Classifying bipolynomial Hopf algebras over graded local rings

Hongzhou (Hazel) Wu

Under the direction of

David Jongwon Lee
Massachusetts Institute of Technology
Department of Mathematics

Research Science Institute
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Abstract

Hopf algebras have been a very important area of research for much of the past century, with people observing and studying such structures in a wide range of fields. Ravenel and Wilson proved that certain bipolynomial Hopf algebras are isomorphic to the Witt Hopf algebra W_R, but only when the underlying rings are $R = \mathbb{Z}_{(p)}$ and $R = \mathbb{F}_p$. We generalize this isomorphism over graded local rings, which creates new possibilities in algebraic topology and other areas of mathematics.

Summary

A Hopf algebra is a complicated algebraic structure that occurs in many different areas of mathematics. We build on previous research to show how different types of Hopf algebras share the same structure for a wider range of conditions. This allows us to simplify and regularise our work by considering more well-studied Hopf algebras rather than less well-understood Hopf algebras, which has implications in different fields of current research.

1 Introduction

Hopf algebras are a type of algebraic structure with applications in different fields of mathematics, such as quantum groups, algebraic geometry and algebraic topology. A Hopf algebra is simultaneously an algebra and a coalgebra [1], which gives them many interesting relationship properties, such as when dualising.

A Hopf algebra H is considered to be bipolynomial if both it and its dual are polynomial algebras. Ravenel and Wilson [2] proved that any bipolynomial Hopf algebra H is isomorphic to the tensor product of Witt Hopf algebras W_R (see Definition 2.5) over the rings $R = \mathbb{Z}_{(p)}$ and $R = \mathbb{F}_p$ for prime p, where $\mathbb{Z}_{(p)} = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } \gcd(b, p) = 1\}$ is the ring of integers localised at p.

Theorem 1.1 (Ravenel and Wilson [2]). *For a graded bicommutative Hopf algebra H over the ring $R = \mathbb{Z}_{(p)}$ or $R = \mathbb{F}_p$, if there are algebra isomorphisms $H \cong R[x_0, x_1, x_2, \dots]$ and $H^* \cong R[y_0, y_1, y_2, \dots]$, where the polynomial algebras $R[x_0, x_1, x_2, \dots]$ and $R[y_0, y_1, y_2, \dots]$ have generators x_i, y_i with $\deg x_i = p^i$ and $\deg y_i = -p^i$, then $H \cong W_R$.*

The above theorem is a special case of Ravenel and Wilson's theorem, but their proof does not immediately generalise if R is a graded local ring (see Definition 2.1), because the dual H^* does not behave well. Over the rings $R = \mathbb{Z}_{(p)}$ and $R = \mathbb{F}_p$, there is a Hopf algebra isomorphism (see Lemma 5.1) between the dual $R[x]^*$ of the Hopf algebra $R[x]$ and the divided power Hopf algebra $\Gamma_R[x^*]$ (see Example 2.1) generated by the dual x^*, but $R[x]^* \cong \Gamma_R[x^*]$ does not hold over graded local rings R in general.

Theorem 1.2. *For a graded bicommutative Hopf algebra H over a graded local ring R, if $H \cong R[x_0, x_1, x_2, \dots]$ as algebras and $H \cong \Gamma_R[z_0, z_1, z_2, \dots]$ as coalgebras, where generators x_i, z_i have $\deg x_i = \deg z_i = p^i$, then $H \cong W_R$.*

We generalise Theorem 1.1 to Theorem 1.2 over graded local rings R, such as $R = \mathbb{Z}_{(p)}[u]$ and $R = \mathbb{F}_p[u]$ where $\deg u = 1$. Rather than consider the algebra isomorphism $H^* \cong R[y_0, y_1, y_2, \dots]$ as Ravenel and Wilson did, we instead consider the coalgebra isomorphism $H \cong \Gamma_R[z_0, z_1, z_2, \dots]$ over graded local rings R. By using induction on different degrees, we show that generators of the same degree in H and W_R map to each other, leading to the Hopf algebra isomorphism $H \cong W_R$.

2 Preliminaries

We define some mathematical terminology involving structures such as algebras, coalgebras and dual spaces. Throughout this paper, we consider only graded structures, and the graded local ring R is assumed to be connected.

Definition 2.1 (Graded local ring). A graded local ring R can be decomposed into the direct sum $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ where $R_0 = \mathbb{Z}_{(p)}$ or $R_0 = \mathbb{F}_p$ and $R_i R_j \subseteq R_{i+j}$.

Let M^* be the dual space of the R-module M.

2.1 Hopf algebras

Definition 2.2 (Algebras [1])**.** We refer to unital associative algebras as algebras. An algebra A is an R-module with multiplication $\mu : A \otimes A \to A$ and unit $\eta : R \to A$ that satisfy the commutative diagrams:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes \mathrm{id}} & A \otimes A \\ \big\downarrow{\scriptstyle \mathrm{id} \otimes \mu} & & \big\downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\quad\mu\quad} & A \end{array} \quad \text{and} \quad \begin{array}{ccccc} A \otimes R & \xrightarrow{\mathrm{id} \otimes \eta} & A \otimes A & \xleftarrow{\eta \otimes \mathrm{id}} & R \otimes A \\ & {\scriptstyle \cong} \searrow & \big\downarrow{\scriptstyle \mu} & \swarrow {\scriptstyle \cong} & \\ & & A & & \end{array} .$$

An augmented algebra A has a morphism of algebras (the counit) $\varepsilon : A \to R$, and the augmentation ideal I is the kernel of ε.

The indecomposables are the elements of the quotient space I/I^2.

Definition 2.3 (Coalgebras [1])**.** A coalgebra C is an R-module with comultiplication $\phi : C \to C \otimes C$ and counit $\varepsilon : C \to R$ that satisfy the commutative diagrams:

$$\begin{array}{ccc} C & \xrightarrow{\quad\phi\quad} & C \otimes C \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \mathrm{id} \otimes \phi} \\ C \otimes C & \xrightarrow{\phi \otimes \mathrm{id}} & C \otimes C \otimes C \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & C & & \\ & {\scriptstyle \cong} \swarrow & \big\downarrow{\scriptstyle \phi} & \searrow {\scriptstyle \cong} & \\ C \otimes R & \xleftarrow[\mathrm{id} \otimes \varepsilon]{} & C \otimes C & \xrightarrow[\varepsilon \otimes \mathrm{id}]{} & R \otimes C \end{array} .$$

An augmented coalgebra C has a morphism of coalgebras (the unit) $\eta : R \to C$.

The primitives are the elements of the set $\{h \in H \mid \phi(h) = h \otimes 1 + 1 \otimes h\}$.

Definition 2.4 (Hopf algebra [1])**.** A Hopf algebra H is defined as a R-module that is both an algebra and a coalgebra.

Example 2.1 (Divided power Hopf algebra [3])**.** The divided power Hopf algebra $\Gamma_R[x]$ has a basis $\gamma_k(x)$ where $\gamma_0(x) = 1$, $\gamma_i(x) = \frac{x^i}{i!}$ and $\gamma_i(x)\gamma_j(x) = \frac{(i+j)!}{i!j!}\gamma_{i+j}(x)$, with comultiplication $\phi\big(\gamma_i(x)\big) = \sum_{j=0}^{i} \gamma_j(x)\gamma_{i-j}(x)$.

In morphisms between Hopf algebras, primitives map to primitives linearly and indecomposables map to indecomposables linearly [3]. We can use such maps to determine morphisms between Hopf algebras, as shown in the following lemma.

Lemma 2.1. *For a coalgebra C, an R-module map $g : C \to R\{x_0, x_1, x_2, \dots\}$ uniquely determines a coalgebra map $G : C \to \Gamma_R[x_0, x_1, x_2, \dots]$.*

Proof. Consider any element $c \in C$. The comultiplication ϕ is coassociative, so $\phi^{n-1}(c) = \sum_{i=1}^{m} c_{1i} \otimes c_{2i} \otimes \cdots \otimes c_{ni}$ where $c_{1i}, c_{2i}, \dots, c_{ni} \in C$ for positive indices i up to non-negative integer m, and

$$g\big(\phi^{n-1}(c)\big) = g\left(\sum_{i=1}^{m} \bigotimes_{j=1}^{n} c_{ji}\right) = \sum_{i=1}^{m} \bigotimes_{j=1}^{n} g(c_{ji}).$$

For a tensor product $c_{1i} \otimes c_{2i} \otimes \cdots \otimes c_{ni}$, if any $g(c_{ji})$ is non-linear, then the tensor product is considered degenerate as it does not contribute towards $G(c)$.

Let $g_n(c) = \sum \prod g(c_{ji})$ for non-degenerate tensor products $c_{1i} \otimes c_{2i} \otimes \cdots \otimes c_{ni}$ be the element(s) in the divided power coalgebra $\Gamma_R[x_0, x_1, x_2, \dots]$ induced by $g\big(\phi^{n-1}(c)\big)$. We write the comultiplication $\phi(c)$ as $\phi(c) = c \otimes 1 + 1 \otimes c + \sum c' \otimes c''$. Since

$$\begin{aligned} G\big(\phi(c)\big) &= G(c \otimes 1 + 1 \otimes c) + G\left(\sum c' \otimes c''\right) \\ &= \big(g_1(c) + g_2(c) + \cdots\big) \otimes 1 + 1 \otimes \big(g_1(c) + g_2(c) + \cdots\big) \\ &\quad + g_1\left(\sum c' \otimes c''\right) + g_2\left(\sum c' \otimes c''\right) + \cdots, \end{aligned}$$

and

$$\begin{aligned} \phi\big(G(c)\big) &= \phi\big(g_1(c) + g_2(c) + \cdots g_n(c)\big) = \phi\big(g_1(c)\big) + \phi\big(g_2(c)\big) + \cdots \\ &= \phi\big(g_1(c) \otimes 1 + 1 \otimes g_1(c)\big) + \phi\big(g_2(c) \otimes 1 + 1 \otimes g_2(c) + \cdots\big) + \cdots, \end{aligned}$$

we prove that $G\big(\phi(c)\big) = \phi\big(G(c)\big)$ by rearranging. Thus, G is a coalgebra map. □

2.2 Witt Hopf algebras

Definition 2.5 (Witt Hopf algebra)**.** The Witt Hopf algebra $W_{Z_{(p)}}$ has generators y_i of $\deg y_i = p^i$, with comultiplication $\phi(z_i) = z_i \otimes 1 + 1 \otimes z_i$ for primitives

$$\begin{aligned} z_0 &= y_0, \\ z_1 &= py_1 + {y_0}^p, \\ z_2 &= p^2 y_2 + p{y_1}^p + {y_0}^{p^2}, \\ &\vdots \\ z_i &= \sum_{j=0}^{i} p^j {y_j}^{p^{p-j}}, \end{aligned}$$

and $W_R = W_{Z_{(p)}} \otimes R$ for the ring R.

Let $W_R(n)$ be the sub-Hopf algebra with generators $y_0, y_1, y_2, \dots, y_n$.

A basis of $W_R(n)$ is the set of monomials $y_0^{i_0} y_1^{i_1} \cdots y_n^{i_n}$ for non-negative exponents i_j, so that there are inclusion maps $W_R(0) \to W_R(1) \to W_R(2) \to \cdots$, and their dual maps $W_R(0)^* \to W_R(1)^* \to W_R(2)^* \to \cdots$ are all onto.

For the graded local ring R, its maximal ideal is $I_m = (p) \oplus R_1 \oplus R_2 \oplus R_3 \oplus \cdots$ where (p) represents the multiples of p. Thus, if $\deg r > 0$ for some element $r \in R$, then $r \in I_m$. According to Nakayama's Lemma (see Lemma 5.2), we construct the quotient ring $R/I_m = \mathbb{F}_p$ to help prove the Hopf algebra isomorphism $H \cong W_R$ over the graded local ring R.

3 Proof of Theorem 1.2

3.1 Key Lemma

Ravenel and Wilson [2] proved the following lemma over the rings $R = \mathbb{Z}_{(p)}$ and $R = \mathbb{F}_p$ for the algebra $H = R[x_0, x_1, x_2, \dots]$ where generators x_i have $\deg x_i = p^i$.

Lemma 3.1 (Ravenel and Wilson [2]). *Given an surjective algebra map $F : H^* \to W_R(n-1)^*$ in degrees $\leq p^n$, there is an algebra map $\tilde{F} : H^* \to W_R(n)^*$ that is isomorphic in degrees $\leq p^n$, such that the following diagram commutes:*

$$\begin{array}{ccc} H^* & \xrightarrow{F} & W_R(n-1)^* \\ & \searrow^{\tilde{F}} & \uparrow \\ & & W_R(n)^* \end{array}.$$

However, because the duals of Hopf algebras do not behave well over graded rings R, we avoid taking duals by using coalgebra maps to prove the following lemma for the coalgebra $H = \Gamma_R[x_0, x_1, x_2, \ldots]$ where generators x_i have $\deg x_i = p^i$.

Lemma 3.2. *Given a coalgebra map $G : W_R(n-1) \to H$ that is injective in degrees $\leq p^n$ mod I_m, we can find a coalgebra map $\tilde{G} : W_R(n) \to H$ that is isomorphic in degrees $\leq p^n$ mod I_m, such that the following diagram commutes:*

$$\begin{array}{ccc} W_R(n-1) & \xrightarrow{G} & H \\ \downarrow & \nearrow_{\tilde{G}} & \\ W_R(n) & & \end{array}.$$

Proof. For $y \in W_R(n-1)$, we let $G(y) = \sum a_i x_0^{i_0} x_1^{i_1} x_2^{i_2} \cdots$ where $a_i \in R$ and i_j are non-negative integers for each $j = 0, 1, 2, \ldots$. We construct a linear map $g : W_R(n-1) \to R\{x_0, x_1, x_2, \ldots\}$ by taking the linear terms of $G(y)$ such that $i_0 + i_1 + i_2 + \cdots = 1$ and letting $g(y) = b_0 x_0 + b_1 x_1 + b_2 x_2 + \cdots$ for $b_i \in R$. As $W_R(n-1)$ is generated by $y_0, y_1, \ldots, y_{n-1}$ while $W_R(n)$ is generated by $y_0, y_1, \ldots, y_n$, we obtain a linear map $\tilde{g} : W_R(n) \to R\{x_0, x_1, x_2, \ldots\}$ by letting

$$\tilde{g}(y_0^{i_0} y_1^{i_1} \cdots y_n^{i_n}) = \begin{cases} g(y_0^{i_0} y_1^{i_1} \cdots y_n^{i_n}) & i_n = 0, \\ 0 & i_n > 0. \end{cases}$$

Thus,

$$\begin{array}{ccc} W_R(n-1) & \xrightarrow{g} & R\{x_0, x_1, x_2, \ldots\} \\ \downarrow & \nearrow_{\tilde{g}} & \\ W_R(n) & & \end{array}.$$

Due to Lemma 2.1, we can define the coalgebra map $\tilde{G} : W_R(n) \to H$ from the linear map $\tilde{g} : W_R(n) \to H$. As G is injective, $\tilde{G}$ is isomorphic in degrees $< p^n$ mod I_m. To prove the isomorphism in degree p^n mod I_m, we need to show that y_n maps to $\gamma_{p^n}(x_0)$ while other y_i do not map to it. However, rather than directly computing $\overbrace{x_0 \otimes x_0 \otimes \cdots \otimes x_0}^{p^n - 1 \text{ times}}$ from the comultiplication $\phi^{p^n-1}(y_n)$, we can use a Hopf algebra mapping from the divided power coalgebra H to the symmetric polynomials to simplify our proof.

Definition 3.1 (Hopf algebra of symmetric polynomials [3])**.** The Hopf algebra S of symmetric polynomials in $s_1, s_2, s_3, \dots$ is $S = R[\sigma_1, \sigma_2, \sigma_3, \dots]$ where generators σ_i are the elementary symmetric polynomials

$$\begin{aligned}\sigma_1 &= \sum s_i,\\ \sigma_2 &= \sum_{1\le i<j} s_i s_j,\\ &\vdots\\ \sigma_n &= \sum_{1\le k_1<k_2<\cdots<k_n} s_{k_1}s_{k_2}\cdots s_{k_n},\end{aligned}$$

with comultiplication $\phi(\sigma_i) = \sum_{j=0}^{i} \sigma_j \otimes \sigma_{i-j}$.

By Newton's identities, the i-th power sum symmetric polynomials $c_i = \sum_{j=1,2,3,\dots} x_j^i$ have comultiplication $\phi(c_i) = c_i \otimes 1 + 1 \otimes c_i$, so c_i are primitives.

According to Husemoller [3], the injective map $K : W_R \to S$ maps the primitives $z_i \in W_R$ to the primitives $c_{p^i} \in S$, so the degrees of elements in W_R and the degrees of elements in S agree with each other.

The primitives of W_R are $z_i = y_0^{p^i} + p y_1^{p^{i-1}} + \cdots + p^i y_i$, and by Newton's identities, K sends y_n to $\sigma_{p^n} + h(\sigma_1, \sigma_2, \dots, \sigma_{p^n-1})$, where the polynomial h is composed of monomials of degree p^n. Because comultiplication is preserved, we can apply ϕ^{p^n-1} for the elements y_n and $\sigma_1, \dots, \sigma_{p^n-1}, \sigma_{p^n}$ on both sides:

$$\begin{aligned}\phi^{p^n-1}(\sigma_1) &= \sigma_1 \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \sigma_1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \sigma_1,\\ &\vdots\\ \phi^{p^n-1}(\sigma_{p^n-1}) &= 1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 + \sigma_1 \otimes 1 \otimes \cdots \otimes \sigma_1 + \cdots + \sigma_1 \otimes \cdots \otimes \sigma_1 \otimes 1\\ &\quad + q_{p^n-1}(\sigma_1, \sigma_2, \dots, \sigma_{p^n-1}),\\ \phi^{p^n-1}(\sigma_n) &= \sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1 + q_{p^n}(\sigma_1, \sigma_2, \dots, \sigma_{p^n}),\end{aligned}$$

where $q_i(\sigma_1, \sigma_2, \dots, \sigma_i)$ are degenerate tensors of degree i.

For a monomial $\sigma_1^{q_1}\sigma_2^{q_2}\cdots\sigma_{p^n-1}^{q_{p^n-1}}$ in the polynomial h, consider its comultiplication

$$\phi^{p^n-1}(\sigma_1^{q_1}\sigma_2^{q_2}\cdots\sigma_{p^n-1}^{q_{p^n-1}}) = \left(\phi^{p^n-1}(\sigma_1)\right)^{q_1}\left(\phi^{p^n-1}(\sigma_2)\right)^{q_2}\cdots\left(\phi^{p^n-1}(\sigma_{p^n-1})\right)^{q_{p^n-1}}.$$

Note the cyclic structure of each comultiplication $\phi^{p^n-1}(\sigma_i)$, which is due to the sum of tensor products being invariant by permutation over σ_i. Only the comultiplication $\phi^{p^n-1}(\sigma_{p^n})$ contains a single linear term $\overbrace{\sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1}^{p^n-1 \text{ times}}$, while all other $\phi^{p^n-1}(\sigma_i)$ have permutations of tensor products that are multiples of p. Thus, in the comultiplication $\phi^{p^n-1}\left(\sigma_1^{q_1}\sigma_2^{q_2}\cdots\sigma_{p^n-1}^{q_{p^n-1}}\right)$, the coefficient of the linear term $\overbrace{\sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1}^{p^n-1 \text{ times}}$ is a multiple of p because of the cyclic structure of the comultiplication for each σ_i.

The sum of the coefficients of $\overbrace{\sigma_1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_1}^{p^n - 1 \text{ times}}$ in $\phi^{p^n-1}\big(K(x_n)\big)$ is indivisible by the prime p, so y_n is the only term among all y_i which maps to σ_n. Thus, $\tilde{G}(x_n)$ contains $\gamma_{p^n}(x_0)$ with an unit coefficient, so $\tilde{G}$ is a coalgebra isomorphism in degree p^n. ☐

3.2 Induction on $W_R(n-1) \to W_R(n)$

Lemma 3.3. *Given an Hopf algebra surjection $F : W_R(n-1) \to H$, there is a Hopf algebra surjection $\tilde{F} : W_R(n) \to H$, such that the following diagram commutes:*

$$\begin{array}{ccc} W_R(n-1) & \xrightarrow{F} & H \\ \downarrow & \nearrow_{\tilde{F}} & \\ W_R(n) & & \end{array}$$

Proof. The Witt Hopf algebra $W_R(n-1)$ has generators $y_0, y_1, y_2, \ldots, y_{n-1}$, and we let $\tilde{F}(y_i) = F(y_i)$ for indices $0 \leq i < n$. Also, the generator $y_n \in W_R(n)$ maps to $\tilde{G}(y_n)$ as a coalgebra, and $\tilde{G}(y_n) = ce_n + f(e_0, e_1, e_2, \ldots, e_{n-1})$ for some unit $c \in R$. Thus, $\tilde{F}(y_n) = \tilde{G}(y_n)$ is an algebra isomorphism in degrees $\leq p^n$, so $\tilde{F}$ is a Hopf algebra surjection. ☐

4 Conclusion

We generalised the first part of Ravenel and Wilson's proof of a Hopf algebra isomorphism between bipolynomial Hopf algebras H whose generators have degrees of prime powers and the Witt Hopf algebra W_R over graded local rings R, which has applications in algebraic topology and other fields of mathematics. A potential path of future research would be to follow through on the second part of Ravenel and Wilson's proof to show that any bipolynomial Hopf algebra H is isomorphic to the tensor product of Witt Hopf algebras W_R over graded local rings R.

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Appendix

Lemma 5.1. *The dual of the bipolynomial Hopf algebra $R[x]$ is the divided power Hopf algebra $\Gamma_R[x^*]$ for rings $R = \mathbb{Z}_{(p)}$ and $\mathbb{F}_p$.*

Proof. By definition of duals, $x^*(x) = 1, (x^2)^*(x^2) = 1, (x^3)^*(x^3) = 1, \ldots, (x^n)^*(x^n) = 1$.

To express the dual $(x^n)^*$, consider the comultiplication $\phi^{n-1}(x^n) = \sum x_1 \otimes x_2 \otimes \cdots \otimes x_n$.

For a tensor product $x_1 \otimes x_2 \otimes \cdots \otimes x_n$, if any x_i is non-linear, then the tensor product is considered degenerate. Thus,

$$\begin{aligned}
\phi(x^2) &= 2\,x \otimes x + \text{degenerate terms } x^2 \otimes 1 + 1 \otimes x^2, \\
\phi^2(x^3) &= 6\,x \otimes x \otimes x + \text{degenerate terms}, \\
&\vdots \\
\phi^{n-1}(x^n) &= n!\overbrace{x \otimes x \otimes \cdots \otimes x}^{n-1 \text{ times}} + \text{degenerate terms},
\end{aligned}$$

so $(x^2)^* = \frac{(x^*)^2}{2}, (x^3)^* = \frac{(x^*)^3}{6}, \ldots, (x^n)^* = \frac{(x^*)^n}{n!}$. □

However, this dualisation does not hold for graded local rings R in general. Because of the following lemma, we mod by I_m so that the quotient space $R/I_m = \mathbb{F}_p$ is concentrated in degree 0, leading to better behavior than over R.

Lemma 5.2 (Nakayama's Lemma)**.** *If there is an isomorphism between finite free R-modules X and Y mod I_m, then there is an isomorphism between X and Y.*