Dynamical Torsion for Surfaces of Constant Negative Curvature

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Abstract

The Ruelle zeta function, originating from the field of dynamical systems and related to differential geometry and spectral theory, is the focus of our study on compact oriented Riemannian surfaces Σ of constant negative curvature. Using a previous result by Chaubet and Dang that links the dynamical torsion to the normalized coefficient of the Ruelle zeta function at zero, we compute the torsion of finite-dimensional resonant states. Specifically, we determine the torsion to be $\det(M)h$, where the automorphism M acts on a basis of the de Rham cohomology group $H^1(\Sigma)$ and h is a fixed element of $\det(H^\bullet)$, independent of surface perturbations. This computation provides a rigorous expression for the torsion in terms of the geometric and dynamical data of the surface, informing future research for surfaces of variable negative curvature.

Summary

We consider surfaces, which are two-dimensional objects that look like a plane in small regions around every point but may have a complex overall structure, like a sphere or a torus. We aim to better understand surfaces with constant negative curvature, which resemble a saddle shape. Using a result that links the Ruelle zeta function evaluated at zero to a property of the surface called dynamical torsion, we directly compute the torsion associated with our surface. This allows us to observe how the coefficient of the Ruelle zeta function changes with different surface modifications. Future directions of our research include applying our methodology for surfaces of variable negative curvature and higher-dimensional objects.

1 Introduction

A common feature of zeta functions is that they encode information about structures and are usually related to the established "points of interest" of various associated objects. In general, zeta functions are special functions that arise in number theory, combinatorics, and mathematical physics, capturing significant properties of these areas. In this paper, we examine the Ruelle zeta function [1], which originates from dynamical systems but also has connections to differential geometry. The Ruelle zeta function is defined analogously to the Riemann zeta function, replacing primes p with *primitive closed geodesics* (Figure 1).

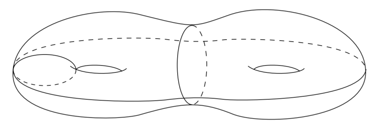

Figure 1: A double torus with examples of primitive closed geodesics [1]

Definition 1.1. Let (Σ, g) be a compact oriented Riemannian surface of negative curvature and let $\mathcal{G}$ be the set of primitive closed geodesics on Σ (counted with multiplicity). For $\gamma \in \mathcal{G}$ denote by l_γ its length. The Ruelle zeta function $\zeta_R : \mathbb{C} \to \mathbb{C}$ is defined as

$$\zeta_R(s) := \prod_{\gamma \in \mathcal{G}} (1 - e^{-sl_\gamma}).$$

We can extract topological information from the Ruelle zeta function, namely, the *Euler characteristic (χ)*, which describes the structure of a topological space. The Euler characteristic is connected to the *genus* g, or the number of "holes" a surface has, by the relationship $\chi = 2 - 2g$. For example, a sphere has $\chi = 2$, while the double torus in Figure 1 has $\chi = -2$.

For a Riemannian surface (Σ, g) of negative curvature, near $s = 0$, the following holds

$$\zeta_R(s) = (c + O(s))s^{-\chi(\Sigma)},$$

where $\chi(\Sigma)$ is the Euler characteristic of the surface Σ. This result was previously shown only for surfaces of constant negative curvature [2] and later extended by Dyatlov and Zworski [3] to all Riemannian surfaces of negative curvature.

Definition 1.2. We define the *dynamical torsion* τ_ϑ [4] as the product

$$\tau_\vartheta(\rho)^{(-1)^q} = \pm \underbrace{\tau(C^\bullet, \Gamma_\vartheta)^{(-1)^q}}_{\text{finite dimensional torsion}} \times \underbrace{\lim_{s \to 0} s^{\chi(\Sigma)} \zeta_R(s)}_{\text{renormalized } \zeta_R \text{ at } s=0},$$

where ϑ is a contact form on Σ, while X is the Reeb vector field of ϑ, and $q = \frac{\dim(M)-1}{2}$ is the dimension of the unstable bundle of X.

The torsion $\tau(C^\bullet, \Gamma_\vartheta)^{(-1)^q}$ of a finite dimensional complex $C^\bullet$, a collection of vector spaces, will be an element of the det $H^\bullet(M, \rho)$ (see Section 2.6) and may vary under perturbations of surface Σ. However, the next result by Chaubet and Dang [4] actually shows that the dynamical torsion remains constant under perturbations.

Theorem 1.1 (Chaubet and Dang [4]). *Let (Σ, ϑ) be a contact manifold such that $(\vartheta_\tau)_{\tau\in(-\varepsilon,\varepsilon)}$ is a smooth family in the space of contact forms on Σ whose Reeb vector field induces an Anosov flow. Then $\partial_\tau \log \tau_{\vartheta_\tau}(\rho) = 0$ for any $\rho \in Rep_{\mathrm{ac}}(\Sigma, d)$.*

We compute the torsion at $t = 0$ for a surface of constant negative curvature, arriving at our main result, Theorem 1.2, which expresses torsion in terms of M, an automorphism of a de Rham cohomology group $H^1(\Sigma)$ of differential 1-forms on Σ, and a fixed element h of the determinant line of the cohomology $H^\bullet(S\Sigma)$. Rigorous definitions will be provided in Section 2.6.

Theorem 1.2. *Let $(\Sigma = \Sigma_0, g)$ be a compact oriented Riemannian surface of constant negative curvature. Then $\tau(C^\bullet, \Gamma_\vartheta)|_{t=0} = \det(M)h$.*

In Section 2, we introduce and formalize definitions relating to Fourier analysis, geodesic flow, differential forms, and dynamical torsion. In Section 3, we perform computations motivated by Section 2.6 to show properties of vector fields related to geodesic flow and ultimately compute how automorphism M acts on a basis of de Rham cohomology group $H^1(\Sigma)$. In Section 4, we present our conclusions and directions for further research.

2 Preliminaries

Let (Σ, g) be a compact oriented Riemannian surface with constant negative curvature. Denote by $S\Sigma$ the sphere bundle of (Σ, g), that is,

$$S\Sigma = \{(x, v) \in T\Sigma : |v|_{g(x)} = 1\},$$

where $T\Sigma$ is the tangent bundle of Σ. Locally, $S\Sigma$ looks like $W \times S^1$, where $W \subset \Sigma$ is a local coordinate chart on Σ and S^1 is the 1-dimensional circle.

2.1 Harmonic analysis

We introduce the notation for Fourier coefficients of a function f on $S\Sigma$.

Definition 2.1. For a function $f : S\Sigma \to \mathbb{C}$, let

$$\hat{f}_k(x) = \int_0^1 f(x, \theta)e^{-ik\theta}d\theta \quad \text{and} \quad f_k(x, \theta) = \hat{f}_k e^{ik\theta}. \tag{1}$$

We thus have

$$f = \sum_{k=-\infty}^{\infty} f_k.$$

2.2 Geodesic flow

Using conventions identical to [5, 6], we consider X to be the vector field that generates geodesic flow on $S\Sigma$. We also consider horocyclic vector fields U_+, U_- on $S\Sigma$. We further define

$$X_\perp := \frac{1}{2}(U_+ + U_-) \quad \text{and} \quad V := \frac{1}{2}(U_+ - U_-).$$

We consider $[A, B] = AB - BA$ to be the commutator of A and B. It has been shown [5] that X, U_+, U_-, $X_\perp$, and V satisfy commutation relations

$$[X, U_\pm] = \pm U_\pm, \quad [X, V] = X_\perp, \quad [X, X_\perp] = V,$$

which will be useful in showing Lemma 3.1.

We also define vector fields $\eta_\pm = \frac{1}{2}(X \pm iX_\perp)$, which behave in a special way in relation to Fourier coefficients. If $D_k = \{g \in \Omega^0(S\Sigma) \mid Vg = ikg\}$ for $k \in \mathbb{Z}$, then $f_k \in D_k$ for all k as seen in [5]. Furthermore, if $f_k \in D_k$, then $\eta_+ f_k \in D_{k+1}$ and $\eta_- f_k \in D_{k-1}$ as shown in [5].

2.3 Hodge star operator and harmonic 1-forms

Consider $z = x_1 + ix_2$. It is then intuitive to define $dz = dx^1 + idx^2$, $d\overline{z} = dx^1 - idx^2$, $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, and $\partial_{\overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$. Then dz and $d\overline{z}$ span $\Omega^1(\Sigma)$.

We define $\star : \Omega^1(\Sigma) \to \Omega^1(\Sigma)$ to be the Hodge star operator, as in [7].

Definition 2.2. Given 1-form $\omega = gdz + hd\overline{z}$, define

$$\star\omega := -igdz + ihd\overline{z}.$$

Remark. In general, for n-dimensional manifolds M, the Hodge star $\star$ takes elements of $\Omega^k(M)$ to $\Omega^{n-k}(M)$.

Example 2.1. On 2-dimensional Σ, it can be shown that $\star dx^1 = dx^2$ and $\star dx^2 = -dx^1$.

Definition 2.3. We say that form ω is *holomorphic* if $\partial_{\overline{z}}\omega = 0$, and that ω is *anti-holomorphic* if $\partial_z\omega = 0$.

From Definition 2.2, we can see that for holomorphic forms ω, we have $\star\omega = -i\omega$, while for anti-holomorphic forms ω, the opposite holds: $\star\omega = i\omega$.

Definition 2.4. A 1-form ω is *harmonic* if and only if $d\omega = d(\star\omega) = 0$.

2.4 Projections, pullback, and pushforward on S and $S\Sigma$

To relate points and differential forms between the manifolds S and $S\Sigma$, we introduce the operators of projection, pullback, and pushforward, which enable the transfer of geometric and analytical information across these surfaces.

Definition 2.5. We define *projection map* $\pi : S\Sigma \to \Sigma$ such that for a point (x, θ) in $S\Sigma$, we have $\pi(x, \theta) = x$.

Definition 2.6. For a 1-form ω on Σ, the *pullback* $\pi^*(\omega)$ of ω is a 1-form on $S\Sigma$ such that

$$(\pi^*\omega)Y_{x_0,\theta_0} = \omega_{x_0}(d\pi|_{x_0,\theta_0} Y_{x_0,\theta_0})$$

for all vector fields Y.

Alternatively, if $\omega = f(x)dx^1 + g(x)dx^2$, we can write

$$(\pi^*\omega) = f(\pi(x, \theta))dx^1 + g(\pi(x, \theta))dx^2.$$

Definition 2.7. For $x \in \Sigma$, the *pushforward* $\pi_* : \Omega^k(S\Sigma) \to \Omega^{k-1}(\Sigma)$ is defined as

$$(\pi_*(\omega))_x = \int_{S^1_x} i_{\partial_{x_1}}(\omega)dx^1 + \int_{S^1_x} i_{\partial_{x_2}}(\omega)dx^2.$$

Definition 2.8. Define 1-form contact form $\alpha \in \Omega^1(S\Sigma)$ as acting on vector fields Y as follows:

$$\alpha_{(x_0,\theta_0)}(Y_{(x_0,\theta_0)}) = \left\langle v(\theta_0), d\pi|_{x_0,\theta_0}\left(Y_{(x_0,\theta_0)}\right)\right\rangle.$$

Alternatively, we can write

$$\alpha = \cos\theta_0 dx^1 + \sin\theta_0 dx^2. \tag{2}$$

2.5 Harmonic 1-forms and resonant states

We know that $\pi^* : H^1(\Sigma) \to H^1(S\Sigma)$ is an isomorphism [3, Lemma 2.4]. Combining this with [3, Lemma 3.4], we can establish a isomorphism between harmonic 1-forms ω on Σ and resonant states u on $S\Sigma$.

Lemma 2.1. *For a harmonic 1-form $\omega \in \Omega^1(\Sigma)$, there exists a unique resonant state $u \in \Omega^1(S\Sigma)$ such that $u(X) = u(U_-) = 0$. Moreover, this resonant state can be written as*

$$u = \pi^*\omega + df$$

for a function f on $S\Sigma$.

2.6 Dynamical torsion and torsion for a complex chain

A finite-dimensional cochain complex $(C^\bullet, \partial)$ can be thought of as a chain of vector spaces and has the form

$$(C^\bullet, \partial) : 0 \xrightarrow{\partial} C^0 \xrightarrow{\partial} C^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} C^n \xrightarrow{\partial} 0,$$

where C^j are finite-dimensional vector spaces.

In Definition 1.2, dynamical torsion is defined as the product of finite-dimensional torsion of the de Rham cochain complex and the renormalized Ruelle zeta function at $s = 0$. From Theorem 1.1, we see that dynamical torsion does not change with small perturbations to the surface Σ. We consider complexes of differential forms on S and $S\Sigma$

$$C_\bullet(\Sigma) : 0 \xrightarrow{d} \Omega_0(\Sigma) \xrightarrow{d} \Omega_1(\Sigma) \xrightarrow{d} \Omega_2(\Sigma) \xrightarrow{d} 0$$

and

$$C_\bullet(S\Sigma) : 0 \xrightarrow{d} \Omega_0(S\Sigma) \xrightarrow{d} \Omega_1(S\Sigma) \xrightarrow{d} \Omega_2(S\Sigma) \xrightarrow{d} \Omega_3(S\Sigma) \xrightarrow{d} 0.$$

Definition 2.9. A form $\omega \in \Omega^k(M)$ is *closed* if $d\omega = 0$ and *exact* if $\omega = df$ for some $f \in \Omega^{k-1}(M)$.

Since $d^2 = 0$, all exact forms are closed.

Let $H^k(M)$ be the de Rham cohomology groups [3, 4], which are the quotient of the space of closed forms by the space of exact forms, so

$$H^k(M) := \frac{\{u \in \Omega^k(M) \mid du = 0\}}{\{dv \mid v \in \Omega^{k-1}(M)\}}.$$

The dimensions of vector spaces H^k,

$$b_k(M) := \dim H^k(M),$$

are called k-th *Betti numbers*. If the genus of Σ is g, then $b_1(\Sigma) = b_1(S\Sigma) = 2g$ holds [7].

Let $H^\bullet(M) = \bigoplus_{j=1}^{2g} H^j(M)$.

The determinant of a vector space V is a one-dimensional vector space $\det(V) = \Lambda^{dimV} V$.[1]

Definition 2.10. We define the *determinant line* of a complex $C^\bullet$ by

$$\det(C^\bullet) = \bigotimes_{j=0}^{n} \det\big(C^j\big)^{(-1)^j}.$$

In the case of complex $H^\bullet(S\Sigma)$,

$$\det(H^\bullet(S\Sigma)) = \det\big(H^0(S\Sigma)\big) \otimes \det\big(H^1(S\Sigma)\big)^{-1} \otimes \det\big(H^2(S\Sigma)\big) \otimes \det\big(H^3(S\Sigma)\big)^{-1}.$$

Define chirality operator $\Gamma : H^k(S\Sigma) \to H^{3-k}(S\Sigma)$ as a linear map:

$$\Gamma : \begin{cases} 1 \to \alpha \wedge d\alpha \\ u \to \alpha \wedge u \\ \alpha \wedge u \to u \\ \alpha \wedge d\alpha \to 1. \end{cases}$$

We have that

- π^* is an isomorphism between $H^1(\Sigma)$ and $H^1(S\Sigma)$,
- Γ is an isomorphism between $H^1(S\Sigma)$ and $H^2(S\Sigma)$,
- π_* is an isomorphism between $H^2(S\Sigma)$ and $H^1(\Sigma)$ (Figure 2).

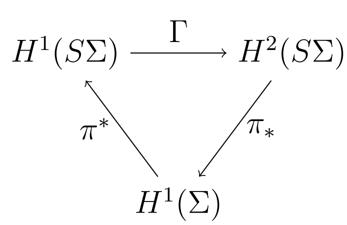

Figure 2: Isomorphisms between de Rham cohomology groups

We start by considering a basis $\{\omega_j\}_{j=1}^{2g}$ for $H^1(\Sigma)$. We can construct $u_j = \pi^*\omega_j + df_j$ in $H^1(S\Sigma)$. Since π^* is an isomorphism, $\{u_j\}_{j=1}^{2g}$ is a basis of $H^1(S\Sigma)$. We can say that $u_j = \pi^*\omega_j$, since if two objects in $H^1(S\Sigma)$ have a difference of df_j, an exact form, they are in the same equivalence class.

[1]Here, we use Λ in place of $\wedge$ to avoid confusion with the notation we later use for the wedge product of differential forms.

Now, $\{\Gamma(u_j)\}_{j=1}^{2g} = \{\alpha \wedge u_j\}_{j=1}^{2g}$ is a basis of $H^2(S\Sigma)$. Finally, $\{\pi_*(\alpha \wedge u_j)\}_{j=1}^{2g}$ is a basis back on $H^1(\Sigma)$, so it must equal to the original basis $\{\omega_j\}_{j=1}^{2g}$.

Using the definition of torsion in [4],

$$\tau(C_\bullet(S\Sigma), \Gamma) = \pm 1 \otimes (u_1 \Lambda \ \ldots \ \Lambda \, u_{2g})^{-1} \otimes ((\alpha \wedge u_1) \Lambda \ \ldots \ \Lambda \, (\alpha \wedge u_{2g})) \otimes (\alpha \wedge d\alpha)^{-1}.$$

We know π_* is an isomorphism, so π_*^{-1} is well-defined. We can then fix an element h of $\det(H^\bullet(S\Sigma))$:

$$h = 1 \otimes (\pi^*\omega_1 \Lambda \ \ldots \ \Lambda \, \pi^*\omega_{2g})^{-1} \otimes ((\pi_*^{-1}\omega_1) \Lambda \ \ldots \ \Lambda \, (\pi_*^{-1}\omega_{2g})) \otimes (\alpha \wedge d\alpha)^{-1}$$

We look at what happens if we deform the surface Σ with time t, giving us surfaces Σ_t for $t \in (-\varepsilon, \varepsilon)$. Since h is an element of the determinant line $\det(H^\bullet(S\Sigma))$, $\tau_t = \lambda_t h$ for some $\lambda_t \in \mathbb{C}$.

We can pick a basis for $H^1(\Sigma)$ to contain g holomorphic and g anti-holomorphic forms $\{dh_1^1, \ldots, dh_1^g, dh_{-1}^1, \ldots, dh_{-1}^g\}$. We can consider how the map $M = \pi_* \circ \Gamma \circ \pi^* : H^1(\Sigma) \to H^1(\Sigma)$ (Figure 2) acts on the basis we just picked. To do this, we will focus on one specific $\pi_*(\alpha \wedge u_j)$, and we express it in terms of dh_1^j and dh_{-1}^j in Proposition 3.4.

3 Computing dynamical torsion

We aim to understand the Fourier coefficients f_1 and f_{-1} of a function f with the properties in Lemma 3.1.

Lemma 3.1. *For a holomorphic $\omega \in \Omega^1(\Sigma)$, the following properties hold:*

(3.1.1) $V\left[(\pi^*\omega)(X)\right] = i(\pi^*\omega)(X)$,

(3.1.2) $\eta_-\left[(\pi^*\omega)(X)\right] = 0$,

(3.1.3) $(\pi^*\omega(U_-) = i(\pi^*\omega)(X)$.

Proof. It is known [6] that for all 1-forms β and vector fields Y_1, Y_2,

$$d\beta = Y_1(\beta Y_2) - Y_2(\beta Y_1) - \beta([Y_1, Y_2]). \tag{3}$$

From taking $(Y_1, Y_2) = (V, X)$ and $\beta = \pi^*\omega$ in (3), we can get $V((\pi^*\omega)(X)) = -(\pi^*\omega)(X_\perp)$.

To show Property (3.1.1), we see

$$V((\pi^*\omega)(X)) = -(\pi^*\omega)(X_\perp) = -(\pi^*(\star\omega))(X) = i(\pi^*\omega)(X).$$

To show Property (3.1.2), from considering $(Y_1, Y_2) = (X, X_\perp)$ and $\beta = \pi^*\omega$ (3), we have

$$X((\pi^*\omega)X_\perp) - X_\perp((\pi^*\omega)(X)) = 0.$$

Applying $\pi^*\omega(X_\perp) = \pi^*(\star\omega)(X)$, we have

$$X((\pi^*(\star\omega))X) - X_\perp((\pi^*\omega)(X)) = 0$$
$$\frac{1}{2}(X - iX_\perp)(\pi^*\omega(X)) = 0$$
$$\eta_-\left[(\pi^*\omega)(X)\right] = 0$$

To show Property (3.1.3),

$$(\pi^*\omega)(U_-)f = (\pi^*\omega)(X_\perp - V)f = -(\pi^*\omega)(X_\perp) = i(\pi^*\omega)(X).$$

□

Recall that there is a one-to-one correspondence betweeen harmonic one-forms on Σ and resonant states on $S\Sigma$ (Lemma 2.1).

Lemma 3.2. *Let $u = \pi^*\omega + df$ be a resonant state for a harmonic 1-form ω on Σ. If $\omega = dh_1 + dh_{-1}$, where h_1 and h_{-1} are holomorphic and anti-holomorphic respectively, then*

$$f_1 = -2(\pi^* dh_1)(X) \quad \text{and} \quad f_{-1} = -2(\pi^* dh_{-1})(X). \tag{4}$$

Proof. We can express f on $S\Sigma$ as an infinite sum using Fourier series (Definition 2.1):

$$f(x,\theta) = \sum_{k\in\mathbb{Z}} \hat{f}_k e^{ik\theta} = \sum_{k\in\mathbb{Z}} f_k. \tag{5}$$

For convenience, define $a = (\pi^*\omega)(X)$. Then, we have $Xf = (\eta_+\eta_-)f = -a$. We start by substituting the Fourier series representation of f in (5) into property $(\eta_+ + \eta_-)f = -a$:

$$\sum_{k\in\mathbb{Z}} \eta_+ f_k + \eta_- f_k = -a.$$

We can conclude

$$\begin{cases} \eta_+ f_0 + \eta_- f_2 = -a & (k=1), & (6a)\\ \eta_+ f_{k-1} + \eta_- f_{k+1} = 0 & k \neq 1. & (6b)\end{cases}$$

By performing the same substitution for property (3.1.3), we can analogously conclude

$$\begin{cases} \eta_- f_2 - \eta_+ f_0 - f_1 = a & (k=1), & (7a)\\ \eta_- f_{k+1} - \eta_+ f_{k-1} - k f_k = 0 & k \neq 1. & (7b)\end{cases}$$

We subtract equalities (6a) and (7a) to get

$$2\cdot \eta_+ f_0 + f_1 = -2a. \tag{8}$$

From adding and subtracting the equalities (6b) and (7b) we get

$$-2\eta_+ f_{k-1} = 2\eta_- f_{k+1} = k f_k.$$

If we take $k = 0$, we get $\eta_- f_1 = \eta_+ f_{-1} = 0$.

Now, we use an important property of η_- and η_+: $\eta_-\eta_+ f_0 = 0 \implies \eta_+ f_0 = 0$. If we apply η_- to Equation 8, we get

$$2\eta_-\eta_+ f_0 + \eta_- f_1 = -2\eta_- a,$$

where we can notice that $\eta_- f_1 = 0$, $\eta_- a = 0$ (Property (3.1.2)), so $\eta_-\eta_+ f_0 = 0$ and we can apply the property. Finally, from cancelling out the $\eta_+ f_0$ term in Equation 8 we get

$$f_1 = -2a.$$

Therefore,

$$f_1 = -2(\pi^*\omega)(X) = -2(\pi^* dh_1)(X).$$

For $\omega = dh_{-1}$, where h_{-1} is anti-holomorphic, by similar computations we can show that

$$f_{-1} = -2a = -2(\pi^* dh_{-1})(X).$$

□

Lemma 3.3. *If f is a function on $S\Sigma$,*

$$\pi_*(\alpha \wedge df) = \frac{i}{2}\hat{f}_1 dz - \frac{i}{2}\hat{f}_{-1} d\overline{z}, \tag{9}$$

where $z = x_1 + ix_2$ and $\hat{f}_k$ is the k-th Fourier coefficient.

Proof. First, we find $\alpha \wedge df$. Recalling the definition of α in Equation 2,

$$\begin{aligned} \alpha \wedge df &= (\cos\theta dx^1 + \sin\theta dx^2) \wedge \big((\partial_{x_1} f)dx^1 + (\partial_{x_2} f)dx^2 + (\partial_\theta f)\big) \\ &= (\cos\theta(\partial_{x_2} f) - \sin\theta(\partial_{x_1}))dx^1 \wedge dx^2 + \cos\theta(\partial_\theta f)dx^1 \wedge d\theta + \sin\theta(\partial_\theta f)dx^2 \wedge d\theta. \end{aligned} \tag{10}$$

To find $\pi_*(\alpha \wedge df)_x$, a 1-form on W, it is sufficient to compute how it acts on ∂_{x_1} and ∂_{x_2}.

$$(\pi_*(\alpha \wedge df))_x (\partial_{x_1}) = \int_{S_x^1} i_{\partial_{x_1}}(\alpha \wedge df)$$

By substituting in the expression for $\alpha \wedge df$ found in Equation 10 and simplifying,

$$= \int_{S_x^1} \cos\theta(\partial_\theta f) d\theta$$

By performing integration by parts and applying the identity $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$, we conclude

$$(\pi_*(\alpha \wedge df))_x (\partial_{x_1}) = \frac{1}{2i}\left(\hat{f}_{-1} - \hat{f}_1\right). \tag{11}$$

For x_2, we can analogously show that

$$\begin{aligned} (\pi_*(\alpha \wedge df))_x (\partial_{x_2}) &= \int_{S_x^1} i_{\partial_{x_2}}(\alpha \wedge df) \\ &= \int_{S_x^1} \sin\theta(\partial_\theta f) d\theta \\ &= -\frac{1}{2}\left(\hat{f}_{-1} + \hat{f}_1\right). \end{aligned} \tag{12}$$

If we combine the results in Equations 11 and 12, and substitute $dz = dx^1 + idx^2$, we get the desired result:

$$\begin{aligned} \pi_*(\alpha \wedge df) &= \frac{1}{2i}(\hat{f}_{-1} - \hat{f}_1)dx^1 - \frac{1}{2}(\hat{f}_{-1} + \hat{f}_1)dx^2 = \\ &= \frac{i}{2}\hat{f}_1 dz - \frac{i}{2}\hat{f}_{-1} d\overline{z}. \end{aligned}$$

□

We combine Lemma 3.2 and Lemma 3.3 to achieve the next result.

Proposition 3.4. *For 1-form α and resonant state u,*

$$\pi_*(\alpha \wedge u) = -idh_1 + idh_{-1}. \tag{13}$$

Proof. We start by reducing $\pi_*(\alpha \wedge u)$ to a known expression,

$$\pi_*(\alpha \wedge u) = \pi_*(\alpha \wedge (\pi^*\omega + df)) = \pi_*(\alpha \wedge \pi^*\omega) + \pi_*(\alpha \wedge df) = \pi_*(\alpha \wedge df).$$

In Lemma 3.3, we found $\pi_*(\alpha \wedge df)$, so we conclude

$$\pi_*(\alpha \wedge u) = \frac{i}{2}\hat{f}_1 dz - \frac{i}{2}\hat{f}_{-1} d\overline{z}.$$

$$\pi_*(\alpha \wedge u) = \frac{i}{2} f_1 e^{-i\theta} dz - \frac{i}{2} f_{-1} e^{i\theta} d\overline{z}.$$

We can finally substitute the coefficients f_1 and f_{-1} found in Lemma 3.2 as below,

$$\begin{aligned}\pi_*(\alpha \wedge u) &= \frac{i}{2}\left(-2(\pi^* dh_1)(X)\right) e^{-i\theta} dz - \frac{i}{2}\left(-2(\pi^* dh_{-1})(X)\right) e^{i\theta} d\overline{z} \\ &= -idh_1 + idh_{-1},\end{aligned}$$

achieving the desired result. □

3.1 Finding the value of $\det(M)$

We have just shown that $M(dh_1) = -idh_1$ for holomorphic dh_1 and $M(dh_{-1}) = idh_{-1}$ for anti-holomorphic dh_{-1}. In the basis $(\{dh_1^1, \ldots, dh_1^g, dh_{-1}^1, \ldots, dh_{-1}^g\})$ of $H^1(\Sigma)$ discussed in Section 2.6, we can consider M to be a $2g \times 2g$ matrix, where g is the genus of our original surface Σ and $2g$ the dimension of $H^1(\Sigma)$. Then,

$$M = \begin{pmatrix} -i & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -i & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & -i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & i & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & i \end{pmatrix},$$

or $M_{i,i} = -i$ for $1 \leq i \leq g$ and $M_{i,i} = i$ for $g+1 \leq i \leq 2g$. From here, we can see that $\det(M) = 1$.

4 Conclusion

In this project, we computed the torsion of finite-dimensional resonant states, linking the dynamical torsion to the coefficient of the Ruelle zeta function at zero for surfaces of constant negative curvature. This understanding not only reveals the specific case of constant negative curvature but also lays the groundwork for extending our methodology to dynamical torsion for surfaces of variable curvature. Additionally, our approach holds potential applicability to higher-dimensional manifolds, informing future research in more complex geometrical and dynamical contexts.

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References

[1] D. Ruelle. Zeta-functions for expanding maps and Anosov flows. *Inventiones mathematicae*, 34:231–242, 1976.

[2] D. Fried. Analytic torsion and closed geodesics on hyperbolic manifolds. *Inventiones mathematicae*, 84(3):523–540, 1986.

[3] S. Dyatlov and M. Zworski. Ruelle zeta function at zero for surfaces. *Inventiones mathematicae*, 210(1):211–229, Apr. 2017.

[4] Y. Chaubet and N. V. Dang. Dynamical torsion for contact Anosov flows, 2020.

[5] C. Guillarmou, J. Hilgert, and T. Weich. Classical and quantum resonances for hyperbolic surfaces. *Mathematische Annalen*, 370(3):1231–1275, 2018.

[6] M. Cekić, B. Delarue, S. Dyatlov, and G. P. Paternain. The Ruelle zeta function at zero for nearly hyperbolic 3-manifolds. *Inventiones mathematicae*, 229(1):303–394, Mar. 2022.

[7] B. Gilligan and O. Forster. *Lectures on Riemann Surfaces*. Graduate Texts in Mathematics. Springer New York, 2012.