On Torus Orbits of Symplectic Leaves from Upper Cluster Algebras with Poisson Structure

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Abstract

Fomin and Zelevinsky introduced the *upper cluster algebras,* defined as an intersection of Laurent polynomial rings generated by the *frozen variables, exchangeable variables,* and the inverses of the exchangeable ones. In suitable cases, they can be geometrically interpreted as the coordinate ring of an affine algebraic variety called a *cluster variety.* Gekhtman, Shapiro, and Vainshtein (GSV) initiated the study of Poisson geometry on upper cluster algebras and defined a compatible *torus action.* There has been interest in the corresponding *symplectic leaves* on these cluster varieties. It has been conjectured that the cluster variety corresponding to any *acyclic* seed, equipped with the mentioned GSV Poisson structure (when it exists), has only finitely many torus orbits of symplectic leaves. Working with a class of type A seeds, we prove the conjecture when all frozen variables are invertible. On the other hand, we disprove the general conjecture by presenting an infinite family of counterexamples arising from the case of two exchangeable variables. We also classified the singular points in that case and showed that every such point is a symplectic point, enabling us to construct the counterexamples.

Summary

Upper cluster algebras are a family of abstract structures which arise naturally from important geometric examples in mathematics. In their full generality, they have applications across many areas of mathematics. Some of them can be equipped with an additional structure called a Poisson bracket. These brackets come from the mathematical theory of quantization, which describes the connection between classical and quantum mechanics, for example. The upper cluster algebra, along with the bracket gives a high-dimensional geometric object equipped with a special flow called the Hamiltonian flow. The flow separates the geometric object into sections called symplectic leaves. There is a way to "jump" from symplectic leaf to symplectic leaf, and the collection of leaves that we can jump between is called an orbit. We are motivated by a conjecture that states that the number of orbits is always finite. In this paper, we prove a special case of the conjecture, but also disprove it in the general case by producing an infinite family of counterexamples.

1 Introduction

Cluster algebras are a family of polynomial algebras generated by distinguished sets of variables called *cluster variables*. They were introduced by Fomin and Zelevinsky in 2002 in the context of dual canonical bases and total positivity in Lie theory [1] but have since played an important role across myriad areas of mathematics [2]. For example, discrete dynamical systems [3, 4, 5], quiver representations[6, 7, 8, 9], algebraic combinatorics, and see [10] for an exposition of some applications to Teichmüller theory and mathematical physics. The cluster variables generating a cluster algebra can be partitioned into finite subsets called *seeds*, and seeds can be obtained from each other through the combinatorial procedure of *mutation*. This separates the cluster variables into *exchangeable* and *frozen*, with the difference being that the frozen ones do not change under mutation. In this way, a cluster algebra can be constructed from the data of an initial seed and an *exchange matrix*, a skew-symmetrizable integer matrix that encodes the data of mutations.

Geometrically, we can interpret the initial seed as a transcendence basis for the function field of some algebraic variety Y known as a *cluster variety*. Its geometry is interesting in its own right, and accordingly, another important object of study is its coordinate ring, known as an *upper cluster algebra*, which can be described combinatorially as an intersection of certain Laurent polynomial rings generated by the cluster variables and their inverses. For example, the coordinate rings of Grassmanians and double Bruhat cells in a semisimple Lie group are upper cluster algebras [11].

In 2002, Gekhtman, Shapiro, and Vainshtein [12] initiated the study of Poisson geometry on these cluster varieties. Under some mild restrictions, they defined a canonical Poisson bracket on certain upper cluster algebras compatible with their cluster structures. These Poisson brackets give rise to Poisson structures, called the GSV Poisson structures, on the corresponding cluster varieties. They are of particular interest in Lie theory and in the study of integrable systems [13]. Gekhtman, Shapiro, and Vainshtein also constructed a torus action on these varieties compatible with their Poisson structures. Namely, this torus action sends symplectic leaves to symplectic leaves, so we can ask about the orbits of symplectic leaves under this action. They are an object of study in their own right, with applications to Lie theory and quantum cluster algebras. In 2022, Muller, Nguyen, Trampel, and Yakimov [14, Theorem 4.1] studied the open subvariety of the cluster variety consisting of nonsingular points where the frozen variables do not vanish, and they showed that it is a single torus orbit consisting of symplectic leaves when it admits the GSV Poisson structure.

In general, there need not be finitely many torus orbits of symplectic leaves on these cluster varieties—an explicit counterexample has been described by [13] in the form of the Belavin–Drinfeld Poisson structures on the coordinate rings of GL_n. Thus, interesting behavior may arise when we consider torus orbits of symplectic leaves in the singular part and in the divisors where the frozen variables vanish. The goal of this paper is to investigate the torus orbits of symplectic leaves in this more general setting.

Our work is motivated by the following conjecture communicated to us by Milen Yakimov, who termed it as a question/expectation.

Conjecture 1.1. *Let (x, B) be an acyclic seed (seed being a pair of cluster variables and exchange matrix). If (x, B) admits a compatible GSV Poisson structure, then the corresponding cluster variety has only finitely many torus orbits of symplectic leaves.*

Working with the specific case of type $\overline{A}_k$ seeds (to be defined in Section 2), we investigate this conjecture from two different avenues. Fix a seed (x, B) of type $\overline{A}_k$. Write $\widetilde{Y}(x, B)$ for the corresponding cluster variety, and let $Y(x, B) \subset \widetilde{Y}(x, B)$ denote the nonvanishing locus of the frozen variables. The result of Muller, Nguyen, Trampel, and Yakimov applies only to the non-singular part of $Y(x, B)$, raising the question of counting orbits in its singular part. First, we show that every point in $Y(x, B)$ is non-singular, so we completely understand the orbits in $Y(x, B)$.

Theorem A. *Let (x, B) be a seed of type $\overline{A}_k$. Then,*

$$Y(x, B) \subset \widetilde{Y}(x, B)^{\text{non-sing}}.$$

Thus, when B admits a compatible pair, $Y(x, B)$ is a single torus orbit of symplectic leaves.

As a consequence of Theorem A, we prove Conjecture 1.1 in the type $\overline{A}_k$ case when all frozen variables are allowed to be inverted. Moreover, Theorem A shows that it suffices to consider the divisors on $Y(x, B)$ where the frozen variables vanish. In contrast to Theorem A, these divisors may contain many singular points, whose symplectic leaves are studied separately. To this end, we focus specifically on the case $k = 2$.

First, we prove the following theorem, whose upshot is an explicit description of the torus orbits of symplectic leaves in the singular part of $\widetilde{Y}(x, B)$.

Theorem B. *Let (x, B) be a seed of type $\overline{A}_2$. Any singular point in $\widetilde{Y}(x, B)$ is a symplectic point, i.e., its symplectic leaf consists of only itself.*

On the other hand, we provide an infinite family of counterexamples to Conjecture 1.1 in the case $k = 2$. That is, we produce infinitely many (non-equivalent) seeds (x, B) of type $\overline{A}_2$ containing infinitely many torus orbits of symplectic leaves.

Theorem C. *Let (x, B) be a seed of type $\overline{A}_2$, and write*

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ a_1 & b_1 \\ \vdots & \vdots \\ a_\ell & b_\ell \end{pmatrix}.$$

Assume $a_q = -b_q > 0$ for some $q \in \{1, \dots, \ell\}$ and $a_i \neq 1, b_i \neq -1$ for all i. Then $\widetilde{Y}(x, B)$ contains infinitely many torus orbits of symplectic leaves, disproving Conjecture 1.1.

Let us give a brief overview of the organization of the paper. In Section 2, we establish the necessary background material and notation on cluster algebras to be used throughout our work. The definitions of the GSV Poisson structure, the torus action, and other related preliminaries on Poisson geometry which are important to our work can be found in the Appendix. In Section 3, we study the non-vanishing locus of the frozen variables and prove Theorem A. Then, in Section 4, we focus on the case $k = 2$, proving Theorems B and C. Finally, in Section 5, we conclude with some potential future directions for investigation.

2 Upper Cluster Algebras

In this section we present neccessary background material on cluster algebras, building on the notation of [14, Section 2].

Definition 2.1. Fix integers $k > 0$ and $\ell \geq 0$. Let $\mathbb{F}$ denote a field of transcendence degree $(k+\ell)$ over $\mathbb{C}$. A *seed* in $\mathbb{F}$ is a pair (x, B), where

(1) $x = \{x_1, \ldots, x_k, z_1, \ldots, z_\ell\}$ is an algebraically independent subset of $\mathbb{F}$ generating $\mathbb{F}$. The variables $x_1, \ldots, x_k$ are referred to as *exchangeable* and $z_1, \ldots, z_\ell$ as *frozen*.

(2) $B = (b_{ij})$ is a $(k+\ell) \times k$ integer matrix with full rank k, whose principal matrix B_p, i.e., the $k \times k$ matrix formed by the first k rows of B, is skew-symmetrizable. That is, there exist positive integers d_i such that $d_i b_{ij} = -d_j b_{ji}$ for all $i, j = 1, \ldots, k$.

Any seed is defined up to a relabeling of elements of x together with the corresponding relabeling of rows and columns of B.

Definition 2.2. The *mutation of a seed* (x, B) *in the direction* $q \in \{1, \ldots, m\}$ is a seed $\mu_q(x, B) := (x', B')$, where $x' = (x \setminus \{x_q\}) \cup \{x'_q\}$. The new cluster variable x'_q is defined by the relation

$$x_q x'_q = \prod_{\substack{i=1 \\ b_{iq}>0}}^{k} x_i^{b_{iq}} \prod_{\substack{s=1 \\ b_{k+s,q}>0}}^{\ell} z_s^{b_{k+s,q}} + \prod_{\substack{i=1 \\ b_{iq}<0}}^{k} x_i^{-b_{iq}} \prod_{\substack{s=1 \\ b_{k+s,q}<0}}^{\ell} z_s^{-b_{k+s,q}},$$

and the entries of the matrix $(k+\ell) \times k$ integer matrix B' are defined by the formula

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = q \text{ or } q = j \\ b_{ij} + \frac{1}{2}(|b_{iq}|b_{qj} + b_{iq}|b_{qj}|) & \text{otherwise.} \end{cases}$$

Observe that mutation is an involution.

Definition 2.3 (Berenstein, Fomin, Zelevinsky [11])**.** Let (x, B) be a seed with exchangeable variables $x_1, \ldots, x_k$ and frozen variables $z_1, \ldots, z_\ell$. For convenience, define the Laurent polynomial ring

$$\widetilde{L}(x, B) := \mathbb{C}[x_1^{\pm 1}, \ldots, x_m^{\pm 1}, z_1, \ldots, z_\ell].$$

The *upper cluster algebra*, or *the cluster algebra of geometric type*, corresponding to the seed (x, B) is defined as the intersection

$$\widetilde{U}(x, B) := \widetilde{L}(x, B) \cap \bigcap_{i=1}^{k} \widetilde{L}(\mu_i(x, B)).$$

Remark. Although it is not clear from the definition, remarkably we have $\widetilde{U}(x, B) = \widetilde{U}(\overline{x}, \overline{B})$ whenever (x, b) and $(\overline{x}, \overline{B})$ can be obtained from the other through a sequence of mutations [11, Theorem 1.5].

Definition 2.4. Given a seed (x, B), we define the affine scheme

$$\widetilde{Y}(x, B) := \mathrm{MaxSpec}\, \widetilde{U}(x, B),$$

where $\mathrm{MaxSpec}(R)$ denotes the set of maximal ideals in R.

Definitions of the upper cluster algebra found in the literature allow for arbitrary subsets of the frozen variables to be inverted. For example, as mentioned in Section 1, Muller, Nguyen, Trampel, and Yakimov studied the Poisson geometry on the nonvanishing locus of the frozen variables.

Definition 2.5. Let (x, B) be a seed with exchangeable variables $x_1, \ldots, x_k$ and frozen variables $z_1, \ldots, z_\ell$. With the notation

$$L(x, B) := \mathbb{C}[x_1^{\pm 1}, \ldots, x_k^{\pm 1}, z_1^{\pm 1}, \ldots, z_\ell^{\pm 1}],$$

the *non-vanishing locus of the frozen variables* corresponding to the seed (x, B) is defined by the intersection

$$U(x, B) = L(x, B) \cap \bigcap_{i=1}^{m} L(\mu_i(x, B)).$$

Similarly, we write

$$Y(x, B) = \mathrm{MaxSpec}\, U(x, B) \subset \widetilde{Y}(x, B)$$

for the corresponding open subscheme.

We are particularly interested in a specific class of cluster varieties defined by *acyclic seeds*. Given a seed (x, B) with k exchangeable variables and ℓ frozen variables, we define a quiver $\Gamma(x, B)$ as follows. The vertices of $\Gamma(x, B)$ are labeled by the exchangeable variables in B, and we draw an edge from the i-th vertex to the j-th vertex if and only if B_{ij} is positive.

Definition 2.6. A seed (x, B) is *acyclic* if its mutation class contains a seed (x', B') such that $\Gamma(x', B')$ is an acyclic quiver. By mutation class, we mean the collection of seeds we can obtain by applying a sequence of mutations to (x, B). In this case, we say that the upper cluster algebra $\widetilde{U}(x, B)$ is acyclic as well.

The following theorem gives an explicit presentation of any acyclic upper cluster algebra as a finitely generated $\mathbb{C}$-algebra. As a consequence, the scheme $\widetilde{Y}(x, B)$ associated with any acyclic seed (x, B) is an honest affine algebraic variety, which is particularly amenable to classical geometric tools. In this case, we refer to $\widetilde{Y}(x, B)$ and $Y(x, B)$ defined above as *cluster varieties* associated to the seed (x, B).

Theorem 2.1 (Berenstein-Fomin-Zelevinsky, [11], Theorem 1.18)**.** *Let (x, B) be an acyclic seed with k exchangeable variables and ℓ frozen variables. Then, we have*

$$\widetilde{U}(x, B) \cong \mathbb{C}[z_1, \ldots, z_\ell, x_1, \ldots, x_k, x_1', \ldots, x_k']/I(x, B),$$

where the ideal $I(x,B)$ *is generated by the polynomials*

$$f_j := x_j x_j' - \prod_{\substack{i=1\\ b_{ij}>0}}^{k} x_i^{b_{ij}} \prod_{\substack{s=1\\ b_{k+s,j}>0}}^{\ell} z_s^{-b_{k+s,j}} - \prod_{\substack{i=1\\ b_{ij}<0}}^{k} x_i^{-b_{ij}} \prod_{\substack{s=1\\ b_{k+s,j}<0}}^{\ell} z_s^{-b_{k+s,j}},\ \textit{for } j = 1,\dots,k$$

. Thanks to Hilbert's Nullstellensatz (maximal ideals are in bijection with points in $\mathbb{C}^n$), Theorem 2.1 above allows us to understand acyclic cluster algebras explicitly as the vanishing sets of polynomials in affine space.

$$\widetilde{Y}(x,B) = \operatorname{MaxSpec}\widetilde{U}(x,B) = \operatorname{MaxSpec}(\mathbb{C}[x_1,\dots,x_k,z_1,\dots,z_\ell/I(x,B)) = V(I(x,B)),$$

where $V(I) = \{a \in \mathbb{C}^{k+\ell} : f(a) = 0 \text{ for all } f \in I\}$ is the *vanishing set of the ideal* I. Similarly,

$$Y(x,B) = V(\langle I(x,B), z_i t_i = 1 | i \in 1,\dots,\ell\rangle)$$

We are particularly interested in a special class of acyclic seeds, which are themselves a special case of the type A finite type seeds (see [11] for further discussion).

Definition 2.7. Let k be a positive integer. A seed (x,B) is of *type* $\overline{A}_k$ if the entries b_{ij} of the principal part of B are defined by the formula

$$b_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ -1 & \text{if } j = i-1 \\ 0 & \text{else} \end{cases}, \qquad \text{for each } i,j = 1,\dots,k.$$

3 The Invertible Case

In this section, we study the geometry of $Y(x,B)$ (the nonvanishing locus of the frozen variables) when (x,B) is a type $\overline{A}_n$ seed. In particular, we demonstrate that every point in $Y(x,B)$ is non-singular. By [14, Theorem 4.1], it follows that $Y(x,B)$ is a single torus orbit of symplectic leaves. Thus, this result confirms Conjecture 1.1 in the type $\overline{A}_n$ case when we allow all frozen variables to be inverted.

Theorem A. *Let* (x,B) *be a seed of type* $\overline{A}_n$*. Then,*

$$Y(x,B) \subset \widetilde{Y}(x,B)^{\textit{non-sing}}.$$

Thus, when B *admits a compatible pair,* $Y(x,B)$ *is a single torus orbit of symplectic leaves.*

Proof. First, recall that the localized upper cluster algebra can be expressed as

$$U(x,B) = \mathbb{C}[z_1,\dots,z_\ell,t_1,\dots,t_\ell,x_1,\dots,x_k,x_1',\dots,x_k']/\langle I(x,B), z_s t_s - 1 \mid s = 1,\dots,\ell\rangle.$$

Consider the corresponding Jacobian matrix (see Section A.3 in the Appendix):

$$J = \begin{pmatrix} x_1' & -\beta_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ -\alpha_2 & x_2' & -\beta_2 & \dots & 0 & 0 & \dots & 0 \\ 0 & -\alpha_3 & x_3' & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & -\beta_{k-1} & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_k' & 0 & \dots & 0 \\ x_1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & x_2 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & x_3 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & x_k & 0 & \dots & 0 \\ \frac{\partial f_1}{\partial z_1} & \frac{\partial f_2}{\partial z_1} & \frac{\partial f_3}{\partial z_1} & \dots & \frac{\partial f_k}{\partial z_1} & 1 & \dots & 0 \\ \frac{\partial f_1}{\partial z_2} & \frac{\partial f_2}{\partial z_2} & \frac{\partial f_3}{\partial z_2} & \dots & \frac{\partial f_k}{\partial z_2} & 0 & \dots & 0 \\ \frac{\partial f_1}{\partial z_3} & \frac{\partial f_2}{\partial z_3} & \frac{\partial f_3}{\partial z_3} & \dots & \frac{\partial f_k}{\partial z_3} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial z_\ell} & \frac{\partial f_2}{\partial z_\ell} & \frac{\partial f_3}{\partial z_\ell} & \dots & \frac{\partial f_k}{\partial z_\ell} & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 & z_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & z_\ell \end{pmatrix}.$$

In particular, the ith column of J represents the gradient of f_i , while the rows represent the partial derivatives with respect to the variables $x_1, \dots, x_k, x_1', \dots, x_k', z_1, \dots, z_\ell, t_1, \dots, t_\ell$, respectively. Moreover, the variables $\alpha_i, \beta_i \neq 0$ represent products of the frozen variables given explicitly by the relations defining $U(x, B)$. That is, for each $i = 1, \dots, k$, we have

$$x_i x_i' = \alpha_{i+1} x_{i+1} + \beta_{i-1} x_{i-1}. \tag{1}$$

For $i = 0$ and $i = k$, the equation above holds with $x_0 = x_{i+1} = 1$.

To prove the theorem, it suffices to show that J has full rank, i.e., $\text{Rank}(J) = k + \ell$. Assume the contrary, that is, there exist $\lambda_1, \dots, \lambda_{k+\ell}$ such that $\sum_{i=1}^{k+\ell} \lambda_i J_i = 0$, where J_i is the i-th column of J and not all of the λ_i are zero.

Examining the last ℓ rows, it follows that $\lambda_{k+s} z_s = 0$, for all $s \in \{1, \dots, \ell\}$. Since $z_i \neq 0$ (it is invertible), it follows that $\lambda_{k+s} = 0$ for all $s \in \{1, \dots, \ell\}$.

Moving on to the $(k+1)$-st to $(2k)$-th rows, we can see that $\lambda_s x_s = 0$ for all $s \in \{1, \dots, k\}$. Since $\lambda_i \neq 0$ for some $i = 1, \dots, k$, it follows that $x_i = 0$ for some $i = 1, \dots, k$.

Suppose for the sake of contradiction that there exists $s \in \{1, \dots, k-1\}$, such that $x_s = 0$ and $x_{s+1} = 0$. From the relations defining $U(x, B)$, we have

$$0 = x_s x_s' = -x_{s-1}\beta_{s-1} - x_{s+1}\alpha_{s+1} = -x_{s-1}\beta_{s-1},$$

$$0 = x_{s+1} x_{s+1}' = -x_{s+2}\alpha_{i+2} - x_s\beta_s = -x_{s+2}\beta_{s+2}.$$

Since the frozen variables are nonzero, the coefficients β_r are nonzero. Thus, $x_{s-1} = x_{s+2} = 0$. Inductively applying this argument shows that $x_s = 0$ for all s. But then, from (1), we see that $0 = x_1 x_1' = x_2\alpha_2 + \beta_0 = \beta_0 \neq 0$, a contradiction. Therefore we know that there cannot exist s such that $x_s = x_{s+1} = 0$.

Now suppose there exists $s \in \{1, \dots, k-1\}$ such that $\lambda_s = 0 = \lambda_{s+1}$. The linear dependence assumption on the columns of J yields

$$0 = \lambda_{s-1}\alpha_s + \lambda_s x_s' + \lambda_{s+1}\beta_s = \lambda_{s-1}\beta_s,$$
$$0 = \lambda_s\alpha_{s+1} + \lambda_{s+1}x_{s+1}' + \lambda_{s+2}\beta_{s+1} = \lambda_{s+2}\beta_{s+1},$$

from which it follows that $\lambda_{s-1} = \lambda_{s+2} = 0$. Applying this argument recursively shows that $\lambda_s = 0$, for all s contradicting the linear dependence assumption.

If $\lambda_1 = 0$ or $\lambda_k = 0$, we have $\beta_1\lambda_2 = 0$ or $\alpha_k\lambda_{k-1} = 0$, which is not possible (since it would imply $\lambda_2 = 0$ or $\lambda_{k-1} = 0$). Thus, $k := 2m+1$ is odd, and moreover, $x_i = 0$ for i odd and $\lambda_j = 0$ for j even. From linear dependence and the relations (1), we have:

$$\lambda_{2i-1}\alpha_{2i} + \lambda_{2i+1}\beta_{2i} = 0, \text{ equivalent to } \frac{\beta_{2i}}{\alpha_{2i}} = -\frac{\lambda_{2i-1}}{\lambda_{2i+1}} \tag{2}$$

$$0 = x_{2i+1}x_{2i+1}' = x_{2i}\beta_{2i} + x_{2i+2}\alpha_{2i+2}. \tag{3}$$

Claim 3.1. *For any $i = 1, \dots, m$, we have* $x_{2i} = -\frac{\lambda_1\beta_0}{\lambda_{2i-1}\alpha_{2i}}$.

Proof. See A.1 in the Appendix. □

For $j \in \{1, \dots, \ell\}$, consider the $(2k+j)$-th row $j \in \{1, \dots, l\}$ of J, corresponding to the partial derivatives of the functions f_i with respect to z_j. From the linear dependence,

$$0 = \sum_{i=1}^{k} \lambda_i \frac{\partial f_i}{\partial z_j} + \sum_{i=k+1}^{k+\ell} \lambda_i \frac{\partial f_i}{\partial z_j} = \sum_{i=1}^{m+1} \lambda_{2i-1}\frac{\partial f_{2i-1}}{\partial z_j} = \sum_{i=1}^{m+1} \lambda_{2i-1}\frac{\partial[x_{2i-1}x_{2i-1}' - x_{2i-2}\beta_{2i-2} - x_{2i}\alpha_{2i}]}{\partial z_j}.$$

Applying Claim 3.1 ,

$$0 = \sum_{i=1}^{e+1}\left(-\lambda_{2i-1}x_{2i-2}\frac{\partial\beta_{2i-2}}{\partial z_j} - \lambda_{2i-1}x_{2i}\frac{\partial\alpha_{2i}}{\partial z_j}\right) = \sum_{i=1}^{m+1}\left(\frac{\lambda_1\lambda_{2i-1}\beta_0}{\lambda_{2i-3}\alpha_{2i-2}}\frac{\partial\beta_{2i-2}}{\partial z_j} + \frac{\lambda_1\beta_0}{\alpha_{2i}}\frac{\partial\alpha_{2i}}{\partial z_j}\right)$$
$$= \lambda_1\beta_0\sum_{i=1}^{m+1}\left(\frac{\lambda_{2i-1}\beta_{2i-2}(\beta_{2i-2})_{z_j}}{\lambda_{2i-3}\alpha_{2i-2}z_j} + \frac{(\alpha_{2i})_{z_j}}{z_j}\right).$$

Here $(\alpha_{2i})_{z_j}$ is the power of z_j in α_{2i} and likewise for β_{2i}. Applying (2), equation (3) becomes

$$0 = -\frac{\lambda_1\beta_0}{z_j}\sum_{i=1}^{m+1}((\beta_{2i-2})_{z_j} - (\alpha_{2i})_{z_j}). \tag{4}$$

By definition, we have

$$\beta_{2i-2} = \prod_{w, b_{2i-1,w}>0} z_w^{b_{2i-1,w}}, \quad \alpha_{2i} = \prod_{w, b_{2i-1,w}<0} z_w^{b_{2i-1,w}}, \text{ and thus}$$

$$(\beta_{2i-2})_{z_j} = \begin{cases} b_{2i-1,k+j} & \text{if } b_{2i-1,w} > 0 \\ 0 & \text{otherwise,} \end{cases} \qquad (\alpha_{2i})_{z_j} = \begin{cases} -b_{2i-1,k+j} & \text{if } b_{2i-1,w} < 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have $(\beta_{2i-2})_{z_j} - (\alpha_{2i})_{z_j} = b_{2i-1,j}$, so equation (4) becomes

$$\sum_{i=1}^{m+1} b_{2i-1,k+j} = 0, \quad \text{for } j = 1, \dots, \ell. \tag{5}$$

Now sum the odd columns of the exchange matrix B.Since k is odd, the entries in the first k rows sum to zero. Relation (5) shows that the entries in the remaining ℓ rows also sum to zero. This conclusion contradicts the assumption that B has full rank. It follows that J has full rank everywhere on $Y(x, B)$, and $Y(x, B)$ has no singular points. The second statement of the theorem follows from a direct application of [14, Theorem 4.1]. □

4 The Non-Invertible Case

In this section, we study the geometry of type $\overline{A}_2$ seeds. We label the following objects as shown and this is how they are going to be referred to for the entirety of the section:

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ a_1 & b_1 \\ \vdots & \vdots \\ a_\ell & b_\ell \end{pmatrix} \qquad \begin{aligned} f_1 &:= x_1x_1' - A^+ - A^-x_2 = 0 \\ f_2 &:= x_2x_2' - B^- - B^+x_1 = 0 \\ A^\pm &:= \prod_{i=1}^{l} z_i^{[\pm a_i]_+} \\ B^\pm &:= \prod_{i=1}^{l} z_i^{[\pm b_i]_+} \end{aligned} \qquad J = \begin{pmatrix} x_1' & -B^+ \\ -A^- & x_2' \\ x_1 & 0 \\ 0 & x_2 \\ -\frac{\partial[A^+ + A^-x_2]}{\partial z_1} & -\frac{\partial[B^- + B^+x_1]}{\partial z_1} \\ \vdots & \vdots \\ -\frac{\partial[A^+ + A^-x_2]}{\partial z_\ell} & -\frac{\partial[B^- + B^+x_1]}{\partial z_\ell} \end{pmatrix},$$

where B is the exchange matrix, $[r]_+ = r$ if r is positive and is zero otherwise, $A^\pm$ and $B^\pm$ help simplify the f_1 and f_2 relations, and J is the Jacobian matrix with columns $(\nabla f_1)_p$ and $(\nabla f_2)_p$, respectively. Recall that the f_i relations come from B, they cut out the variety $\widetilde{Y}(x, B)$ and we construct J from them (see Section A.3 in the Appendix). Note that B forms a compatible pair (see A.3) with the $(2 + \ell) \times (2 + \ell)$ matrix Λ such that the top left 2×2 submatrix has the form $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and the remainder of the matrix consists of zeros. Thus, every Type $\overline{A}_2$ variety can be equipped with GSV Poisson structure.

Making some assumptions on the entries of B, we classify singular points on $\widetilde{Y}(x, B)$.

Proposition 4.1. *Suppose $a_i, b_i \neq \pm 1$ for all $i = 1, \dots, \ell$. Then, the set of singular points on $\widetilde{Y}(x, B)$ have the form $(x_1, x_2, x_1', x_2', z_1, \dots, z_\ell)$, where the coordinates satisfy one of the following sets of equations:*

Type 1: $x_1 = x_1' = A^- = A^+ = 0$
Type 2: $x_2 = x_2' = B^- = B^+ = 0$
Type 3: $x_1 = x_2 = 0, \quad x_1'x_2' = A^-B^+, \quad A^+ = B^- = 0.$

Proof. If a point $p \in \widetilde{Y}(x, B)$ is non-singular, then the Jacobian matrix at p, J_p does not have full rank. The i-th column of J at p is given by the gradient $(\nabla f_i)_p$ for $i = 1, 2$. Therefore, if p is a singular point, the columns of J_p fall into one of the following three cases:

$$\begin{aligned} &\text{(i)} \quad (\nabla f_1)_p = 0, \\ &\text{(ii)} \quad (\nabla f_2)_p = 0, \text{ or} \\ &\text{(iii)} \quad (\nabla f_1)_p = \lambda (\nabla f_2)_p, \text{ for some nonzero } \lambda \in \mathbb{C}. \end{aligned}$$

From the expression for J_p, we see that Cases (i), (ii), (iii) give rise to points of Types 1, 2, 3, respectively. Thus, it remains to show that a point p being of Types 1, 2, or 3 implies J_p does not have full rank.

Suppose first that p is of Type 3, and consider the first entry of the $(j+4)$-th row of J_p:

$$-\frac{\partial [A^+ + A^- x_2]}{\partial z_j} = -\frac{\partial A^+}{\partial z_j} - x_2 \frac{\partial A^-}{\partial z_j} = \left(\prod_{s=1}^{j-1} z_s^{[a_s]_+} \right) [a_j]_+ z_j^{[a_j]_+ - 1} \left(\prod_{s=j+1}^{\ell} z_s^{[a_s]_+} \right). \tag{6}$$

We know that $A^+ = 0$ and $a_j \neq 1$. Therefore there is a frozen variable equal to 0 appearing in A^+ as well as its derivative, making the latter 0. Similarly, the second entry of this row is also 0. Assuming $x_1' \neq 0$, we have $(\nabla f_1)_p = -\frac{A^-}{x_1'} (\nabla f_2)_p$. Otherwise, the first column of J_p is zero. Therefore, J_p is not full rank when p is Type 3. The Type 1 and Type 2 cases are handled similarly. □

Studying the matrix MH_p, the matrix whose column space is H_p (see Section A.2 in the Appendix), and using the relations from 4.1 yields the following theorem.

Theorem B. *Let (x, B) be a seed of type $\overline{A}_2$. Every singular point in $\widetilde{Y}(x, B)$ is a symplectic point, i.e., its symplectic leaf consists of only itself. As a consequence, torus orbits of symplectic leaves in the singular part are orbits of singular points.*

Proof. We begin by determining the explicit form of MH_p. Consider the derivations

$$D_1 := \{x_1, \cdot\}, \quad D_2 := \{x_2, \cdot\}, \quad D_3 := \{x_1', \cdot\}, \quad D_4 := \{x_2', \cdot\}.$$

Note that the fibers of the corresponding vector fields at p span H_p, due to the entries of Λ.

$$\{x_1, x_1'\} = A^- x_2, \quad \{x_2, x_2'\} = -B^+ x_1, \quad \{x_2, x_2'\} = -B^+ x_1,$$

$$\{x_2, x_1'\} = x_1' x_2, \quad \{x_1', x_2'\} = x_1' x_2' - A^- B^+$$

Using the natural identification $\mathrm{Der}_p \simeq \mathrm{T}_p V(I)$ see that the derivations above correspond to the vectors

$$D_1 \mapsto \begin{pmatrix} 0 \\ x_1 x_2 \\ x_2 A^- \\ -x_1 x_2' \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad D_2 \mapsto \begin{pmatrix} -x_1 x_2 \\ 0 \\ x_1' x_2 \\ -x_1 B^+ \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad D_3 \mapsto \begin{pmatrix} -x_2 A^- \\ -x_1' x_2 \\ 0 \\ x_1' x_2' - A^- B^+ \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad D_4 \mapsto \begin{pmatrix} x_1 x_2' \\ x_1 B^+ \\ A^- B^+ - x_1' x_2' \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus, the matrix MH_p whose columns are the vectors above has zeros everywhere, except possibly the top left 4×4 sub-matrix, Now consider a singular point $p \in \widetilde{Y}(x, B)$. There are three cases for the columns of J_p—either one is a scalar multiple of the other or one of them is 0. For the first two cases, by symmetry, we assume without loss of generality that the first column is 0. Then we have $x_1 = x_1' = A^- = 0$,and direct substitution shows that D_1, D_2, D_3 and D_4 are 0. For the third case, we have $x_1 = x_2 = 0$ and $x_1'x_2' = A^-B^+$, where again substitution shows that $\dim H_p = 0$, concluding the proof. □

Combining the ideas behind the previous results, we proceed to Theorem C.

Theorem C. *Assume $a_q = -b_q > 0$ for some $1 \leq q \leq \ell$ and $a_i \neq 1, b_i \neq -1$ for all i. Then $\widetilde{Y}(x, B)$ contains infinitely many torus orbits of symplectic leaves, disproving Conjecture 1.1.*

Proof. We will show that the points $p(x_1') = \left(0, 0, x_1', -\frac{A^-B^+}{x_1'}, 1, \ldots, 1, 0, 1, \ldots, 1\right)$, where $x_1' \neq 0$ and z_q is the only frozen variable equal to zero, belong to pairwise distinct torus orbits. Notice that the points depend only on x_1' since A^- and B^+ are determined by the frozen variables. Since $a_q = -b_q > 0$ and $z_q = 0$, it follows that $A^+ = B^- = 0$. By an identical argument to the Type 3 case in the proof of Proposition 4.1, we can deduce that $(\nabla f_1)_{p(x_1')} = 0$, meaning that $J_{p(x_1')}$ is not full rank and thus $p(x_1')$ is singular. From Proposition B it follows that each $p(x_1')$ is a symplectic point. Now, we write the action of the torus $\mathbb{T}$ explicitly. Pick the following basis for $\ker B^\top$:

$$v_1 = \begin{pmatrix} -b_1 \\ a_1 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -b_2 \\ a_2 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad v_\ell = \begin{pmatrix} -b_\ell \\ a_\ell \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

From the definition of the torus action in Section A.4 (see Appendix),

$$(\vec{\omega}, x_1) = \prod_{i=1}^{\ell} \omega_i^{-b_i} x_1, \quad (\vec{\omega}, x_2) = \prod_{i=1}^{\ell} \omega_i^{a_i} x_2, \quad (\vec{\omega}, z_j) = \omega_j z_j.$$

Then, from the relations for $\widetilde{U}(x, B)$, we compute

$$(\vec{\omega}, x_1') = \left(\vec{\omega}, \frac{\prod_{i=1}^{\ell} z_i^{[a_i]_+} + x_2 \prod_{i=1}^{\ell} z_i^{[-a_i]_+}}{x_1}\right) = \frac{\prod_{i=1}^{\ell} (\vec{\omega}, z_i)^{[a_i]_+} + (\vec{\omega}, x_2) \prod_{i=1}^{\ell} (\vec{\omega}, z_i)^{[-a_i]_+}}{(\vec{\omega}, x_1)}$$

$$= \frac{\prod_{s=1}^{\ell} \omega_s^{[a_s]_+} \prod_{i=1}^{\ell} z_i^{[a_i]_+} + x_2 \prod_{s=1}^{\ell} \omega_s^{a_s + [-a_s]_+} \prod_{i=1}^{\ell} z_i^{[-a_i]_+}}{x_1 \prod_{i=1}^{\ell} \omega_i^{-b_i}} = \prod_{s=1}^{\ell} \omega_s^{[a_s]_+ - b_s} \prod_{i=1}^{\ell} \omega_i^{-b_i} x_1'.$$

Suppose there exist $\alpha, \beta \in \mathbb{C}^\times$ such that $p(\alpha)$ and $p(\beta)$ belong to the same orbit under $\mathbb{T}$. That is, there exists $\vec{\omega} \in \mathbb{T}$ so $(\vec{\omega}, p(\alpha)) = p(\beta)$. Since $(\vec{\omega}, z_j) = \omega_j z_j$, it follows that $\omega_j = 1$ for each $j \neq q$. Hence, from the explicit description of the torus action above, it follows that the third coordinate of $(\vec{\omega}, p(\alpha))$ is given by

$$\omega_q^{[a_q]_+ - b_q} \alpha = \alpha,$$

where we used the assumption that $a_q = -b_q > 0$. Since $(\vec{\omega}, p(\alpha)) = p(\beta)$, it follows that $\alpha = \beta$. In particular, $p(\alpha)$ and $p(\beta)$ belong to the same torus orbit if and only if $\alpha = \beta$. The result follows from Theorem B, since each point is its own symplectic leaf. □

5 Conclusion and Future Directions

Our work can be summarized in three main results. First, we showed that the non-vanishing locus of the cluster variety for any acyclic seed, when equipped with the GSV Poisson structure, is a single torus orbit of symplectic leaves – this proves a special case of Conjecture 1.1. Second, we classified the singular points of the cluster variety in the case of two exchangeable variables and showed that they each belong to a separate zero-dimensional symplectic leaf. Finally, we disproved the full statement of Conjecture 1.1 by presenting an infinite family of seeds in the type $\overline{A}_k$ case for which the cluster variety consists of infinitely many torus orbits of symplectic leaves.

Our next steps are continuing to study orbits of symplectic leaves in the Type $\overline{A}_2$ varieties. More generally we hope to establish a classification result for the Type $\overline{A}_k$ case for any k.

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A Appendix

A.1 Proof of Claim 3.1

Claim A.1. *For any $i = 1, \ldots, m$, we have*

$$x_{2i} = -\frac{\lambda_1 \beta_0}{\lambda_{2i-1}\alpha_{2i}}. \tag{7}$$

Proof. We prove this claim by induction with base $i = 1$, which follows from relation (3):

$$x_2 = -\frac{\beta_0}{\alpha_2} = -\frac{\lambda_1 \beta_0}{\lambda_1 \alpha_2}.$$

Now suppose that it holds for i. Then from relations (2) and (3) for $i+1$, we have

$$x_{2i+2} = -\frac{x_{2i}\beta_{2i}}{\alpha_{2i+2}} = \frac{\lambda_1 \beta_0 \beta_{2i}}{\lambda_{2i-1}\lambda_{2i}\alpha_{2i+2}} = -\frac{\lambda_1 \beta_0}{\lambda_{2i-1}\alpha_{2i+2}}.$$ □

A.2 Poisson Geometry

We introduce the Poisson structure defined by Gekhtman, Shapiro, and Vainshtein [12].

Definition A.1. A *Poisson algebra* is an algebra A equipped with a bilinear form $\{\cdot,\cdot\} : A \times A \to A$, which satisfies the properties

$$\{x, x\} = 0, \quad \{x, \{y, z\}\} + \{y, \{z, x\}\} + \{z, \{x, y\}\} = 0$$

$$\{x, yz\} = \{x, y\}z + \{x, z\}y,$$

for all $x, y, z \in A$, called the *Poisson bracket.*

In the following exposition, let (x, B) be any seed. Under some mild restrictions on the seed (x, B), Gekhtman, Shapiro, and Vainshtein [12] have defined a Poisson structure on $\widetilde{U}(x, B)$ compatible with its cluster structure.

Definition A.2. Let B be a $(k+\ell) \times k$ skew-symmetrizable integer matrix, and let Λ be a $(k+\ell) \times (k+\ell)$ skew symmetric integer matrix. We say that (B, Λ) is a *compatible pair* if $B^\top \Lambda$ has the form $[D\ 0]$, where D is a $k \times k$ diagonal matrix whose diagonal entries are positive integers.

Suppose the seed (x, B) satisfies the property that the exchange matrix B belongs to a compatible pair (B, Λ), and say x consists of the k exchangeable variables $x_1, \ldots, x_k$ and the ℓ frozen variables $z_1, \ldots, z_\ell$. For any integers $i, j \in \{1, \ldots, k\}$ and $r, s \in \{1, \ldots, \ell\}$ define

$$\{x_i, x_j\} := \Lambda_{ij} x_i x_j, \quad \{x_i, z_r\} := \Lambda_{i,r+k} x_i z_r, \quad \{z_r, z_s\} := \Lambda_{r+k,s+k} z_r z_s.$$

Since the matrix Λ is skew-symmetric and the cluster variables x_i generate the ambient field $\mathbb{F}$, this pairing extends to a Poisson bracket on $\mathbb{F}$. For each $i = 1, \ldots, k$, recall that the Laurent polynomial algebra $\widetilde{L}(\mu_i(x, B))$ is also a subalgebra of $\mathbb{F}$. Thus, this Poisson bracket on $\mathbb{F}$ also descends to a Poisson bracket on their common intersection $\widetilde{U}(x, B)$. One can also show that this Poisson bracket descends to a bracket on any of the localizations $U_{\mathcal{S}}(x, B)$.

Definition A.3. Let (x, B) be a seed such that B belongs to a compatible pair (B, Λ). The Poisson structure defined above is called the *Gekhtman–Shapiro–Vainshtein (GSV) Poisson structure* on $\widetilde{U}(x, B)$.

Remark. The GSV Poisson structure defined above has many mathematically significant properties, detailed in [12]. For instance, they showed that the Poisson bracket is *log-canonical* with respect to any seed in the mutation class of (x, B). That is, if y and y' are cluster variables belonging to the same seed, then $\{y, y'\} = cyy'$ for some constant $c \in \mathbb{C}$.

Thus, in the case where (x, B) is a seed of type $\overline{A}_k$, the cluster variety $\widetilde{Y}(x, B)$ becomes a Poisson variety. We investigate properties of the *symplectic leaves* on this Poisson variety.

A.3 Symplectic Leaves

Let us recall some necessary facts from the Poisson geometry of affine algebraic varieties. In what follows, let $A := \mathbb{C}[y_1, \ldots, y_n]/I$ denote an arbitrary finitely generated $\mathbb{C}$-algebra, and let $V = \operatorname{MaxSpec}(A)$ denote the corresponding affine algebraic variety.

Definition A.4. A $\mathbb{C}$-linear derivation on A is a linear map $D : A \to A$ satisfying the Leibniz rule:

$$D(ab) = D(a)b + aD(b)$$

for any $a, b \in A$. The A-module of all $\mathbb{C}$-linear derivations on A is denoted $\operatorname{Der} A$.

On the other hand, fix a maximal ideal $\mathfrak{m} \subset A$ and let $p \in V$ denote the corresponding point. A derivation of A at p is a linear map $D_p : A \to \mathbb{C}$ satisfying the Leibniz rule:

$$D_p(ab) = a(p)D_p(b) + D_p(a)b(p)$$

for all $a, b \in A$. We write $\operatorname{Der}_p A$ for the $\mathbb{C}$-vector space of all derivations of A at p.

We can also understand the definitions above from a geometric perspective.

Definition A.5. Fix a point $p \in V$. The *tangent space* T_pV to V at p is defined as the $\mathbb{C}$-vector space $\operatorname{Der}_p A$. An element of T_pV is called a *tangent vector* to V at p. A *vector field* on V is an element $X \in \operatorname{Der} A$.

Note that composing any derivation $X \in \operatorname{Der} A$ with the evaluation map at p gives rise to a derivation $X_p \in \operatorname{Der}_p A = T_pV$. That is, a vector field is a collection of tangent vectors for every point in V. Given a vector field $X \in \operatorname{Der} A$ and a point $p \in V$, the corresponding tangent vector $X_p \in T_pV$ is called its *fiber* at p.

If $V = \mathbb{C}^n$, then T_pV can be identified with $\mathbb{C}^n$ for any $p \in \mathbb{C}^n$. In particular, T_pV has a basis given by the derivations

$$\left.\frac{\partial}{\partial x_i}\right|_p \text{ for } i = 1, \ldots, n,$$

and we can identify $T_p\mathbb{C}^n$ with $\mathbb{C}^n$ by mapping $\left.\frac{\partial}{\partial x_i}\right|_p$ to the ith standard basis vector.

More generally, if V is arbitrary, we can view T_pV as a subspace of $T_p\mathbb{C}^n = \mathbb{C}^n$ consisting of derivations D_p such that $D_p(f) = 0$ for all $f \in I$. Thus, if I is generated by polynomials

$f_1, \ldots, f_k$, we obtain a natural identification of T_pV with the orthogonal complement of the subpsace of $\mathbb{C}^n$ spanned by the vectors

$$(\nabla f_i)_p := \left(\frac{\partial f}{\partial x_1}\Big|_p \quad \frac{\partial f}{\partial x_2}\Big|_p \quad \ldots \quad \frac{\partial f}{\partial x_n}\Big|_p \right)^\top$$

The vector $(\nabla f_i)_p$ is called the *gradient* of f_i at p, and the $n \times k$ matrix $J(V)_p$ whose columns are given by $(\nabla f_i)_p$ for $i = 1, \ldots, k$ is called the *Jacobian matrix* of V at p. When the variety V is understood from context, we write J_p for $J(V)_p$.

Thus, if we identify V with a closed subvariety of $\mathbb{C}^n$, i.e., as the vanishing set of the polynomials in I, the tangent space T_pV can be identified with the vectors in $\mathbb{C}^n$ tangent to V at p in the usual geometric sense.

Before we proceed, we must establish what it means for a point to be singular.

Definition A.6. If the ideal I is generated by a regular sequence of polynomials $f_1, \ldots, f_k$, then we say that a point $p \in V$ is *non-singular* if $\dim_{\mathbb{C}} T_pV = n - k$. Otherwise, we say that $p \in V$ is *singular*.

Assume now that A admits a Poisson bracket $\{-,-\}$. For any $f \in A$, the map $A \to A$ given by $a \mapsto \{f, a\}$ is a derivation on A. Thus, it defines a vector field X_f on V.

Definition A.7. The vector field X_f is called the *Hamiltonian vector field* corresponding to $f \in A$. The collection of all Hamiltonian vector fields defines a distribution H on V, i.e., a collection of subspaces $H_p \subset T_pV$ spanned by the fibers $(X_f)_p$ for all $f \in A$. Note that H_p is in fact spanned by $(X_{x_i})_p$ for $i = 1, \ldots, n$. We write MH_p for the $n \times n$ matrix whose columns are given by the vectors $(X_{x_i})_p$ for $i = 1, \ldots, n$.

We are now equipped to introduce our main object of interest, the symplectic leaves.

Definition A.8. Assume that V is a non-singular Poisson variety. A *symplectic leaf* on a V is a maximal connected subvariety $L \subset V$ such that $T_pL = H_p \subset T_pV$ for all $p \in L$. If V is a general variety, then its symplectic leaves are defined recursively as the union of the symplectic leaves of its non-singular part and the symplectic leaves of its singular part (viewed as a variety in its own right).

In fact, the Poisson variety V is partitioned into its symplectic leaves, i.e., every point of V belongs to exactly one symplectic leaf. In other words, the distribution H defined by the Hamiltonian vector fields is integrable, and the symplectic leaves are the leaves of the corresponding foliation.

We now return to the setting of cluster varieties. As before, fix a seed (x, B) of type $\overline{A}_k$ such that B belongs to some compatible pair (B, Λ). Assume that x contains k exchangeable variables and ℓ frozen variables. The upper cluster algebra $\widetilde{U}(x, B)$ becomes a Poisson algebra via the GSV Poisson structure defined earlier. We aim to study the symplectic leaves on the Poisson variety $\widetilde{Y}(x, B)$, and their orbits under the torus action to be defined in the following section.

A.4 Torus Action

In this section, we describe the torus action on the cluster variety defined by Gekhtman Shapiro, and Vainshtein [12]. Once again, fix a seed (x, B) of type $\overline{A}_k$ such that B belongs to some compatible pair (B, Λ), so that $\widetilde{Y}(x, B)$ is an affine Poisson variety. Assume that x contains k exchangeable variables and ℓ frozen variables.

As the $(k+\ell) \times k$ matrix B has full rank, its transpose B^T has kernel with dimension ℓ. Let us fix a basis $\{v_1, \ldots, v_\ell\}$ for this kernel. Then, for each $i = 1, \ldots, \ell$ and each $j = 1, \ldots, k+\ell$, write v_{ij} for the jth entry of the vector v_i. Consider the torus $\mathbb{T} := (\mathbb{C}^\times)^\ell$. For any $\vec{\omega} := (\omega_1, \ldots, \omega_\ell) \in \mathbb{T}$ and any $j = 1, \ldots, k$ or $r = 1, \ldots, \ell$, define

$$(\vec{\omega}, x_j) := \prod_{i=1}^{\ell} \omega_i^{v_{ij}} x_j, \quad (\vec{\omega}, z_r) := \prod_{i=1}^{\ell} \omega_i^{v_{i,r+k}} z_r.$$

Gekhtman, Shapiro, and Vainshtein [12] showed that this definition extends to an action of $\mathbb{T}$ on the ambient field $\mathbb{F}$ and hence, gives rise to an action of $\mathbb{T}$ on $\widetilde{U}(x, B)$. They also showed that this action is compatible with the Poisson bracket on $\widetilde{U}(x, B)$ [12]. Thus, the induced action on the cluster variety $\widetilde{Y}(x, B)$ sends symplectic leaves to symplectic leaves, and it makes sense to speak of orbits of symplectic leaves under this action.

A.5 Quivers

In this section, we offer some motivation for the definitions of a seed and its mutations through quivers. The definition of a seed and its mutations can be visually interpreted as a special type of directed graphs called *quivers* together with a set of operations, depending on the choice of a vertex — *mutating on vertex q.* We construct a quiver from a given seed (x, B) by assigning each exchangeable variable a *mutable vertex* and every frozen one — a *frozen vertex,* with the difference being that we cannot mutate on the frozen ones. Now we can complete the quiver by adding edges using the information in B, serving as the incidence matrix. B might not be a square matrix, but since there are no edges in between frozen vertices we do not consider them and it is enough to define the quiver. We mutate the quiver on a given vertex q as follows:

(1) For each subquiver $i \to q \to j$, add a new arrow $i \to j$.

(2) Reverse all allows with source or target q.

(3) Remove the arrows in a maximal set of pairwise disjoint 2-cycles.

Then if we have a quiver Q_1 with incidence matrix B_1 and a quiver Q_2 with incidence matrix B_2, $\mu_q(B_1) = \mu_q(B_2)$ is equivalent to $\mu_q(Q_1) = \mu_q(Q_2)$. Intuitively in this context, we can consider the mutations on the variables as a way of "algebraically recording" the changes in the quiver.

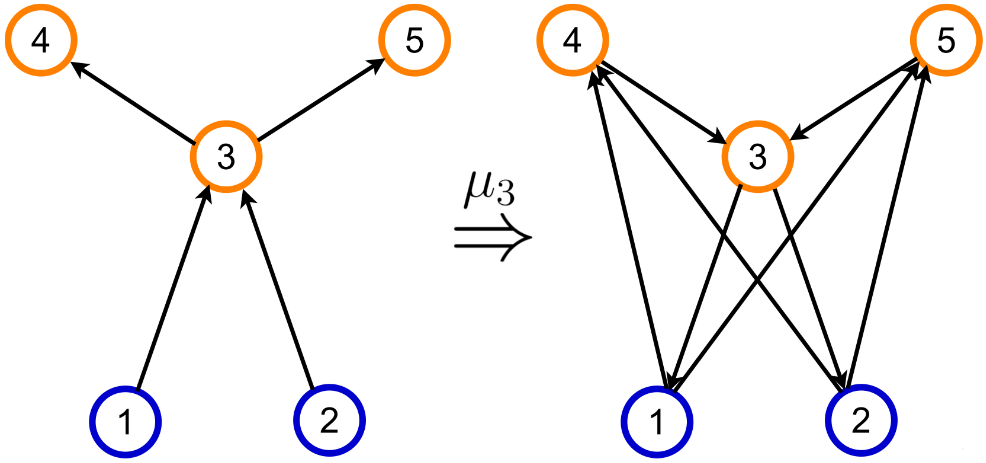

Figure 1: Example of the mutation on vertex 3 for a quiver with 3 mutable (orange) and 2 frozen (blue) vertices