An Isoperimetric Problem for the First Neumann Eigenvalue on the Sphere

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Abstract

The eigenvalue problem for the Laplacian has gained a lot of attention from its application to physics such as in the wave equation and heat equation. While eigenvalues are hard to compute for general regions, they can be bounded by geometric quantities of the region. Bounding eigenvalues for volume constraint has already been extensively explored, both in the plane and on the sphere, as well as in higher dimensions. However, bounds of eigenvalues by geometric quantities other than volumes have not been deeply studied.

In this paper, we prove an upper bound on the first nonzero Neumann eigenvalue of the Laplacian in terms of width and area for domains on a sphere. We also pose a conjecture for an upper bound of the first nonzero Neumann eigenvalue under perimeter constraint for convex domains on a sphere. Our results are spherical analogs of results of Henrot, Lemenant, Lucardesi (2022) for domains on the plane.

Summary

Eigenvalues of regions are quantities that are vital to physics because they connect the geometry of a region to physical phenomena, such as frequency of vibration or heat distribution of the region. Neumann eigenvalues are a special type of these quantities that can be interpreted physically as the frequency of a freely vibrating region, such as ripples on the surface of water, as opposed to the vibration of a region with a fixed boundary, such as the membrane of a drum. While eigenvalues are hard to compute for general regions, they can be bounded by geometric quantities of the region. In our paper, we bound the smallest positive Neumann eigenvalue of regions on the sphere by the width and area of the region. We also state a conjecture on bounding the smallest positive Neumann eigenvalue of regions on the sphere by the perimeter of the region.

1 Introduction

The eigenvalue problem for the Laplacian over a region is important because of the connection between eigenvalues and the shape of the region. One of the most famous problems that explored this connection was, "Can you hear the shape of a drum?" which was popularized by Mark Kac [1]. The eigenvalue problem for the Laplacian is also important to physics as it appears in the heat equation and the wave equation. Thus, eigenvalues have many physical interpretations in terms of wavelength of vibration and heat distribution.

Due to the significance of eigenvalues, they have raised a variety of questions: examining the size of the gaps between consecutive eigenvalues [2], optimizing the upper bound on the harmonic mean of the eigenvalues [3], and optimizing the upper bound on the first eigenvalue given regions of a fixed volume [4, 5]. Most of these problems have been extensively studied for regions on the plane and surface of a sphere, as well as in higher dimensions.

In our paper, we look at a different problem: bounding the first nonzero Neumann eigenvalue by the width and area of a region, which was recently done in the case of planar regions in [6] by Henrot, Lemenant, and Lucardesi. We use methods from their paper to prove analogs of their results for regions on the surface of the sphere. First, in Section 2 we explain the Neumann eigenvalue problem and discuss the mathematical techniques behind our results. Then, in Section 3 we compute the first nonzero Neumann eigenvalue of some simple regions on the sphere which help provide a sanity check for our main result. Next, in Section 4 we prove our main result, which bounds the first nonzero Neumann eigenvalue by the width and area of regions on the surface of a sphere. Then, in Section 5 we state a conjecture that bounds the first nonzero Neumann eigenvalue of regions on the surface of a sphere under perimeter constraints. Finally, in Section 6 we present ideas for the future direction of our work.

2 Preliminaries

2.1 Neumann problem

The Neumann eigenvalue problem has two parts. The first is the eigenvalue problem for the Laplacian and the second is the Neumann boundary condition. The former involves solving the partial differential equation

$$-\Delta u = \mu u,$$

over the domain Ω for functions $u : \Omega \to \mathbb{R}$. Any real numbers μ that admit a solution u are called eigenvalues. Note that Δ is the Laplacian, that is

$$\Delta u = \sum_{i=1}^{n} \partial_i^2 u.$$

Meanwhile, the second part of the problem is the Neumann boundary condition [7], which requires that

$$\langle \nabla u, \nu \rangle = 0$$

on the boundary $\partial\Omega$ of Ω, where $\langle \cdot, \cdot \rangle$ is the dot product, ∇ is the gradient, and ν is the unit normal vector to the boundary of Ω. In other words, the normal derivative on the boundary $\partial\Omega$ is 0. There are numerous other boundary conditions for the eigenvalue problem, but in our paper we focus on the Neumann boundary conditions.

Just as the eigenvalue problem has several different boundary conditions, a given region has several different eigenvalues. In fact, there are an infinite number of eigenvalues, all of which are nonnegative real numbers. Note that zero is always a Neumann eigenvalue with constant eigenfunctions u, and thus is not interesting. In our paper we focus on smallest nonzero Neumann eigenvalue, which we denote by μ_1. We also refer to μ_1 as the "first Neumann eigenvalue."

2.2 Riemannian metrics

Since we are working on the sphere, a Riemannian manifold, we need a Riemannian metric to carry out our computations. For our purposes, the Riemannian metric can be thought of as a 2×2 matrix. There are a few common metrics we use that come from different coordinate systems.

The first coordinate system is geodesic polar coordinates, where

$$x = \sin r \cos \theta$$

$$y = \sin r \sin \theta$$

$$z = \cos r$$

for a unit sphere. The parameter r is the distance from the north pole and θ is the longitudinal angle.

Geodesic polar coordinates admit the Riemannian metric

$$g = \mathrm{d}r^2 + \sin^2 r \, \mathrm{d}\phi^2 = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 r \end{pmatrix}. \tag{1}$$

The second coordinate system is geodesic normal coordinates, where

$$x = \cos s \cos \phi$$

$$y = \cos s \sin \phi$$

$$z = \sin s$$

for a unit sphere. The parameter ϕ is the angle around the equator while s is the angle we deviate from the equator. While both coordinate systems have a longitudinal angle ϕ, geodesic polar coordinates measure latitude in terms of distance from the north pole, whereas geodesic normal coordinates measure latitude in terms of deviation from the equator.

Geodesic normal coordinates admit the Riemannian metric

$$g = \cos^2 s \, \mathrm{d}\phi^2 + \mathrm{d}s^2 = \begin{pmatrix} \cos^2 s & 0 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

Using these Riemannian metrics, we can compute various quantities we need, such as the length, volume form, gradient, and Laplacian. The formulas for these quantities are in Appendix A.

2.3 Rayleigh quotient

For the proof of our result Theorem 4.1 later on, we use a different form of the first Neumann eigenvalue μ_1:

$$\mu_1 = \inf_{u \in H^1(\Omega) \, : \, \int_\Omega u \, \mathrm{d}x = 0} \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}x}{\int_\Omega u^2 \, \mathrm{d}x}. \tag{3}$$

This expression comes from a generalized version of the Rayleigh quotient from linear algebra. In linear algebra, the Rayleigh quotient comes from taking the dot product of a vector with its image and then normalizing it by the magnitude of the vector v:

$$\frac{\langle Av, v \rangle}{\langle v, v \rangle}.$$

The maximal value of this quantity is the largest eigenvalue of A and is achieved if and only if v is in the direction of the corresponding eigenvector. However, if v is restricted to be orthogonal to the eigenvector of the largest eigenvalue, then the maximal value of the Rayleigh quotient is the second largest eigenvalue of A, which is achieved when v is in the direction of the corresponding eigenvector. If v is restricted to be orthogonal to the eigenvectors of the largest and second largest eigenvalues, then the maximal value of the Rayleigh quotient is the third largest eigenvalue, and so on. In this way one can find all the eigenvalues one by one.

In spectral geometry, the Rayleigh quotient comes from the "weak formulation" of our partial differential equation, which is obtained through integration by parts. This involves multiplying the PDE by a function and then integrating:

$$-\Delta u = \mu u \implies -\Delta u \cdot \phi = \mu u \cdot \phi \implies \int -\Delta u \cdot \phi = \int \mu u \cdot \phi.$$

Using integration by parts on the left side gives

$$\int_\Omega -\Delta u \cdot \phi = -\int_\Omega \mathrm{div}(\nabla u) \cdot \phi = \int_\Omega \nabla u \cdot \nabla \phi - \int_{\partial\Omega} \langle \nabla u, \nu \rangle u = \int_\Omega \nabla u \cdot \nabla \phi.$$

Notice that the integral over the boundary $\partial\Omega$ of Ω equals zero under Neumann boundary conditions. Thus, letting $\phi = u$ gives

$$\mu = \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}x}{\int_\Omega u^2 \, \mathrm{d}x}$$

as the expression for the Rayleigh quotient.

To get the smallest nonzero eigenvalue, one can take the infinimum over all $u \in H^1(\Omega)$ such that $\int_\Omega u \, \mathrm{d}x = 0$. The notation $H^1(\Omega)$ represents the Sobolev Space [7]. Meanwhile the condition $\int_\Omega u \, \mathrm{d}x = 0$ can be interpreted as $\int_\Omega u \cdot C \, \mathrm{d}x = 0$, where C is a constant function. In the function space L^2, this means u is orthogonal to constants, which is imposed to remove the eigenvalue $\mu = 0$ whose solutions are constant functions.

We use this form of μ_1 to find an upper bound by evaluating the Rayleigh quotient for a "test function" u that satisfies the conditions.

2.4 Prior Work

Our paper follows the work of Henrot, Lemenant, and Lucardesi. In [6] they prove an upper bound on the first Neumann eigenvalue μ_1 in terms of minimal width and area for regions in $\mathbb{R}^2$. They also pose a conjecture for an upper bound in terms of perimeter for regions in $\mathbb{R}^2$ which they prove for some special cases.

Lemma 2.1 (Henrot, Lemenant, Lucardesi)**.** *For all planar bounded (Lipschitz) open sets*

Ω we have:

$$\mu_1(\Omega) \leq \pi^2 \cdot \frac{\omega(\Omega)^2}{|\Omega|^2},$$

where $\mu_1(\Omega)$ is the first Neumann eigenvalue of Ω, $|\Omega|$ is the area of Ω, and $\omega(\Omega)$ is the minimal width of Ω. The inequality becomes an equality if and only if Ω is a rectangle.

Conjecture 2.2 (Henrot, Lemenant, Lucardesi)**.** *For all planar convex domains Ω, we have*

$$P(\Omega)^2 \mu_1(\Omega) \leq 16\pi^2,$$

where $P(\Omega)$ is the perimeter of Ω and $\mu_1(\Omega)$ is the first nontrivial Neumann eigenvalue of Ω. The equality case is achieved only when Ω is a square or an equilateral triangle.

Theorem 2.3 (Henrot, Lemenant, Lucardesi)**.** *For all $\Omega \subset \mathbb{R}^2$ with two axes of symmetry,*

$$P(\Omega)^2 \mu_1(\Omega) \leq 16\pi^2$$

where $P(\Omega)$ is the perimeter of Ω and $\mu_1(\Omega)$ is the first nontrivial Neumann eigenvalue of Ω. The equality case is achieved only when Ω is a square or an equilateral triangle.

In our paper, we first compute the first Neumann eigenvalue, perimeter, width, and area for some simple regions on the sphere. We then use similar methods as in [6] to prove an upper bound on $\mu_1(\Omega)$ in terms of width and area for regions on the sphere. Finally, we pose a conjecture for an upper bound for $\mu_1(\Omega)$ under perimeter constraints for convex domains on the sphere.

3 Computing eigenvalues of simple regions on the sphere

In this section we prove that $\mu_1 = 6$ for a 1/8 sphere and $\mu_1 = 2$ for a lune. Note that since the hemisphere is a special lune, we prove that $\mu_1 = 2$ for a hemisphere as well.

Proof. First recall that the Neumann eigenvalue problem for any eigenvalue μ involves a differential equation and a boundary condition:

$$\Delta u + \mu u = 0$$
$$\partial_\nu u = 0.$$

We focus on solving the differential equation first and deal with the boundary conditions for specific shapes later.

For this computation, we use geodesic polar coordinates, as described in Section 2.2. Let r be the distance from the north pole and θ be the longitudinal angle. Then, on a unit sphere,

$$x = \sin r \cos\theta$$
$$y = \sin r \sin\theta$$
$$z = \cos r.$$

Using the formula for the Laplacian in geodesic polar coordinates gives

$$u_{rr} + \frac{\cos r}{\sin r} u_r + \frac{1}{\sin^2 r} u_{\theta\theta} + \mu u = 0$$

and applying the separation of variables $u = X(r)Y(\theta)$ yields

$$X''(r)Y(\theta) + \frac{\cos r}{\sin r} X'(r)Y(\theta) + \frac{1}{\sin^2 r} X(r)Y''(\theta) + \mu X(r)Y(\theta) = 0. \tag{4}$$

Moving forward, we make a few assumptions about the form of the eigenfunctions, meaning we cannot use completeness of eigenfunctions to check that we the eigenvalue we find is necessarily the smallest. However, for both the 1/8 sphere and the lune, we use other methods to bypass this issue and confirm that the eigenvalues we compute are indeed the first Neumann eigenvalues of those regions.

We assume that

$$Y''(\theta) + k^2 Y(\theta) = 0 \tag{5}$$

for some real k, which has the solution

$$Y(\theta) = A\sin(k\theta) + B\cos(k\theta),$$

where A and B are real numbers [7]. Then plugging Equation (5) into Equation (4) gives

$$X''(r) + \cot(r)X'(r) + \left(\mu - \frac{k^2}{\sin^2(r)}\right)X(r) = 0. \tag{6}$$

Next perform the change of variables $z = \cos r$, letting $v(z) = X(r)$. Computing derivatives of X gives

$$\begin{aligned}
X'(r) &= \frac{\mathrm{d}X(r)}{\mathrm{d}r} = \frac{\mathrm{d}v(z)}{\mathrm{d}z}\cdot\frac{\mathrm{d}z}{\mathrm{d}r} = -(\sin r)\cdot v'(z) \\
X''(r) &= \frac{\mathrm{d}X'(r)}{\mathrm{d}r} = -(\cos r)\cdot v'(z) - (\sin r)\cdot\frac{\mathrm{d}v'(z)}{\mathrm{d}z}\cdot\frac{\mathrm{d}z}{\mathrm{d}r} \\
&= -(\cos r)\cdot v'(z) + (\sin^2 r)\cdot v''(z) = (1-z^2)\cdot v''(z) - z\cdot v'(z).
\end{aligned}$$

Plugging these derivatives into Equation (6) gives

$$(1-z^2)\cdot v''(z) - z\cdot v'(z) + \frac{\cos r}{\sin r}\left(-(\sin r)\cdot v'(z)\right) + \left(\mu - \frac{k^2}{\sin^2(r)}\right)v(z) = 0$$

$$(1-z^2)\cdot v''(z) - 2z\cdot v'(z) + \left(\mu - \frac{k^2}{1-z^2}\right)v(z) = 0, \tag{7}$$

which is in a special form called the general Legendre equation, where $\mu = m(m+1)$ for some real number m.

Because the general Legendre equation has been extensively studied, assuming m and k are integers such that $0 \leq k \leq m$, it is known that the solutions to Equation 7 are given by special functions called the *associated Legendre polynomials* $P_m^k(z)$ [8]. The first few associated Legendre polynomials as well as their respective eigenvalues are given in Table 1.

m	k	$P_m^k(z)$	μ
0	0	$P_0^0(z) = 1$	0
1	0	$P_1^0(z) = z$	2
1	1	$P_1^1(z) = -(1-z^2)^{\frac{1}{2}}$	2
2	0	$P_2^0(z) = \frac{1}{2}(3z^2-1)$	6
2	1	$P_2^1(z) = -3z(1-z^2)^{\frac{1}{2}}$	6
2	2	$P_2^2(z) = 3(1-z^2)$	6

Table 1: The first few associated Legendre polynomials

For a brief recap, we have

$$u = X(r)Y(\theta)$$

$$X(r) = v(\cos(r)) = P_m^k(\cos r)$$

$$Y(\theta) = A\sin(k\theta) + B\cos(k\theta),$$

where r is the distance from the north pole, θ is the longitudinal angle, $0 \leq k \leq m$ are integers, and $\mu = m(m+1)$. The exact functions $X(r)$ and $Y(\theta)$ are determined by the boundary conditions, which we now take into consideration.

Case 1: Consider the 1/8 sphere defined by $0 < r < \pi/2$ and $0 < \theta < \pi/2$. The boundary conditions are

$$u_\theta(r,0) = u_\theta(r,\pi/2) = u_r(\pi/2,\theta) = 0.$$

From $u_\theta(r,0) = u_\theta(r,\pi/2) = 0$, we get that $Y(\theta) = \cos(k\theta)$ where k is even. Then, we check $u_r(\pi/2,\theta) = 0$ for potential $X(r)$ starting with those that give the smallest eigenvalue, as depicted in Table 1. First, we check the function corresponding to $m = 1$, $k = 0$ and see it fails:

$$X(\pi/2) = \left(P_1^0(\cos r)\right)_r\Big|_{r=\pi/2} = (\cos r)_r\Big|_{r=\pi/2} = (-\sin r)\Big|_{r=\pi/2} = -1.$$

Next, the function corresponding to $m = 1$, $k = 1$ is not a candidate since k must be even. Finally, the condition holds for the next potential function, which corresponds to $m = 2$,

$k = 0$:

$$X(\pi/2) = \left(P_2^0(\cos r)\right)_r \Big|_{r=\pi/2} = \left(\frac{1}{2}(3\cos^2 r - 1)\right)_r \Big|_{r=\pi/2} = \left(\frac{3}{2} \cdot 2\cos r(-\sin r)\right) \Big|_{r=\pi/2} = 0,$$

so thus for the 1/8 sphere, $\mu_1 = 6$.

As mentioned previously, we made assumptions such as the fact that $Y(\theta)$ has to satisfy a particular form, and that k is an integer. As such, we are not guaranteed completeness of the eigenfunctions and must verify that our computed eigenvalue is the smallest in a different way. We do so using a reflection method that is explained in Appendix B.

Case 2: For a lune of angle β, the boundary conditions are

$$u_\theta(r, 0) = u_\theta(r, \beta) = 0,$$

from which we get that $Y(\theta) = \cos(k\theta)$ where $k = \frac{k_0\pi}{\beta}$ for integer k_0. We can take $k = k_0 = 0$ and $m = 1$ and refer to Table 1 to get the function $X(r) = P_1^0(\cos r)$ works. Thus, for the lune, $\mu_1 = 2$.

Once again, we are not guaranteed completeness of eigenfunctions, but to confirm that $\mu_1 = 2$ is indeed the smallest nonzero eigenvalue, we refer to the lower bound $\mu_1 \geq 2$ on the sphere, proved by Escobar in [9].

□

4 Bounding μ_1 by minimal width and area

In this section, we prove our main result, which bounds the first Neumann eigenvalue μ_1 of a region by its minimal width and area.

We first define minimal width.

Definition 4.1. For a connected open subset $\Omega \subset \mathbb{S}^2$ where $\mathbb{S}^2$ is the unit sphere, the *minimal width* is defined as the smallest positive ω such that Ω can be contained completely within a strip defined by $z^2 = \sin^2(\omega/2)$ in some Cartesian coordinate system where $\mathbb{S}^2$ is centered at the origin.

For example, Figure 1 shows a region trapped by the two latitudes that determine the minimal width.

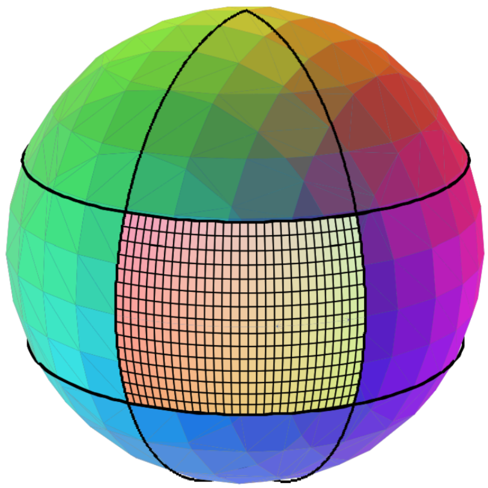

Figure 1: A "rectangle" on the sphere trapped by two latitudes and two geodesics through the poles

We now state our main result.

Theorem 4.1. *Let Ω be a connected, open subset of the unit sphere $\mathbb{S}^2$ such that the diameter of Ω is less than or equal to π. Then*

$$\mu_1 \leq \frac{4\pi^2 \sin\left(\frac{\omega}{2}\right) \ln\left(\sec\left(\frac{\omega}{2}\right) + \tan\left(\frac{\omega}{2}\right)\right)}{|\Omega|^2} \approx \frac{4\pi^2}{|\Omega|^2}\left(\frac{1}{4}\omega^2 + \frac{1}{2880}\omega^6 + \cdots\right)$$

where ω is the minimal width of Ω and $|\Omega|$ is the area. Equality holds if and only if Ω is a "rectangle" with two opposite sides that are geodesics going through the poles, and the other two sides are latitudes of the same length, as depicted in Figure 1.

Proof. We start with geodesic normal coordinates (ϕ, s) where the parameter ϕ is the distance around the equator and s is the distance we deviate from the equator. These coordinates have the associated metric

$$g = \cos^2(s)\mathrm{d}\phi^2 + \mathrm{d}s^2.$$

Since ω is the minimal width of Ω, we can trap it in the strip $S = (-\pi, \pi) \times (-\omega/2, \omega/2)$, as seen in Figure 1, where the first interval is over ϕ and the second is over s. We define a test function u on S and use the Rayleigh quotient to bound μ_1. Consider the test function

$$u = \begin{cases} -1 & -\pi \leq \phi \leq -L/2 \\ \sin((\pi/L)\phi) & -L/2 \leq \phi \leq L/2 \\ 1 & L/2 \leq \phi \leq \pi. \end{cases}$$

Before doing calculations, we need to make sure that the condition $P = \int_\Omega u \mathrm{d}V = 0$ is satisfied, as required by Equation 3. This can be done by considering the "translations" of Ω in the ϕ-coordinate. If Ω is translated such that it is contained in $(-\pi, 0) \times (-\omega/2, \omega/2)$, then P is negative. Meanwhile if Ω is contained in $(0, \pi) \times (-\omega/2, \omega/2)$ then P is positive. Since P varies continuously, by intermediate value theorem, there is a position of Ω where $P = 0$ is satisfied. Notice this argument makes use of the condition that the diameter of Ω is at most π.

We then compare this intermediate position of Ω to the "rectangle" of the same width and area: $R = (-L/2, L/2) \times (-\omega/2, \omega/2)$ as depicted in Figure 1. This rectangle is the equality case of our bound and has area

$$|\Omega| = |R| = \int_R \mathrm{d}V = \int_R \cos s \, \mathrm{d}\phi \mathrm{d}s = 2L \sin(\omega/2).$$

Next, we compute the different parts of the Rayleigh quotient using formulas dependent on the Riemannian metric, as detailed in Appendix C to get

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}V = \frac{\pi^2}{L} \cdot \ln\left(\sec\left(\frac{\omega}{2}\right) + \tan\left(\frac{\omega}{2}\right)\right)$$

and

$$\int_\Omega u^2\,\mathrm{d}V = \int_{R\cap\Omega} u^2 \mathrm{d}V + \int_{\Omega\setminus R} u^2\,\mathrm{d}V$$
$$= \left(2\sin\left(\frac{\omega}{2}\right)\right)\cdot\left(\frac{L}{2}\right) = L\cdot\sin\left(\frac{\omega}{2}\right).$$

Thus, the Rayleigh quotient gives

$$\mu_1 = \inf_{\substack{u\in H^1(\Omega)\\ \int_\Omega u\,\mathrm{d}x=0}} \frac{\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\int_\Omega u^2\,\mathrm{d}x} \le \frac{4\pi^2\cdot\sin\left(\frac{\omega}{2}\right)\cdot\ln\left(\sec\left(\frac{\omega}{2}\right)+\tan\left(\frac{\omega}{2}\right)\right)}{|\Omega|^2}.$$

Finally, the equality case of all of the inequalities in the full calculation in Appendix C are satisfied if and only if $\Omega = R$ is a "rectangle" as desired.

□

Remark. Notice that the proof of Theorem 4.1 does not require Ω to be convex, and in fact for the equality case, the "rectangle" R is actually not convex unless the width is π in which case R would be the hemisphere.

Remark. Notice the first term of the Taylor expansion of our bound in Theorem 4.1 is

$$\frac{4\pi^2}{|\Omega|^2}\left(\frac{1}{4}\omega^2\right) = \frac{\pi^2\omega^2}{|\Omega|^2},$$

which matches the bound in [6] for the Euclidean plane. This means that if we shrink a shape on the sphere, the limit of our eigenvalue bound approaches the bound for the Euclidean plane. Such an observation should not be surprising because as a manifold, the sphere looks locally like the plane.

5 A conjecture for the sphere

In this section we propose a conjecture for an upper bound of $P(\Omega)\mu_1(\Omega)$ for the sphere, where $P(\Omega)$ is the perimeter of the region Ω. Since the sphere looks locally like the plane, one might intuitively expect that shrinking a shape on the sphere will make it asymptotically approach the upper bound proposed for the planar case, which motivates the statement of our conjecture.

Conjecture 5.1. *For a convex set in the unit sphere* $\mathbb{S}^2$*,*

$$P(\Omega)^2\mu_1(\Omega) < 16\pi^2,$$

where $P(\Omega)$ *is the perimeter of* Ω *and* $\mu_1(\Omega)$ *is the first nonzero Neumann eigenvalue of* Ω*. The upper bound is approached asymptotically by geodesic triangles and geodesic squares.*

We can easily check that our conjecture holds for the simple shapes considered in Section 3, for which $P(\Omega)$ and $\mu_1(\Omega)$ are listed in Table 2.

	Eigenvalue μ_1	Perimeter	Width	Area
Hemisphere	2	2π	π	2π
Lune	2	2π	β	2β
1/8 Sphere	6	$3\pi/2$	$\pi/2$	$\pi/2$

Table 2: A chart of μ_1, perimeter, width, and area for simple shapes on the sphere

We also check our conjecture for another special shape called a geodesic ball, which looks like a polar ice cap. In geodesic polar coordinates, a geodesic ball of radius R is described by $r \leq R$. Langford and Laugesen proved in [10] that for a geodesic ball $\mathbb{B}_r$ of radius r, the quantity

$$\mu_1(\mathbb{B}_r)\sin^2(r)$$

is strictly decreasing with respect to r and has a limit of $\mu_1(\mathbb{D})$ as r goes to 0, where $\mu_1(\mathbb{D}) \approx (1.84)^2$ is the first Neumann eigenvalue of a unit disk. Using the fact that the perimeter of a geodesic ball is

$$P(\mathbb{B}_r) = 2\pi \sin r,$$

we can take a limit to get

$$\lim_{r\to 0}\left(P^2\left(\mathbb{B}_r\right)\mu_1\left(\mathbb{B}_r\right)\right)=\lim_{r\to 0}\left((2\pi\sin r)^2\mu_1(\mathbb{B}_r)\right)=\lim_{r\to 0}\left(4\pi^2\mu_1(\mathbb{B}_r)\sin^2(r)\right)$$
$$=4\pi^2\mu_1(\mathbb{D})\approx 4\pi^2(1.84)^2<16\pi^2.$$

6 Conclusion

Our project examined the the problem of bounding the first Neumann eigenvalue μ_1 of regions on the sphere. We first computed μ_1 for simple shapes such as a spherical lune and a 1/8 sphere, for which we found that $\mu_1 = 2$ and $\mu_1 = 6$, respectively. Then, we proved a sharp upper bound on μ_1 in terms of minimal width and area. For shapes with very small width, our upper bound asymptotically approaches the upper bound proven by Henrot, Lemenant, and Lucardesi in the planar case which matches our intuition given that the sphere looks locally like $\mathbb{R}^2$. Finally, we gave a conjecture for a strict upper bound on μ_1 under perimeter constraint, where the upper bound is reached asymptotically by small squares and equilateral triangles, the same shapes that are proposed to be maximizers for the conjecture in the plane.

Future work on bounding μ_1 for regions on the sphere under perimeter constraint would likely continue to follow the same path Henrot, Lemenant, and Lucardesi took for the planar case. The bound using width and area could potentially be extended to higher dimensional spheres and other Riemannian manifolds. As for the conjecture, confirming that shrinking geodesic squares and equilateral triangles asymptotically approach the proposed upper bound would be an important first step. Then, one might try to prove the conjecture under some assumptions of regularity or symmetry on the region. Another potential path would be to take shape derivatives of the quantity $P(\Omega)^2\mu_1(\Omega)$, and examine how it changes under perturbations of the region. All of these directions provide ways to generalize this new and intriguing isoperimetric problem for eigenvalues.

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A Formulas involving the Riemannian metric

In this appendix, we introduce a few formulas involving the Riemannian metric g.

The formula for the length of a vector v in the tangent space of a point on the manifold is

$$|v|_g = v^\intercal g v,$$

where v is written in the same coordinates that determine the metric g.

The formula for the volume form, the differential form over which we are integrating, is

$$\omega_g = \sqrt{\det(g)} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n,$$

where $\wedge$ is the wedge product, and x^i for $1 \leq i \leq n$ are the coordinates that determine the metric.

The formula for the gradient (in Einstein notation) is

$$\nabla f = g^{ij} \frac{\partial f}{\partial x^i} \partial x^j,$$

where g^{ij} are the elements of the inverse matrix of g, x^i is the i-th coordinate, and ∂x^j is the x^j component of the vector.

The formula for the Laplacian (in Einstein notation) is

$$\Delta u = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(g^{ij} \sqrt{\det g} \frac{\partial u}{\partial x^j} \right),$$

where g^{ij} are the elements of the inverse matrix of g and x^i is the i-th coordinate.

B Confirming $\mu_1 = 6$ for a 1/8 sphere

In our computations for μ_1 of a 1/8 sphere, we made assumptions such as the fact that $Y(\theta)$ has to satisfy a particular form, and that k is an integer. As such, we are not guaranteed completeness of the eigenfunctions and must verify that our computed eigenvalue is the smallest in a different way.

To do so, notice that for any function $u(r,\theta)$ that satisfies the Neumann problem on the 1/8 sphere defined by $0 < r < \pi/2$ and $0 < \theta < \pi/2$, we can extend the function to the whole sphere by

$$u(r,\theta) = \begin{cases} u(r,\theta) & (r,\theta) \in [0,\pi/2] \times [0,\pi/2] \\ u(r,\pi-\theta) & (r,\theta) \in [0,\pi/2] \times [\pi/2,\pi] \\ u(r,\theta-\pi) & (r,\theta) \in [0,\pi/2] \times [\pi,3\pi/2] \\ u(r,2\pi-\theta) & (r,\theta) \in [0,\pi/2] \times [3\pi/2,2\pi] \\ u(\pi-r,\theta) & (r,\theta) \in [\pi/2,\pi] \times [0,\pi/2] \\ u(\pi-r,\pi-\theta) & (r,\theta) \in [\pi/2,\pi] \times [\pi/2,\pi] \\ u(\pi-r,\theta-\pi) & (r,\theta) \in [\pi/2,\pi] \times [\pi,3\pi/2] \\ u(\pi-r,2\pi-\theta) & (r,\theta) \in [\pi/2,\pi] \times [3\pi/2,2\pi] \end{cases}$$

which essentially involves "reflecting" the function to cover the entire sphere. Due to the Neumann boundary condition, our extended function is twice-differentiable everywhere and thus satisfies the Neumann problem for the whole sphere.

Eigenfunctions u for the whole sphere are known to be complete under spherical harmonics [8]. These functions are exactly the functions

$$u = X(r)Y(\theta),$$

where

$$X(r) = v(\cos(r)) = P_m^k(\cos r)$$

$$Y(\theta) = A\sin(k\theta) + B\cos(k\theta),$$

that we derived in Section 3 above and thus the only ones we need to check. Upon checking, we find that $\mu_1 = 6$ is the smallest nonzero Neumann eigenvalue for a 1/8 sphere.

C Computations for Theorem 4.1

Here we show the computations for the proof of Theorem 4.1 in full.

Using formulas dependent on the Riemannian metric, we compute the different parts of the Rayleigh quotient to get

$$\nabla u = \sum_{i,j} g^{ij} \frac{\partial u}{\partial x^i} \partial x^j = \frac{1}{\cos^2(s)} \frac{\partial u}{\partial \phi} \partial \phi + \frac{\partial u}{\partial s} \partial s$$

$$|\nabla u|^2 = (\nabla u)^T g u = \frac{1}{\cos^2(s)} \left(\frac{\partial u}{\partial \phi}\right)^2 + \left(\frac{\partial u}{\partial s}\right)^2$$

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}V = \int_\Omega |\nabla u|^2 \cdot \sqrt{\det g} \, \mathrm{d}s \mathrm{d}\phi = \int_\Omega \frac{1}{\cos(s)} \left(\frac{\partial u}{\partial \phi}\right)^2 + \cos(s) \left(\frac{\partial u}{\partial s}\right)^2 \mathrm{d}s \mathrm{d}\phi$$

and

$$\int_\Omega u^2 \, \mathrm{d}V = \int_\Omega u^2 \cdot \cos s \, \mathrm{d}s \mathrm{d}\phi.$$

Plugging our test function

$$u = \begin{cases} -1 & -\pi \leq \phi \leq -L/2 \\ \sin((\pi/L)\phi) & -L/2 \leq \phi \leq L/2 \\ 1 & L/2 \leq \phi \leq \pi \end{cases}$$

into the expressions gives

$$\begin{aligned}
\int_\Omega |\nabla u|^2 \, \mathrm{d}V &= \int_\Omega \frac{1}{\cos(s)} \left(\frac{\partial u}{\partial \phi}\right)^2 + \cos(s) \left(\frac{\partial u}{\partial s}\right)^2 \mathrm{d}s \mathrm{d}\phi \\
&= \int_{R \cap \Omega} \frac{1}{\cos(s)} \left(\cos\left(\frac{\pi}{L}\phi\right) \cdot \frac{\pi}{L}\right)^2 \mathrm{d}s \mathrm{d}\phi \\
&\leq \int_R \frac{1}{\cos(s)} \left(\cos\left(\frac{\pi}{L}\phi\right) \cdot \frac{\pi}{L}\right)^2 \mathrm{d}s \mathrm{d}\phi \\
&= \left(\frac{\pi}{L}\right)^2 \cdot \int_{-L/2}^{L/2} \int_{\omega/2}^{\omega/2} \frac{1}{\cos(s)} \left(\cos\left(\frac{\pi}{L}\phi\right)\right)^2 \mathrm{d}s \mathrm{d}\phi \\
&= \left(\frac{\pi}{L}\right)^2 \cdot \left(\int_{-\omega/2}^{\omega/2} \frac{1}{\cos(s)} \, \mathrm{d}s\right) \left(\int_{-L/2}^{L/2} \left(\cos\left(\frac{\pi}{L}\phi\right)\right)^2 \mathrm{d}\phi\right) \\
&= \left(\frac{\pi}{L}\right)^2 \cdot \left(2 \ln\left(\sec\left(\frac{\omega}{2}\right) + \tan\left(\frac{\omega}{2}\right)\right)\right) \cdot \left(\frac{L}{2}\right)
\end{aligned}$$

$$= \frac{\pi^2}{L} \cdot \ln\left(\sec\left(\frac{\omega}{2}\right) + \tan\left(\frac{\omega}{2}\right)\right)$$

and

$$\begin{aligned}
\int_\Omega u^2 \,\mathrm{d}V &= \int_{R\cap\Omega} u^2 \mathrm{d}V + \int_{\Omega\setminus R} u^2 \,\mathrm{d}V \\
&= \int_R \cos(\phi)^2 \mathrm{d}V - \int_{R\setminus\Omega} \cos(\phi)^2 \mathrm{d}V + \int_{\Omega\setminus R} 1 \,\mathrm{d}V \\
&\geq \int_R \cos(\phi)^2 \mathrm{d}V - \int_{R\setminus\Omega} 1 \mathrm{d}V + \int_{\Omega\setminus R} 1 \,\mathrm{d}V \\
&= \int_R \cos(\phi)^2 \mathrm{d}V - |R\setminus\Omega| + |\Omega\setminus R| \\
&= \int_R \cos(\phi)^2 \mathrm{d}V = \int_{-L/2}^{L/2} \int_{\omega/2}^{\omega/2} \cos(\phi)^2 \cos(s) \,\mathrm{d}s\mathrm{d}\phi \\
&= \left(\int_{-\omega/2}^{\omega/2} \cos(s) \,\mathrm{d}s\right) \left(\int_{-L/2}^{L/2} \cos^2(\phi) \ \mathrm{d}\phi\right) \\
&= \left(2\sin\left(\frac{\omega}{2}\right)\right) \cdot \left(\frac{L}{2}\right) = L \cdot \sin\left(\frac{\omega}{2}\right).
\end{aligned}$$

Thus, we can use the Rayleigh quotient to bound the first Neumann eigenvalue as

$$\begin{aligned}
\mu_1 &= \inf_{\substack{u\in H^1(\Omega) \\ \int_\Omega u \,\mathrm{d}x = 0}} \frac{\int_\Omega |\nabla u|^2 \,\mathrm{d}x}{\int_\Omega u^2 \,\mathrm{d}x} \\
&\leq \frac{\frac{\pi^2}{L} \cdot \ln\left(\sec\left(\frac{\omega}{2}\right) + \tan\left(\frac{\omega}{2}\right)\right)}{L \cdot \sin\left(\frac{\omega}{2}\right)} \\
&= \frac{\pi^2}{\sin\left(\frac{\omega}{2}\right)} \cdot \frac{1}{L^2} \cdot \ln\left(\sec\left(\frac{\omega}{2}\right) + \tan\left(\frac{\omega}{2}\right)\right) \\
&= \frac{\pi^2}{\sin\left(\frac{\omega}{2}\right)} \cdot \left(\frac{2\sin\left(\frac{\omega}{2}\right)}{|\Omega|}\right)^2 \ln\left(\sec\left(\frac{\omega}{2}\right) + \tan\left(\frac{\omega}{2}\right)\right) \\
&= \frac{4\pi^2 \cdot \sin\left(\frac{\omega}{2}\right) \cdot \ln\left(\sec\left(\frac{\omega}{2}\right) + \tan\left(\frac{\omega}{2}\right)\right)}{|\Omega|^2}.
\end{aligned}$$