Slice Genus Bounds for Knots using Grid Diagrams

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Abstract

Knots that are topologically but not smoothly slice are of great interest to geometric topologists. To determine the smooth sliceness of a knot, we study its slice genus, which is related to knot invariants like the grid index $g(K)$ and the maximal Thurston–Bennequin number $\overline{\text{tb}}(K)$. It has been conjectured that $g(K)$ and $\overline{\text{tb}}(K)$ detect roughly the same information. In this paper, we establish a connection between $g(K)$ and $\overline{\text{tb}}(K)$ and give an explanation as to why the Thurston-Bennequin number gives a better approximation of the grid index when compared to other knot invariants and strategies. We also find an infinite sequence of topologically slice knots with slice genus approaching infinity. As such knots are topologically slice but not smoothly slice, applying knot surgery to each knot gives rise to an exotic $\mathbb{R}^4$.

Summary

Geometric topologists like to study knots that are topologically slice but not smoothly slice, as such knots give rise to exotic $\mathbb{R}^4$'s. To determine the smooth sliceness of a knot, we study its slice genus, which is related to knot invariants like the maximal Thurston–Bennequin number and the grid index. Existing knot invariants and strategies give weak bounds on the grid index. Instead, we use the Thurston-Bennequin number to study the grid index, as they detect similar features of a knot. Additionally, we use the Thurston-Bennequin number to construct a sequence of knots whose slice genera approach infinity.

1 Introduction

Generalizations of the Poincaré Conjecture are of great interest to mathematicians studying geometric topology. The generalized Poincaré Conjecture roughly states that any manifold that is homotopy equivalent to a sphere is equivalent to a sphere, where the notion of equivalence can be taken as topological or smooth. The conjecture has been solved for dimensions 5 and above by the work of Milnor [1] and Kervaire-Milnor [2] (smooth) and Smale [3] (topological) up to determining the stable homotopy groups of spheres. However, for lower dimensions, the conjecture is much harder to solve. In the topological case, the work of Freedman [4] (dimension 4) and Perelman [5, 6, 7] (dimension 3) confirms that the conjecture is true. Moreover, Perelman's proof also works for the smooth three-dimensional case. However, it remains unknown whether the smooth four-dimensional Poincaré conjecture holds.

Since the work of Freedman, topologists have tried to use knot surgery to construct exotic 4-spheres, counterexamples to the smooth Poincaré conjecture in dimension 4 [8, 9]. This technique relies on the notion of *smooth sliceness* of a knot, which determines if a knot embedded in S^3 is the boundary of a smoothly embedded disc in a 4-ball. However, it turns out that proving a certain knot is not smoothly slice is also very difficult: for example, a proof that the Conway knot is not smoothly slice has only been recently obtained by Piccirillo [10] using sophisticated tools such as Khovanov homology [11] and the Rasmussen s-invariant [12]. In [9], Manolescu and Piccirillo pursued this approach, and found five knots that yield exotic four-spheres if slice. Manolescu and Piccirillo computed some properties of these knots of interest, shown in Table 3.

In this paper, we introduce a new method that can determine non-sliceness of a knot using a simple knot representation called the *grid diagram*. The crucial observation is that the minimal grid diagram of a knot is closely related to its maximal Thurston-Bennequin number (which is an invariant), giving a lower bound on the slice genus.

The main machinery we use in this paper are the Slice-Bennequin inequality and an inequality due to Matsuda [13]. The Slice-Bennequin inequality provides a lower bound for the slice genus of a knot, and Matsuda's inequality relates the grid index of a knot with its maximal Thurston-Bennequin number. With these tools, we demonstrate that our method recovers the slice genus bound for torus knots obtained by Kronheimer-Mrowka [14] and Rasmussen [15], and we construct a family of topologically slice but not smoothly slice knots with arbitrarily large slice genus. In addition, applying knot surgery to any knot in this family gives rise to an exotic $\mathbb{R}^4$ [16].

Knot Invariant	K_1	K_2	K_3	K_4	K_5
Crossing number $\mathrm{cr}(K)$	29	29	32	29	32
Alexander polynomial	1	1	1	1	1
τ ϵ, and ν invariants	0	0	0	0	0
Rasmussen s invariant	0	0	0	0	0
Lipshitz-Sarkar Sq^1 s-invariants	0	0	0	0	0
Seifert genus	2	2	2	2	2
Total rank	65	65	193	65	193

Table 1: Properties of Manolescu and Piccirillo knots [9].

2 Preliminaries

In this section, we define some necessary concepts. First, we must define the concept of knot invariants and sliceness, and then we introduce grid index, Thurston-Bennequin number, and results used in our research involving these quantities.

2.1 Knots, Invariants, and Sliceness

Let $B^n = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq 1\}$ denote the n-dimensional ball and $S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}$ denote its boundary.

Definition 2.1 (knot)**.** A *knot* is a smooth embedding of S^1 into $\mathbb{R}^3$.

In other words, a knot is a closed non-self-intersecting curve in three dimensions. The inclusion of the smoothness constraint excludes "wild knots", which exhibit pathological behavior and cannot be represented using grid diagrams. A *link* is a collection of non-intersecting knots that could be intertwined but do not intersect. To visualize knots, we tend to think of them in terms of their projections onto a plane. Since knots can yield many different representations by projection onto different planes, it is hard to distinguish knots using solely their knot representations. Naturally, mathematicians want to find a way to determine when two representations of knots are actually topologically distinct. The main tools used to distinguish knots based on their representations are *knot invariants*.

Definition 2.2 (knot invariant)**.** A *knot invariant* is an expression $\varphi(K)$ in terms of a knot K such that if the knots K_1 and K_2 are equivalent by ambient isotopy, then $\varphi(K_1) = \varphi(K_2)$.

For a knot K, we mainly study three knot invariants: the slice genus $g_s(K)$, the grid index $g(K)$, and the maximal Thurston-Bennequin number $\overline{\mathrm{tb}}(K)$.

Definition 2.3 (slice genus)**.** The *slice genus* of a knot $K \subset S^3$ is the smallest integer $g_s(K)$ such that K is the oriented boundary of a smooth oriented surface Σ of genus $g_s(K)$ embedded in B^4.

We define the notions of smoothly and topologically slice knots.

Definition 2.4 (smooth sliceness)**.** A knot $K \subset S^3$ is *smoothly slice* if it is the boundary of some smoothly embedded disk in B^4.

By definition, a knot is smoothly slice if and only if its slice genus is 0.

Definition 2.5 (topological sliceness)**.** A knot $K \subset S^3$ is *topologically slice* if it is the boundary of some locally flat embedded disk in B^4.

By definition, any smoothly slice knot must also be topologically slice. In [17], the converse is shown to be false. We are interested in finding new families of knots that are topologically slice but not smoothly slice, as such knots can be used to construct exotic smooth structures in $\mathbb{R}^4$ [18] and could potentially be used to construct exotic 4-spheres [8].

The Alexander polynomial (see [19] for more details) is a knot invariant that is useful in detecting topological sliceness of knots.

Theorem 2.6 ([4])**.** *If a knot K has Alexander polynomial 1, then K is topologically slice.*

On the other hand, it is notoriously difficult to detect the smooth sliceness of a given knot [17]. Even for simple knots like the Conway knot or the $(2,1)$-cable of the figure-eight knot, it was only determined recently that they were not smoothly slice [10, 20]. For many other simple examples like the Whitehead double of the figure-eight knot, it is still not known whether they are smoothly slice.

2.2 Grid Index and its Relations to Other Knot Invariants

Knot invariants can be used to dinstinguish knots, but each knot invariant is used to detect different properties of a knot. In this paper, we primarily study three knot invariants: the grid index $g(K)$, the arc index $\alpha(K)$, and the maximal Thurston-Bennequin number $\overline{\text{tb}}(K)$. The first is the grid index $g(K)$, based on the minimal size of a knot representation known as the planar grid diagram.

Definition 2.7 (grid diagram)**.** A *planar grid diagram* is an $n \times n$ grid with n distinct cells labelled with an X and n distinct cells labelled with an O such that the following conditions are satisfied.

- No cell has both an X and an O.
- Each row has exactly one X and exactly one O.
- Each column has exactly one X and exactly one O.

If we connect all X's and O's in the same row or column and place the vertical strand on top of the horizontal strand at each crossing, we obtain a knot K. We call $\mathbb{G}$ the (planar) grid diagram of K.

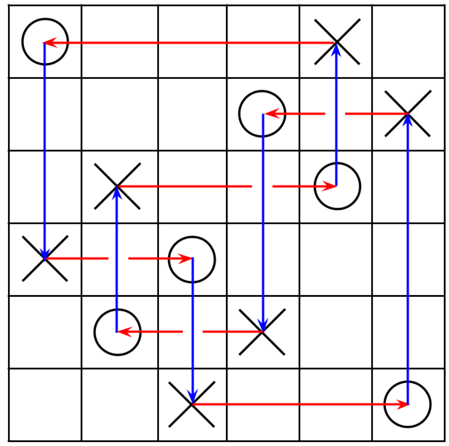

Figure 1: Grid diagram of the figure-eight knot.

Definition 2.8 (grid index)**.** The *grid index* of a knot K is the minimum grid size among all grid diagrams of K.

Grid diagrams are related to another type of knot representation known as an arc presentation.

Definition 2.9 (open book decomposition)**.** An *open book decomposition* of S^3 is a binding of half-planes using the Cartesian z-axis as the binding axis.

Definition 2.10 (arc presentation)**.** An *arc presentation* of a knot K is an embedding of K into the pages of an open book decomposition of the sphere S^3, with exactly one arc on each page. The *arc index* $\alpha(K)$ is the smallest number of pages necessary in an arc presentation of knot K.

There exists a well-known bijection between arc presentations and grid diagrams of a knot implying that $g(K) = \alpha(K)$ for all knots K [21]. This result allows us to use historical bounds involving the grid index and arc index interchangeably. For the rest of the paper, we will use the terms "grid index" and "arc index" interchangeably unless otherwise stated.

A front projection of a Legendrian knot is a knot representation with no vertical tangents, no non-smooth points other than the cusps, and at every crossing, the understrand has positive slope and the overstrand has negative slope.

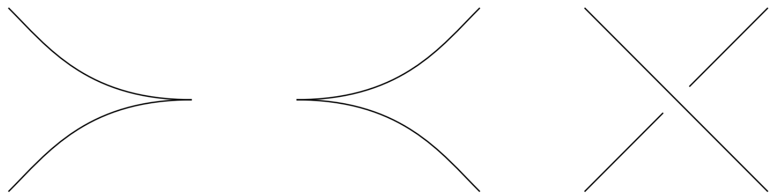

Figure 2: A front projection of a Legendrian knot has three types of *singularities*. From left to right, they are *right cusps*, *left cusps*, and *crossings* [22].

For simplicity, we call a front projection of a Legendrian knot a Legendrian knot representation. The main property of Legendrian knot representations we study is the Thurston-Bennequin number.

Definition 2.11. The *Thurston-Bennequin number* of a Legendrian knot representation L is defined as

$$\mathrm{tb}(L) = \mathrm{wr}(L) - \#(\text{right cusps}),$$

where the writhe $\mathrm{wr}(L)$ is the number of positive crossings minus the number of negative crossings.

Definition 2.12. Denote by $\mathcal{L}(K)$ the set of all Legendrian representations of a knot K. Define the maximal Thurston-Bennequin number to be

$$\overline{\mathrm{tb}}(K) = \max_{L \in \mathcal{L}} \mathrm{tb}(L).$$

Legendrian knots have a convenient relationship to grid diagrams. One could obtain a Legendrian representation of a knot K embedded in a grid diagram by rotating it 45° clockwise and smoothing out the corners (using cusps when necessary).

Theorem 2.13 ([13])**.** *The Thurston-Bennequin number of a knot is related to the grid index by the following equation:*

$$-\alpha(K) \leq \overline{tb}(K) + \overline{tb}(\overline{K}),$$

where $\overline{K}$ denotes the mirror image of K.

Moreover, equality is conjectured to hold.

Conjecture 2.14 ([23]). *For every knot K, $-\alpha(K) = \overline{tb}(K) + \overline{tb}(\overline{K})$.*

There are no counterexamples to this conjecture up to crossing number 11 [23]. If this conjecture is true, Thurston-Bennequin number should detect roughly the same properties of a knot as the arc index. A lower bound on the slice genus can be given using the Thurston-Bennequin number of any representation of the knot.

Theorem 2.15 ([22]). *Given a Legendrian representation L of a knot K, the Thurston-Bennequin number is related to the slice genus by the following equation:*

$$g_s(K) \geq \frac{1}{2}(\mathrm{tb}(L) + |r(L)| + 1),$$

where $r(L)$ is half the difference between the number of left cusps and right cusps.

This inequality is derived from Lisca and Matić [24] and Akbulut and Matveyev [25]'s work on the Thurston-Bennequin number. We will refer to Theorem 2.15 as the Slice-Bennequin Inequality.

3 Methods to Lower Bound the Grid Index

Another common way knots can be represented in a grid is by using a knot mosaic. These knot representations are related to grid diagrams, but they have looser conditions.

Definition 3.1. A *knot mosaic* of a knot K or link L is an $n \times n$ grid filled with the following tiles such that the tiles form K or L (see Figure 3).

Figure 3: Tiles in a knot mosaic [26].

The main difference between knot mosaics and grid diagrams is that unlike grid diagrams, knot mosaics allow multiple strands of a knot to be in the same row or column and the overstrand at each crossing can be either horizontal or vertical.

In [9], Manolescu and Piccirillo gave the mosaic diagrams of five knots that, if smoothly slice, were candidate knots for a construction that could yield exotic 4-spheres, potential counterexamples to the smooth four-dimensional Poincaré conjecture. These diagrams are shown in Figure 4. All five knots are hyperbolic knots and are thus prime [9]. We give bounds for the grid indices of these knots. Given the mosaic diagrams of the Manolescu-Piccirillo knots, each crossing can be thought of as a horizontal or vertical bridge based on which strand lies on top. The goal is to use local modifications to correct all horizontal bridges to vertical bridges while maintaining as few distinct horizontal strands as possible.

See Table 3 in Appendix A for the enumeration of all prime knots with grid index less than 13 or crossing number less than 14. Since no prime knots with grid index less than 13 have a crossing number less than 29, the Manolescu-Piccirillo knots must all have grid index at least 13. The grid diagrams achieving the upper bounds of the Manolescu-Piccirillo knots in Theorem 3.2 are shown in Figures 8, 9, 10, 11, and 12. The upper bound constructions can be found in Appendix A.

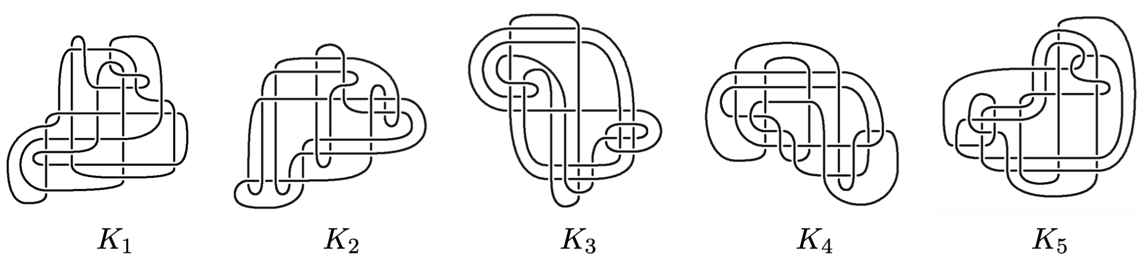

Figure 4: Manolescu and Piccirillo's knots [9].

Theorem 3.2. *The grid index of the Manolescu-Piccirillo knots can be bounded as follows:*

$$13 \leq g(K_1) \leq 20,$$

$$13 \leq g(K_2) \leq 20,$$

$$13 \leq g(K_3) \leq 21,$$

$$13 \leq g(K_4) \leq 22,$$

$$13 \leq g(K_5) \leq 25.$$

We also calculated lower bounds on the grid index based on several other strategies, including several knot invariants. Let $c(K)$ denote the crossing number of knot K. A square-root bound is obtained by recognizing that no crossings can occur on the boundary of the grid diagram, yielding the inequality $c(K) \leq (g(K)-2)^2$, which is weaker than the lower bound of 13. In addition, grid diagrams give rise to a knot Floer homology. A deduction from the work of Ozsváth, Stipsicz, and Szabó [27] shows that the quantity $(g(K)+1)!$ is greater than the rank of the knot Floer homology of K. This bound is also weaker than the lower bound of 13. Morton and Beltrami [28] gave a lower bound on the grid index using the Kauffman polynomial.

Theorem 3.3 ([28]). *Let the spread* spr_v *denote the difference between the maximum and minimum exponents of* v. *Let* $f_L(v,z)$ *denote the Kauffman polynomial of the link* L. *Then*

$$\alpha(L) \geq \mathrm{spr}_v(f_L(v,z)) + 2.$$

The Kauffman polynomial is computationally expensive to calculate for the large number of crossings in the Manolescu-Piccirillo knots, as it relies on recursive skein relations [29]. The last method is to write the knot as a connected sum of two smaller knots and use the grid indices of each portion to find a lower bound on the grid index. This method does not work for the Manolescu-Piccirillo knots, as they are prime and any way to express these knots as a connect sum must have the unknot as a summand. See Table 2 for a table outlining the drawbacks of the lower bounds on $g(K)$ due to these knot invariants and strategies.

Strategy Used	Drawback
Square-root bound	Too weak
Rank of Heegaard Floer homology	Too weak
Kauffman Polynomial	Computationally expensive
Cutting the knot	Weak for prime knots

Table 2: Methods used for the lower bound.

The weakness of these bounds suggests that some property of the Manolescu-Piccirillo knots is captured by the grid index but not other canonical knot invariants like the Heegaard Floer homology and the Kauffman polynomial. Although grid diagrams are simple ways to depict knots, they reflect subtle features of certain knots, as demonstrated by their relation to knot Floer homology and the maximal Thurston-Bennequin number, which detect subtler properties of knots.

4 Bounds on the Slice Genus of Potentially Slice Knots

In this section, we use the Slice-Bennequin Inequality to generate knots with large slice genus. Since the slice genus of a knot is bounded below by half the Thurston-Bennequin number, it is natural to consider grid diagrams where the Thurston-Bennequin number of a knot K is maximized. This method could be used to find bounds on the slice genera of torus knots and knots with small crossing number. For such knots, one way to achieve a large slice genus is to consider the relationship between grid index and Thurston-Bennequin number.

Theorem 4.1 ([30])**.** *The Thurston-Bennequin number of a knot K is maximized by a grid diagram of K with minimal grid size.*

Conversely, it is also natural to study the maximum possible value of the Thurston-Bennequin number given a fixed grid size. The maximum is conjectured to be achieved by a special type of knot.

Definition 4.2. Let p and q be positive integers. The (p, q)-*torus knot* $T(p, q)$ (or *link* if $\gcd(p, q) \neq 1$) is a knot embedded on the surface of a torus that revolves p times around the axis of symmetry of the torus and q times around the interior of the torus.

The Slice-Bennequin Inequality gives a lower bound on slice genus, and thus can be used to find slice genera of certain families of knots. One such case is the positive torus knot $T(p, q)$ with $p, q > 0$, in which the slice genus was conjectured to be $\frac{1}{2}(p-1)(q-1)$ [31]. In

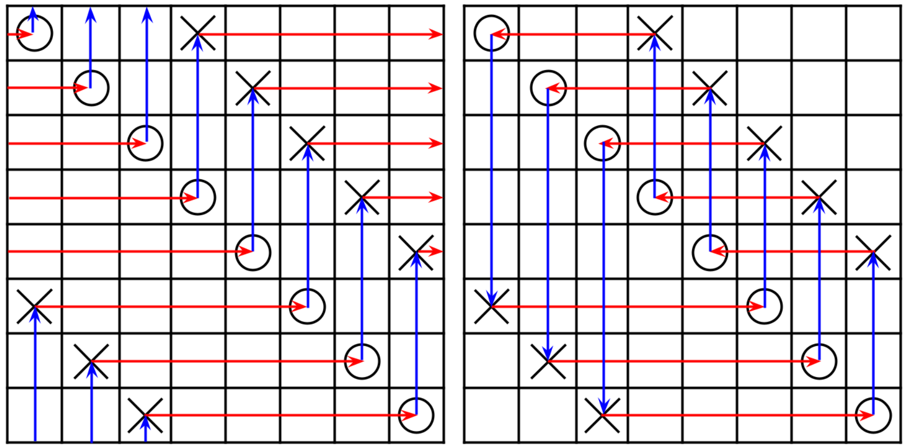

Figure 5: Gluing together opposite edges of the figure on the left yields the torus knot $T(5,3)$. The figure on the right is a grid diagram of $T(5,3)$.

1993, Kronheimer and Mrowka [14] used gauge theoretic methods to prove this conjecture. In 2004, Rasmussen [15] found a purely combinatorial proof using Khovanov homology. To gauge the effectiveness of using the grid index to give a bound on this slice genus, we give a simpler combinatorial proof of the Milnor Conjecture.

Matsuda [13] and Etnyre and Honda [32] showed that the grid index of a torus knot $T(p,q)$ with $p,q>0$ is $p+q$ and the maximal Thurston-Bennequin number of its mirror is $-pq$, respectively. The maximal Thurston-Bennequin number of $T(p,q)$ then satisfies

$$\begin{aligned}\overline{\mathrm{tb}}(T(p,q)) &\geq -\alpha(T(p,q)) - \overline{\mathrm{tb}}\left(\overline{T(p,q)}\right)\\ &= -(p+q)-(-pq)\\ &= pq-p-q.\end{aligned}$$

Note that this lower bound matches the lower bound obtained by taking the Lee et al. [26] construction of a minimal grid diagram of $T(p,q)$, shown in Figure 6. Applying the Slice-Bennequin Inequality, we obtain

$$g_s(T(p,q)) \geq \frac{1}{2}(pq-p-q+1),$$

confirming the slice genus calculated by Kronheimer and Mrowka [14] and Rasmussen [15]. This shows that our method is effective at giving strong bounds on the slice genus.

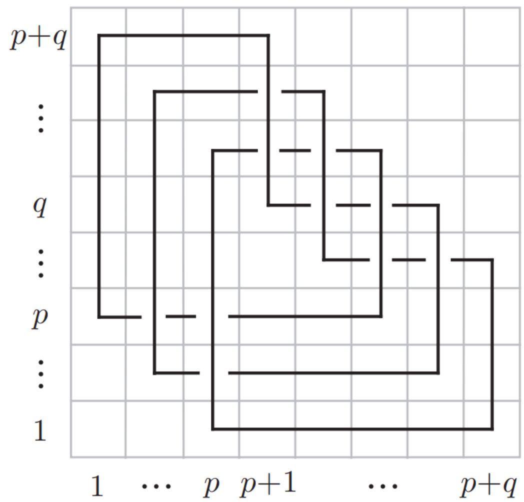

Figure 6: A grid diagram for $T(p,q)$ of size $p+q$ [26]. Notice that the rotation number is zero, achieving equality.

The second strategy is to find a knot K with very small maximal Thurston-Bennequin number. If K satisfies Conjecture 2.14, $\overline{\text{tb}}(\overline{K})$ will have a very large Thurston-Bennequin number, from which $\overline{K}$ must have large slice genus. This method is useful because it could be used to characterize knots with positive slice genus among all knots with small crossing numbers. Another potential strategy is to find knots K such that the maximal Thurston-Bennequin numbers of K and $\overline{K}$ are as far apart as possible, as such knots give a large positive maximal Thurston-Bennequin number and thus a large slice genus. If Conjecture 2.14 is true, then torus knots should achieve the largest difference between $\overline{\text{tb}}(K)$ and $\overline{\text{tb}}(\overline{K})$.

Moreover, we use the relationship between the grid diagram and the Thurston-Bennequin number to construct an infinite sequence of knots whose slice genera approach infinity. Given a knot K, let K^n denote the connected sum of K^{n-1} and K, where $K^1 = K$.

Theorem 4.3. *If a knot K has a Legendrian knot representation L satisfying* $\text{tb}(L)+|r(L)| \geq -\frac{1}{2}$*, the sequence*

$$g_s(K^1), g_s(K^2), g_s(K^3), \ldots$$

approaches infinity. Specifically, $g_s(K^n) \geq \frac{1}{4}(n-2)$ for all positive integers n.

Proof. Let L^n denote the connected sum of n copies of the Legendrian knot L using the recursive definition $L^n = L^{n-1} \# L$, where # denotes the connected sum operation. Observe that the connected sum of two Legendrian knots L_1 and L_2 satisfies the following two properties [22].

- $\mathrm{tb}(L_1 \# L_2) = \mathrm{tb}(L_1) + \mathrm{tb}(L_2) + 1$.
- $r(L_1 \# L_2) = r(L_1) + r(L_2)$.

By induction, $\mathrm{tb}(L^n) = n\,\mathrm{tb}(L) + (n-1)$ and $|r(L^n)| = n|r(L)|$. Using the Slice-Bennequin Inequality, we obtain

$$g_s(K^n) \geq \frac{1}{2}\left(\mathrm{tb}(L^n) + |r(L^n)| + 1\right) = \frac{1}{2}\left(n\,\mathrm{tb}(L) + n|r(L)| + (n-1)\right).$$

Since $\mathrm{tb}(L) + |r(L)| \geq -\frac{1}{2}$,

$$\frac{1}{2}\left(n\,\mathrm{tb}(L) + n|r(L)| + (n-1)\right) \geq \frac{1}{2}\left(-\frac{1}{2}n + (n-1)\right) = \frac{1}{4}(n-2).$$

Hence, the lower bound for $g_s(K^n)$ is at least $\frac{1}{4}(n-2)$, which approaches infinity as n approaches infinity. □

This sequence is especially interesting when the knot K has Alexander polynomial 1. The Alexander polynomial of a connected sum of two Legendrian knots L_1 and L_2 is equal to the product of the Alexander polynomials of each individual knot. Therefore, the connected sums of any knot K with Alexander polynomial 1 would also have Alexander polynomial 1 and are thus topologically slice.

Using Knotscape [33], we checked all knots with up to 16 crossings that have Alexander polynomial 1. Among these knots, there is one prime knot $16n_{196836}$ whose corresponding grid diagram in [34] gives rise to a Legendrian knot representation satisfying the condition for Theorem 4.3.

Theorem 4.4 (Main result)**.** *The sequence*

$$(16n_{196836})^3, (16n_{196836})^4, (16n_{196836})^5, \ldots$$

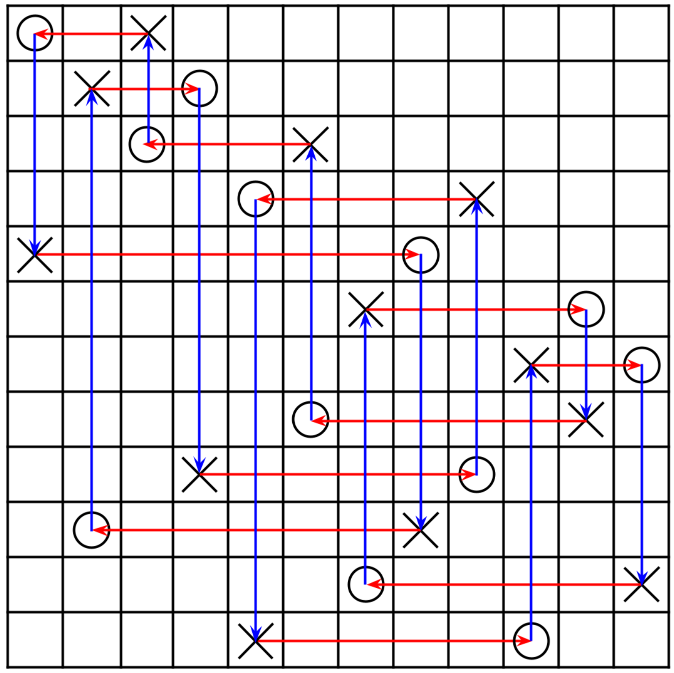

Figure 7: The grid diagram of $16n_{196836}$.

consists of knots that are topologically slice and not smoothly slice. Moreover, the slice genera of the knots in the sequence approach infinity.

Proof. A grid diagram for $16n_{196836}$ is shown in Figure 7. The Legendrian knot representation $16n'_{196836}$ obtained by rotating it 45° clockwise and smoothing out the corners has 5 right cusps and a writhe of 4, from which we obtain $\text{tb}(16n'_{196836}) = -1$. Additionally, $|r(16n'_{196836})| = \frac{1}{2}$, so the condition for Theorem 4.3 is satisfied. Applying Theorem 4.3, we found that for $n \geq 3$, the slice genus of $(16n_{196836})^n$ is at least 1 and therefore $(16n_{196836})^n$ is not smoothly slice. Moreover, since $16n_{196836}$ has Alexander polynomial 1, every knot in the sequence also has Alexander polynomial 1. By Theorem 2.6, all of these knots are topologically slice. Hence, the knots in this sequence are topologically slice but not smoothly slice, and they have slice genera that approach infinity. □

There are several important implications of our main result. Applying knot surgery on these knots yield exotic $\mathbb{R}^4$ [16]. Mathematicians have known of the existence of exotic $\mathbb{R}^4$'s for decades using gauge theoretic techniques, but our recursive construction using our strategy of connecting the slice genus to the grid index via the Thurston-Bennequin number is novel in finding infinite families of these exotic structures that can only exist in four-dimensional space [35].

5 Conclusion

In our project, we devised a new strategy to provide lower bounds on the slice genera of knots by using known minimal grid diagrams. we also determined a relation between the size of a grid diagram and its maximal Thurston-Bennequin number and explored the effectiveness of this bound in relation to other bounds and knot invariants. By constructing a sequence of knots via connected sums, we found an infinite sequence of topologically slice knots with Alexander polynomial 1 that have slice genera approaching infinity. Since the knots in this infinite sequence are topologically slice but not smoothly slice, they generate a new infinite family of exotic $\mathbb{R}^4$'s [18] and could potentially be used to construct exotic 4-spheres [8].

Our method can also be extended to recursive constructions other than the connected sum. For example, *knot infection* (see [36] for more details) could be used in place of connected sums. Setting the original knot to have Alexander polynomial 1 and repeatedly applying infection gives an infinite sequence of knots with a fixed Alexander polynomial and slice genera approaching infinity. If the fixed Alexander polynomial is 1 as well, this construction could also potentially yield exotic $\mathbb{R}^4$'s.

It would be interesting to consider using existing slice genus data in [37] and explore the maximal Thurston-Bennequin numbers of knots similar to torus knots. To this extent, it could be possible to find a metric that measures how similar a knot is to a torus knot. Knots similar to a torus knot should have Thurston-Bennequin numbers close to the maximum and thus must pass the conditions in Theorem 4.3. As a result, Theorem 4.3 could also be applied to such knots.

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Appendix A Constructions for Theorem 3.2

	Grid Index											
Crossings	5	6	7	8	9	10	11	12	13	14	15	Subtotal
3	1											1
4		1										1
5			2									2
6				3								3
7					7							7
8			1	2		18						21
9				2	6		41					49
10				1	9	32		123				165
11					4	46	135		367			552
12					2	48	211	627		1288		2176
13						49	399	1412	3250		4878	9988
14						17	477	3180				
15					1	22	441	6216				
16						7	345	7955				
17						1	192	10283				
18							75	8584				
19							12	6063				
20							3	3540				
21							3	1284				
22								761				
23								124				
24							1	132				
25								39				
26								3				
27												
28								1				
Subtotal	1	1	3	8	29	240	2335	50327				

Table 3: Enumeration of prime knots K with $g(K) < 13$ or $c(K) < 14$ [34].

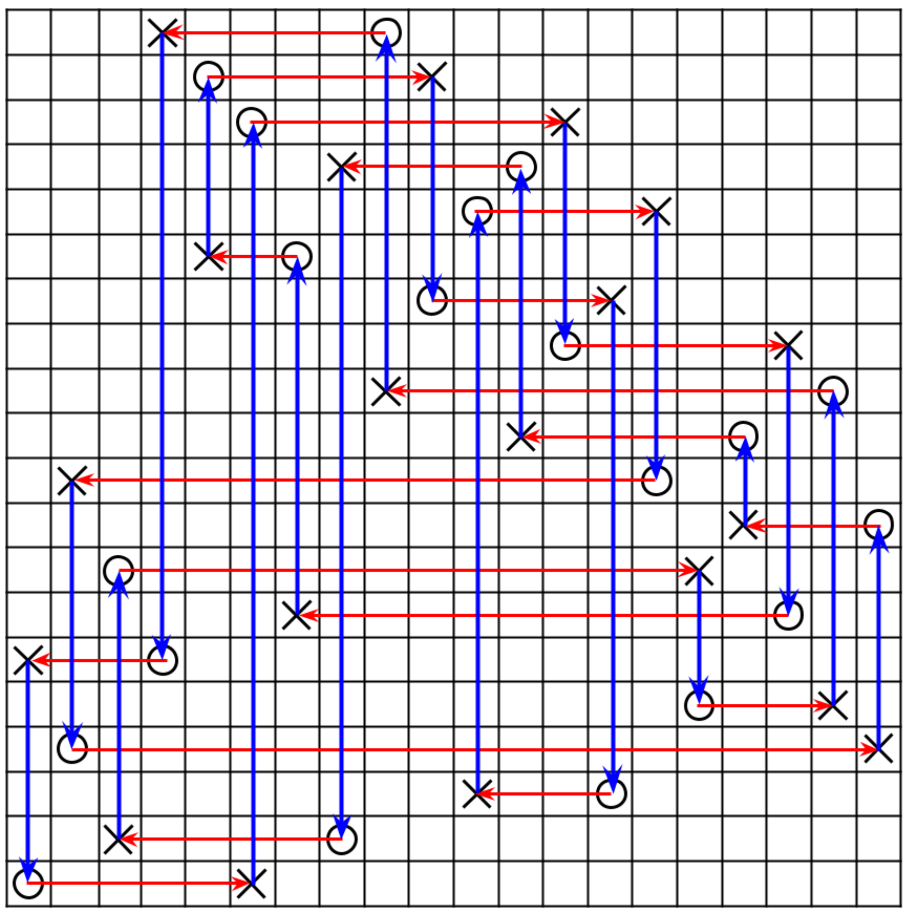

Figure 8: Grid diagram of K_1 with size 20.

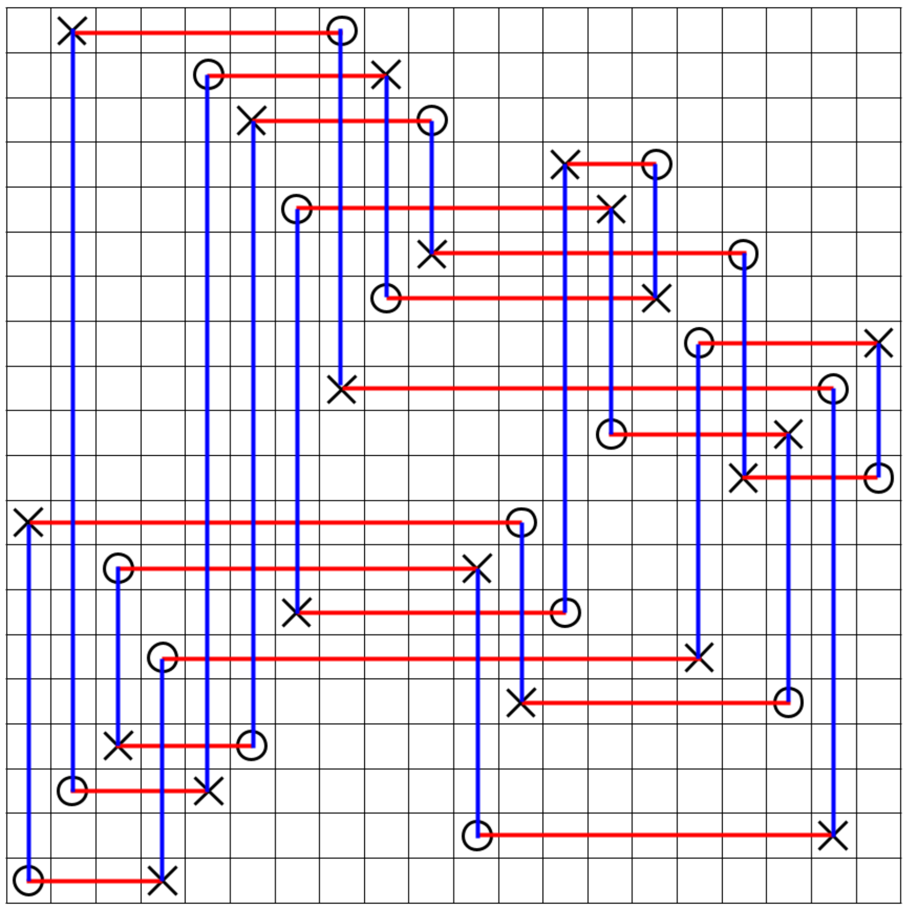

Figure 9: Grid diagram of K_2 with size 20.

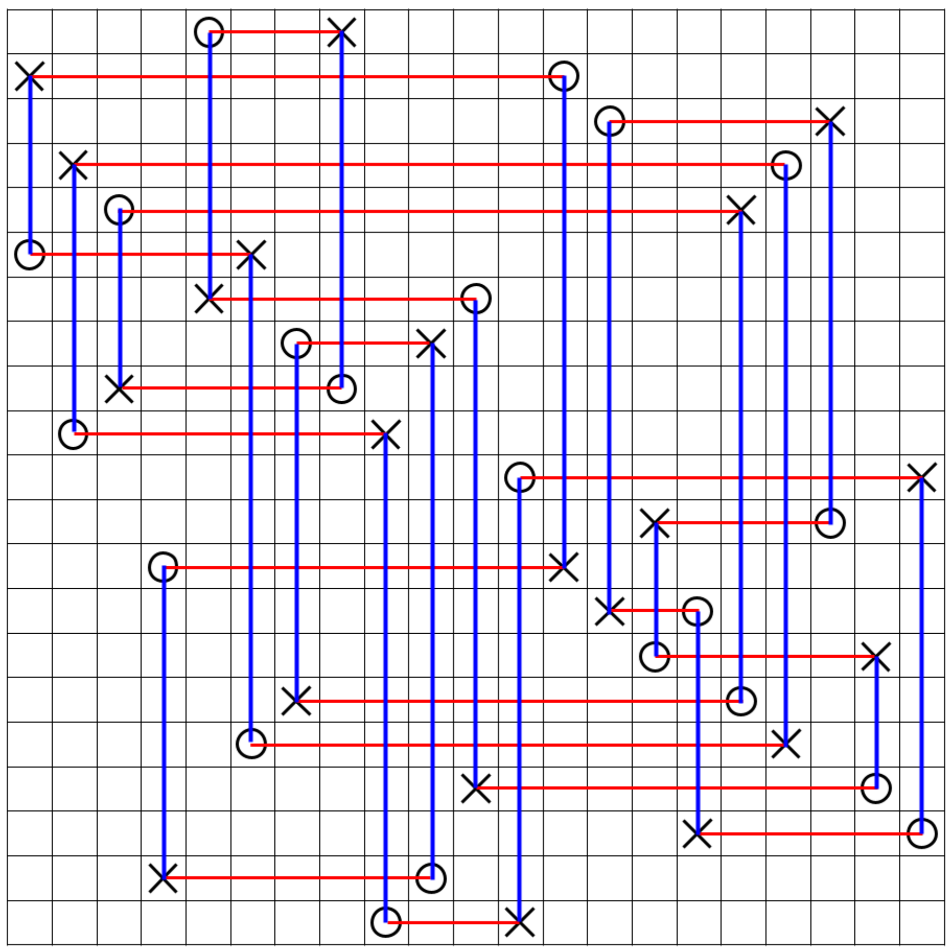

Figure 10: Grid diagram of K_3 with size 21.

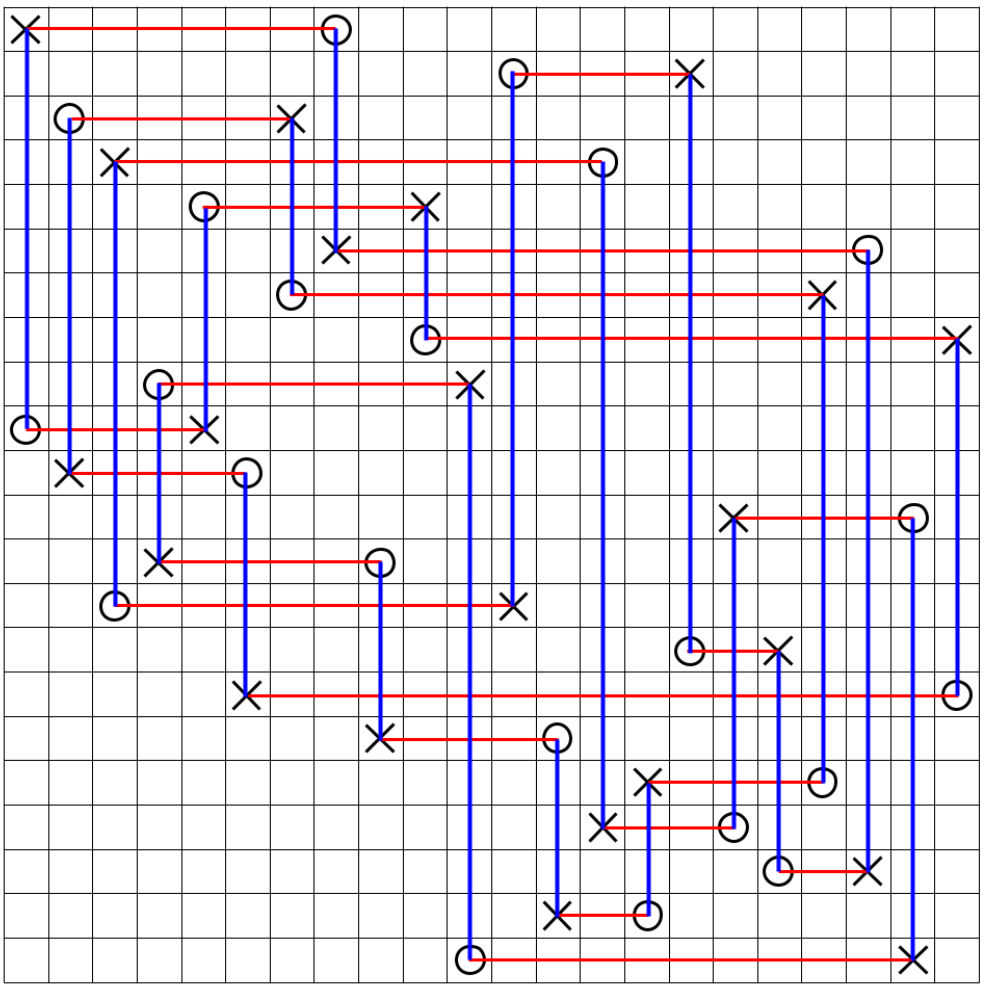

Figure 11: Grid diagram of K_4 with size 22.

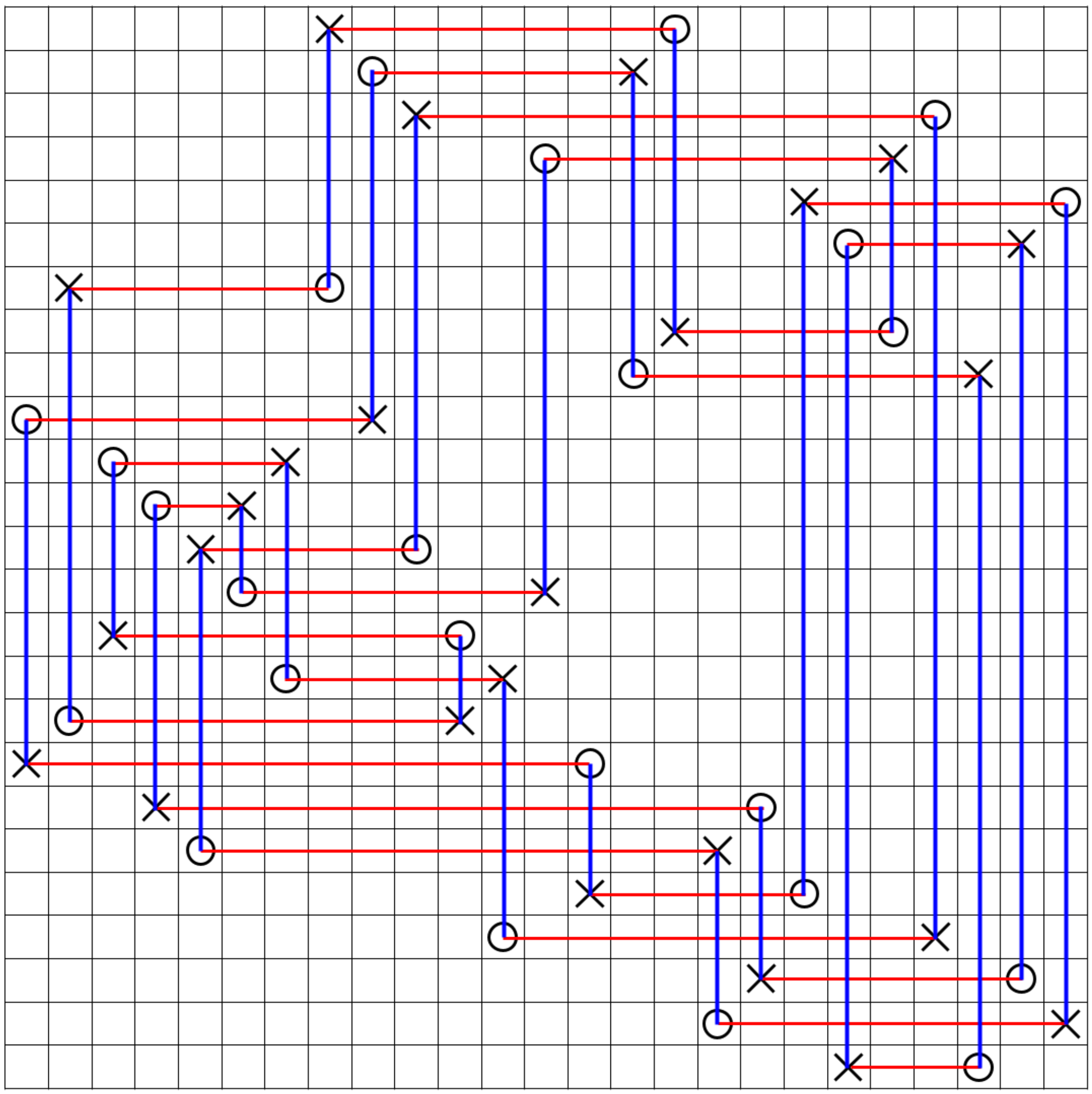

Figure 12: Grid diagram of K_5 with size 25.