loops generated by the equation

$$y^2 = (1 - x^2)(x^2 - a^2)^2, \tag{1}$$

where $0 < a < 1$.They then showed that a 3-loop curve for which intersections were preserved existed in this family. In this section, we will show that the value of a which preserves intersections is unique.

Lemma 3.1. *Let $r(X, t)$ denote the rightmost point of a closed curve X. We consider the first time t_1 such that two solutions to Curve Shortening Flow $X_1(s, t)$ and $X_2(s, t)$ meet at a point $p \neq r(X_1, t_1) \neq r(X_2, t_1)$. Then, given that $X_1(s, 0)$ and $X_2(s, 0)$ are generated by inputting a_1 and a_2 respectively into* (1) *such that $a_1 < a_2$, the inequality $r(X_2, t) \leq r(X_1, t)$ holds for all $t < t_0$.*

Proof. For the sake of contradiction, let us assume otherwise. Then, let us consider the first time t at which the $r(X_2, t) = r(X_1, t)$ and directly after, at time $t + \varepsilon$, $r(X_2, t + \varepsilon) > r(X_1, t + \varepsilon)$. We note that this implies that $r(X_1, t)$ has higher velocity than $r(X_2, t)$. However, the curvature κ of X_2 at point $r(X_2, t)$ is higher than the curvature κ of X_1 at point $r(X_1, t)$, so this is a contradiction. □

This lemma then allows us to show the following proposition, which describes the behavior of the center lobe of a curve under Curve Shortening Flow as we increase the parameter a:

Proposition 3.2. *Given two initial curves X_1 and X_2 generated by a_1 and a_2 respectively such that $a_1 < a_2$, under Curve Shortening Flow, the center lobe in the curve X_1 will not ever touch the center lobe in the curve X_2.*

Proof. Let us consider the graph $y(x, a) = (\sqrt{1 - x^2})(x^2 - a^2)$ for $x \in [-1, 1]$ and $a \in (0, 1)$. Then we see that

$$\begin{aligned} X_1 &= -y(x, a_1) \cup y(x, a_1), \\ X_2 &= -y(x, a_2) \cup y(x, a_2). \end{aligned} \tag{2}$$

We also see that

$$y(x, a_2) < y(x, a_1) < -y(x, a_1) < -y(x, a_2)$$

holds for all $x \in [-a_1, a_1]$, so the center lobe of X_1 is contained within that of X_2.

We will now show that under Curve Shortening Flow, the center lobe of X_1 will at no point meet the center lobe of X_2. The proof of the Avoidance Principle [**?**] for simple closed curves motivates us to consider, for the sake of contradiction, the first time t_1 at which the lobes meet. Say that the two lobes meet at some point u not on the x-axis. Then, the two lobes must be tangent at point u and time t_1, so $\kappa(X_2, u, t_1) < \kappa(X_1, u, t_1)$. However, in order for the two lobes to meet at time t_1, the normal speed of X_1 at point u and time $t_1 + \varepsilon$ must be greater than the normal speed of X_2, so $\kappa(X_1, u, t_1 + \varepsilon) < \kappa(X_2, u, t_1 + \varepsilon)$ must hold for every $\varepsilon > 0$.

The normal speed of a curve $-\kappa(u, t)$ under Curve Shortening Flow is a continuous function, so this is a contradiction. It is thus sufficient to show that the first point at which X_1 and X_2 meet cannot be on the x-axis.

Let us assume for the sake of contradiction that X_1 and X_2 meet at time t_2 and some point m on the x-axis. We see that in order for X_1 and X_2 to meet at point m and time t_2, it must be the case that $\alpha(X_1, m, t_2) < \alpha(X_2, m, t_2)$. However, by Lemma 3.1, we see that for this to occur, the outer lobe of X_1 must first meet the outer lobe of X_2 at some point not on the x-axis. At the first point at which it does so, the two lobes will be tangent, so by the same argument as above, we are done. □

It is natural to attempt to extend this result to outer lobes. We find that we can similarly show the following proposition:

Proposition 3.3. *Given two initial curves X_1 and X_2 generated by a_1 and a_2 respectively such that $a_1 < a_2$, the outer lobes in the curve X_2 will not ever touch the outer lobes in the curve X_1 at any points other than their original tangency points.*

Proof. It is simple to show that the outer lobes of X_2 are *nearly* contained within the outer lobes of X_1: that is, the rightmost outer lobe of X_2 is tangent to the rightmost outer lobe of X_1 at $(1,0)$ but otherwise contained within it. This must be the case because similarly considering the graph $y(x,a) = (\sqrt{1-x^2})(x^2-a^2)$, we see that

$$-y(x,a_1) < -y(x,a_2) < y(x,a_2) < y(x,a_1)$$

holds for all $x \in (-1,-a_2] \cup [a_2,1]$.

Without loss of generality, we will consider the right outer lobe of each curve. We claim that the right outer lobe of X_2 can never meet the right outer lobe of X_1 at a point not on the x-axis. We note that if this statement is not true, then the right outer lobe of X_2 must at some point become tangent to the right outer lobe of X_1, and an identical argument to that of the Avoidance Principle, which is outlined above, applies.

Then, it is sufficient to show that the right outer lobes of X_1 and X_2 will never meet at a point on the x-axis other than their rightmost points. However, this is certainly true by Proposition 3.2, since if they meet at such a point, then the center lobes will also meet. □

Given these propositions, we may now prove the following theorem:

Theorem 3.4. *There exists a unique a such that as Curve Shortening Flow is applied to the equation $y^2 = (1-x^2)(x^2-a^2)^2$, intersections are preserved.*

Proof. We will assume for the sake of contradiction that there exist two values a_1 and a_2 such that Curve Shortening Flow preserves intersections for the associated curve of each variable. We will also assume without loss of generality that $a_1 < a_2$. Let us refer to the initial curves associated with a_1 and a_2 as $X_1(0)$ and $X_2(0)$ respectively.

We consider the time t_1 at which $X_1(t)$ reaches singularity and the time t_2 at which $X_2(t)$ reaches singularity. Then we see that both the center and outer lobes of X_1 disappear at time t_1. Similarly, both the center and outer lobes of X_2 disappear at time t_2. It then follows directly from Proposition 3.2 that $t_1 \leq t_2$. However, it also follows from Proposition 3.3 that $t_1 \geq t_2$, so we see that the time of singularity for both curves is $T = t_1 = t_2$.

Let us then consider some times t and s such that $0 \leq s \leq t < T$. We note that if we refer to the area of the center lobe of a curve X at time t as $A_c(X,t)$ and the area of the outer lobes of the curve as $A_o(X,t)$. Let us consider the points of intersection on X_1 at time t, $p_{11}(t)$ and $p_{12}(t)$. Then we may also consider $\alpha_1(t) = \alpha(X_1, p_{11}, t) = \alpha(X_1, p_{12}, t)$. We analogously define $\alpha_2(t)$ for X_2.

Then it must be the case that $A_c'(X_1,t) = -2\pi + 2\alpha_1(t)$ and $A_o'(X_1,t) = -2\pi + \alpha_1(t)$. We thus see that $A_c'(X_1,t) = 2A_o'(X_1,t) + 2\pi$, so the following is true:

$$\begin{aligned} A_c(X_1,t) - A_c(X_1,s) &= \int_s^t A_c'(X_1,t)\,dx \\ &= 2\int_s^t (A_o'(X_1,t) + \pi)\,dx \\ &= 2(A_o(X_1,t) - A_o(X_1,s) + \pi(t-s)). \end{aligned}$$

We note that taking the limit as $t \nearrow T$ gives

$$-A_c(X_1,s) = 2(-A_o(X_1,s) + \pi(T-s))$$

for $s \in [0, T)$. It similarly must be true that

$$-A_c(X_2, s) = 2(-A_o(X_2, s) + \pi(T - s))$$

for $s \in [0, T)$. However, it follows directly from Proposition 3.2 that $A_c(X_1, s) < A_c(X_2, s)$, and it similarly follows from Proposition 3.3 that $A_o(X_2, s) < A_o(X_1, s)$. Then $-A_c(X_2, s) < -A_c(X_1, s)$, but $2(-A_o(X_1, s)+\pi(T-s)) < 2(-A_o(X_2, s)+\pi(T-s))$. This is a contradiction, so we are done. □

We have therefore shown that the value of a which corresponds to a curve such that intersections are preserved under Curve Shortening Flow is unique.

4 Parameter Space of n-Loop Curves

We are motivated by the proofs of the 3-loop and 4-loop conjectures to consider in particular the family of curves given by

$$y^2 = (1 - x^2)(x^2 - a^2)^2(x^2 - b^2)^2, \tag{3}$$

where $0 < a < b < 1$. In this section, we will conduct numerical studies in order to identify and formally conjecture likely properties of the parameter space of such curves.

4.1 Numerical Studies on the Parameter Space of 5-Loop Curves

Using methods outlined in Angenent et al.'s [7] paper on singularities in Curve Shortening Flow, we construct a program simulating Curve Shortening Flow for any given curve. We then build off of the program, allowing it to identify the points at which each curve intersects itself at each point in time, as well as which lobes are the first to have their area shrink to zero.

We run our program for a range of a and b satisfying $0 < a < b < 1$ in order to approximate the shapes of regions within the parameter space of 5-loop curves. In particular, we examine three regions: that in which the *center lobe*, or the lobe containing the origin, is the first to disappear, that in which the *inner lobes*, adjacent to the center lobe, are the first to disappear, and that in which the *outer lobes*, adjacent to the inner lobes but not the center lobe, are the first to disappear. These regions are shown in Figure 4.

There several are notable properties of the parameter space of the 5-loop curve, many of which we will prove in this paper and others which we will formally conjecture.

It is mathematically interesting that the parameter space of 5-loop curves generated by this family of curves appears to be divided into three continuous regions. Additional numerical simulations show us that it is likely the case that those points at which two different regions appear to meet correspond with curves such that under Curve Shortening Flow, two different sets of lobes vanish simultaneously and before the third set. Similarly, numerical simulations suggest that the point at which all three regions appear to meet corresponds with the curve such that all lobes disappear simultaneously. Equivalently, it is the point at which all intersections are preserved under Curve Shortening Flow.

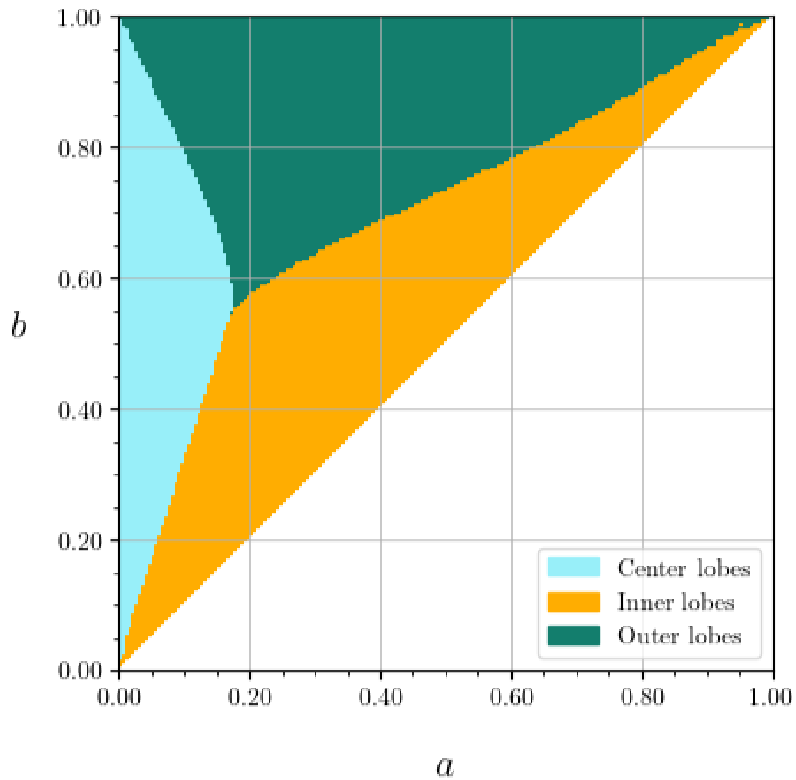

Figure 4: The parameter space of the 5-loop curve, generated by a Python program. Here, the teal region represents the parameters for which the center lobe is the first to disappear, the yellow region represents the parameters for which the inner lobes are the first to disappear, and the green region represents the parameters for which the outer lobes are the first to disappear.

4.2 Generalizing to n-Loop Curves

We note that the parameter space of 5-loop curves generated by (3) is composed of regions of respective co-dimensions of zero, one, and two, in which one, two, and three sets of lobes are simultaneously the first to disappear. It is then natural to extend this formulation to n-loop curves. In particular, we consider the family of $(2k+1)$-loop curves given by the equation

$$y^2 = (1-x^2)\prod_{m=1}^{k}(x^2-\lambda_m^2)^2 \tag{4}$$

where $0 < \lambda_1 < \lambda_2 \cdots \lambda_k < 1$.

Then it is also natural to formulate a conjecture extending our description of the parameter space of 5-loop curves, as below. For the sake of our conjecture, we will define a *k-simplex* in the following way. For some $A \subset \mathbb{R}^k$, a *q-simplex* is the image of a continuous embedding $f : T^q \to A$ where T^q is equal to

$$\{x \in \mathbb{R}^q \mid x_1 + x_2 + \cdots x_q < 1,\ x_i > 0\},$$

or equivalently, the interior of the standard q-simplex. Similarly, we define an *l-face* of a q-simplex in A as the image under f of an l-face in T^q. Additionally, we will refer to the center lobe as the 1st lobe, the lobes directly bordering those as the 2nd lobes, and generally to the set of lobes w lobes away from the center lobe as $w-1$th lobes. We then have the following:

Conjecture 4.1. *Given any $(2k+1)$-loop generated by (4), we consider the parameter space*

$$P = \{(a_1, ..., a_k) \in \mathbb{R}^k \mid 0 < a_1 < \cdots < a_k < 1\}.$$

Then the following is true:

(a) P allows a triangulation by open k-simplices $\Omega_1, \Omega_2, \cdots \Omega_{k+1}$ such that if some point $(\lambda_1, \lambda_2, \cdots \lambda_{k+1}) \in \Omega_i$, then in the curve corresponding with those parameters, the ith set of lobes is the first to disappear under Curve Shortening Flow. Further, each Ω_i has one face coincide with a distinct face P_i of P.

(b) Then

$$\partial\,\Omega_{i_1} \cap \partial\,\Omega_{i_2} \cap \cdots \cap \partial\,\Omega_{i_l}$$

is a $(k-l+1)$-simplex which is a common $(l-1)$-side of each Ω_i and for all

$$(a_1, a_2, \cdots, a_k) \in \partial\,\Omega_{i_1} \cap \partial\,\Omega_{i_2} \cap \cdots \cap \partial\,\Omega_{i_l},$$

in the curve corresponding with those parameters, the i_1th, i_2th, $\cdots i_k$th lobes vanish first and simultaneously.

(c) In particular,

$$\partial\,\Omega_1 \cap \partial\,\Omega_2 \cap \cdots \cap \partial\,\Omega_k$$

is nonempty and represents the parameters for which all lobes vanish simultaneously, or equivalently, intersections are preserved up to the point of singularity.

(d) *Moreover,*

$$\partial\Omega_1 \cap \partial\Omega_2 \cap \cdots \cap \partial\Omega_k$$

consists of exactly one point.

5 Behavior of 5-Loop Curve Under Curve Shortening Flow

The behavior of n-loop curves under Curve Shortening Flow is highly related to the relative sizes of the lobes. The following lemmas expand on and generalize Grayson's [5] theorem for figure-8 curves comparing the behavior of curves with lobes of unequal area and curves with lobes of equal area [5]. Corollaries from these lemmas allow us to prove some properties of Curve Shortening Flow on 5-loop curves that we can see from our numerical code.

Lemma 5.1. *Given a 5-loop in the family of curves generated by the equation* (3) *where* $0 < a \leq b < 1$, *there exists for every positive* b *a positive value* λ_1 *such that if* $a < \lambda_1$, *then the center lobe is the first to disappear. Likewise, there exists for every positive* a *a positive value* λ_2 *such that if* $b > \lambda_2$, *then the outer lobes are the first to disappear.*

Proof. We note that for any given value of b, we may consider the curve for which $a = 0$. Then the resulting curve is not a 5-loop, but can essentially be treated as one. It is natural to consider the first times at which the outer lobes, the inner lobes, and the center lobe reach an area of zero. We note that these times can respectively be denoted as $t_1(a)$ where $t_1(0) > 0$, $t_2(t)$ where $t_2(0) > 0$, and $t_3(t)$ where $t_3(0) = 0$.

Let us then consider the following values of a:

$$a_1 = \inf\{a_1 > 0 \mid t_1(a_1) = t_3(a_1)\}$$

and

$$a_2 = \inf\{a_2 > 0 \mid t_2(a_2) = t_3(a_2)\}.$$

We note that a curve is continuously dependent on its initial data. Then, if a_1 does not exist, the center lobe always disappears before the outer lobes. The same is true of a_2 and the inner lobes. We may then assume that a_1 and a_2 exist.

We also see that a_1, $a_2 \neq 0$ because $t_1(0)$, $t_2(0) > t_3(0)$. Then, it must be the case that a_1, $a_2 > 0$.

We can prove by contradiction that the center lobe disappears before the outer lobes at every time $t < a_1$. Let us assume for the sake of contradiction that this is not the case; then there exists a time $t < T_1$ such that $t_1(a) > t_3(a)$. However, t_1 and t_3 are continuous, so by the Intermediate Value Theorem, there exists a value a for which $t_1 = t_3$. This is a contradiction because a_1 is the first value at which $t_1 = t_3$. Then, it is true that the center lobe disappears before the outer lobes at every value $a < a_1$. We can similarly prove that the center lobe disappears before the inner lobes at every value $a < a_2$.

Then, it is clear that setting $\lambda_1 = \min(a_1, a_2)$ is sufficient.

The proof for the existence of λ_2 follows similarly. □

Corollary 5.1.1. *There exist open and continuous regions bounded by the lines* $x = 0$ *and* $y = 1$ *which correspond with the center and outer lobes respectively disappearing first.*

Lemma 5.2. *Given a 5-loop in the family of curves generated by the equation* (3) *where* $0 < a \leq b < 1$, *for all* $\varepsilon > 0$ *there exists a* $\varsigma > 0$ *such that if* $b - a < \varsigma$ *and* $a > \varepsilon, b < 1 - \varepsilon$, *then the inner lobes are the first to disappear.*

Proof. We note that for any given value of ε, we may consider the curves generated by setting $a = b$. Once again, it is natural to consider the first times at which the outer lobes, the inner lobes, and the center lobe reach an area of 0. Let us describe these times as $t_1(a, b)$ where $t_1(a, a) > 0$, $t_2(a, b)$ where $t_2(a, a) = 0$, and $t_3(a, b)$ where $t_3(a, a) > 0$.

Let us then consider the following values of ς:

$$\varsigma_1(a) = \inf\{\varsigma_1 > 0 \mid t_1(a, a + \varsigma_1) = t_2(a, a + \varsigma_1)\}$$

and

$$\varsigma_2(a) = \inf\{\varsigma_2 > 0 \mid t_3(a, a + \varsigma_2) = t_2(a, a + \varsigma_2)\}.$$

We note that by continuity of Curve Shortening Flow based on initial data, for every value of ε and every value of a, there exists a finite value of $\varsigma_1(a) > 0$ unless $t_1(a, a+\varsigma_1) < t_2(a, a+\varsigma_1)$ for all ς_1. If this is the case, we can take the value of ς_1 to be ∞. We then note a value of $\varsigma_1(a) > 0$ exists for every a on the range $(\varepsilon, 1-\varepsilon)$, so there must exist a value of $0 < \varsigma_1 < \varsigma_1(a)$.

We can see that there must similarly exist a value of ς_2 defined in the same way. Then it is sufficient to take $\varsigma = \min(\varsigma_1, \varsigma_2)$, so we are done. □

Corollary 5.2.1. *There exists an open and continuous region bounded by the line* $x = y$ *which corresponds with the inner lobes disappearing first.*

6 5-Loop Conjecture

Our numerical studies demonstrate that nearly all 5-loop curves which reach singularity when they collapse to a point become embedded before singularity. Interestingly, however, there does appear to exist a 5-loop curve such that all intersections are preserved up to the point of singularity as Curve Shortening Flow is applied. This statement is equivalent to the n-loop conjecture for the case of $n = 5$. In this section, we will make progress on proving for the 5-loop conjecture by extending statements which are true of 3-loop curves.

We examine particularly the family of 5-loops generated by the equation (3) for $0 < a \leq b < 1$.

The following proposition is similar to Proposition 3.2. However, there are several steps which differ, so we will include it for completeness.

Proposition 6.1. *Given two initial curves* X_1 *and* X_2 *generated by* (a_1, b_1) *and* (a_2, b_1) *respectively such that* $a_1 < a_2$, *the center lobe in the curve* X_1 *disappears in less time than the center lobe in the curve* X_2.

Proof. Let us consider the graph $y(x, a, b) = (\sqrt{1 - x^2})(x^2 - a^2)(x^2 - b)^2$. Then, we see that

$$X_1 = y(x, a_1, b_1) \cup -y(x, a_1, b_1)$$

and

$$X_2 = y(x, a_2, b_1) \cup -y(x, a_2, b_1)$$

.It is clearly the case that

$$0 < -(x^2 - a_1^2)(x^2 - b_1^2)\sqrt{1 - x^2} < -(x^2 - a_2^2)(x^2 - b_1^2)\sqrt{1 - x^2}$$

for all $x \in \{-a_1, a_1\}$. Then, in this range, it must be the case that

$$y(x, a_2, b_1) < y(x, a_1, b_1) < -y(x, a_1, b_1) < -y(x, a_2, b_1),$$

so the center lobe of X_1 is contained within the center lobe of X_2. We can similarly show that the inner lobes of X_2 are *nearly* contained within the inner lobes of X_1, meeting them at $-b_1$ and b_1 respectively.

We will now show by contradiction that the two lobes can never intersect each other. Let us assume for the sake of contradiction that the lobes do at some point intersect. The proof of the Avoidance Principle for simple closed curves motivates us to consider the unique time t at which they first meet. If we assume that the lobes first meet at a point not on the x-axis, it is simple to see that they become tangent to each other at the moment they meet. Let us denote their point of tangency as Q. Then the center lobe of X_1 must have higher curvature, and therefore higher speed, at pont Q and time t. However, in order for the center lobe of X_2 to at some point meet the center lobe of X_1, its speed at point Q prior to time t must have been higher than that of the inner lobe. The speed of Q on each lobe over time must be continuous, so this is a contradiction. Then it is sufficient to show that the two lobes cannot intersect each other for the first time on the x-axis.

Let us assume for the sake of contradiction that the inner lobe and outer lobe intersect each other for the first time on the x-axis. Then, let us again say that they first meet at point Q at time t. Let us denote by $\alpha(X, P, t)$ the angle between the tangent vectors to point P in the curve X at time t. Then we see that $\alpha(X_1, Q, t) > \alpha(X_2, Q, t)$. From here, it is clear that the inner lobe of X_1 which contains Q must meet the inner lobe of X_2 which contains Q at some point prior to t. At the point which they meet, this inner lobe of X_1 is necessarily tangent to this inner lobe of X_2. From here, we proceed as above. □

The proof above provides us with useful insight on how we might begin to generalize results regarding n-loops for $n = 3$ to larger n. Doing so is likely useful to proving the n-loop conjecture for $n = 5$ and perhaps for general n.

7 Numerical Computations

We used similar techniques to those outlined in Angenent et al.'s paper in order to simulate the path of an n-loop curve under Curve Shortening Flow. We then looked specifically at the family of 5-loop curves generated by (3) for $0 < a < b < 1$.

We used the values of a and b at which there seemed to be a *triple point* in our parameter space. We then ran the program until the values it was calculating grew too large for the simulation to be accurate.

We found that for the pair $(a, b) = (0.1720900, 0.5360760)$ the intersections were preserved as the curve shrunk under Curve Shortening Flow, and the graph appeared to approach the graph of $y = x^{n-1}$. Figure 5 shows this.

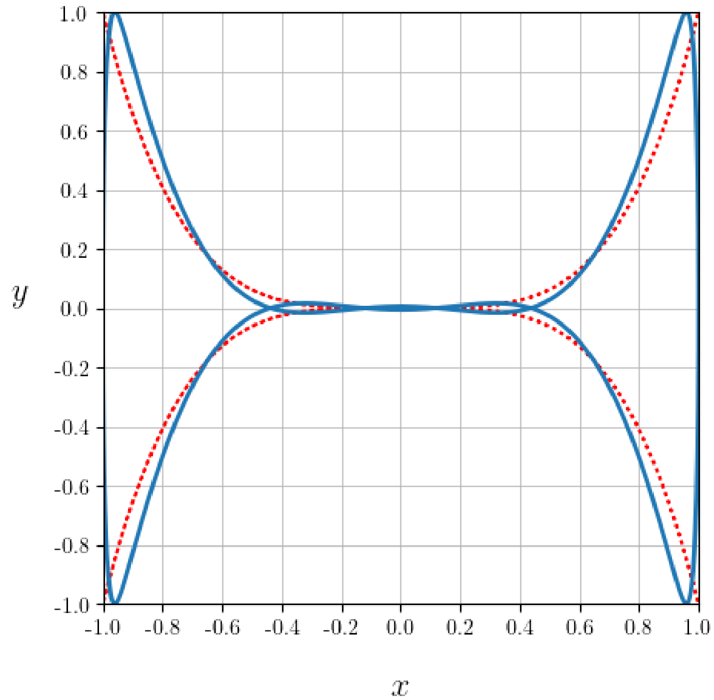

Figure 5: A 5-loop curve after near the point of singularity. It begins to approach the graph of $y = x^4$. The blue line represents the graph of the curve, while the red line represents the graph of $y = x^4$.

8 Conclusion and Future Directions

Our results on the behaviors of 3-loops within a certain family give us insight on how to show similar properties exist for n-loops with $n \geq 4$. In particular, Theorem 3.4 is useful in understanding the structure of the parameter spaces of families of n-loops, as well as the properties of curves such that intersections are preserved under Curve Shortening Flow.

Our results on the properties of 5-loop curves also make meaningful progress towards proving the n-loop conjecture for the case of $n = 5$. A key avenue we could consider in proving this conjecture is first proving the one-dimensionality of the regions in which two sets of lobes disappear first and simultaneously. A proof for this statement might be similar to the proof Theorem 3.4. Proving the one-dimensionality of these regions would allow us to consider whether they are open on their endpoints within the region in which no unique set of lobes is the first to disappear. From here, we could show the existence of a curve for which all lobes disappear simultaneously, or equivalently, for which intersections are preserved up to the point of singularity.

Applying these techniques to n-loops for $n > 5$ might also lead to a proof of Conjecture 4.1. In turn, characterizing the parameter spaces of n-loop curves in (4) would allow for a broader understanding of the way in which Curve Shortening Flow operates on n-loop curves.

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A Numerical Computations Code

```
#!/usr/bin/env python3
# -*- coding: utf-8 -*-
"""
Created on Tue Jul  9 22:34:00 2024

@author: ashleyzhu
"""


import matplotlib.pyplot as plt
from matplotlib.animation import FuncAnimation
import numpy as np
import scipy as sp

ax = plt.axes()

N = 600

j = 600

def t(j):
    return j/10000

x_coords = np.empty(N)
fun_x = [x_coords]

for i in range(N):
        x_coords[i] = np.sin(2*np.pi*(i+1)/N)


y_coords = np.empty(N)
fun_y = [y_coords]

for i in range(N):
    y_coords[i] = (np.sin(2*np.pi*(i+1)/N)**2-0.17**2)*(np.sin(2*np.pi*(i
   +1)/N)**2-0.53**2)*np.cos(2*np.pi*(i+1)/N)

for q in range(j):

    def x(i):
        if i<1:
            return x_coords[i+N-1]
        elif i<=N:
            return x_coords[i-1]
        else:
            return x_coords[i-1-N]
    def y(i):
        if i<1:
            return y_coords[i+N-1]
        elif i<=N:
            return y_coords[i-1]
```

```
        else:
            return y_coords[i-1-N]

    def g(i):
        return (x(i), y(i))

    def K(i):
        return 2*(t(q+1)-t(q))/(np.linalg.norm(np.subtract(g(i+1), g(i)))**2 + np.linalg.norm(np.subtract(g(i), g(i-1)))**2)

    alpha = np.zeros((N, N))
    for p in range(N):
        alpha[p, p] = 1 + 2 * K(p+1)
        if p > 0:
            alpha[p, p-1] = -K(p+1)
        if p < N-1:
            alpha[p, p+1] = -K(p+1)
    alpha[0, N-1] = -K(1)
    alpha[N-1, 0] = -K(N)

    print(np.max(alpha))

    beta_x = np.empty(N)
    beta_y = np.empty(N)
    for p in range(N):
        weird = g(p+1)
        beta_x[p] = weird[0]
        beta_y[p] = weird[1]

    x_coords = np.linalg.solve(alpha, beta_x)
    fun_x.append(x_coords)
    y_coords = np.linalg.solve(alpha, beta_y)
    fun_y.append(y_coords)

    plt.subplots()
    for i in range(N):
        if i>0:
            x1 = x_coords[i-1]
            y1 = y_coords[i-1]
        else:
            x1 = x_coords[N-1]
            y1 = y_coords[N-1]
        plt.plot(x1, y1, color="0", marker = ",")
    plt.show()
```