Computing the Mosaic Number of Reduced Projections of Knots and Links

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Abstract

In 2008, Lomonaco and Kauffman introduced the concept of knot mosaics, which are diagrams representing knots on $n \times n$ grids. They defined an invariant, called the knot mosaic number, which is the smallest n such that the knot can be represented on an $n \times n$ mosaic. We provide elementary ways of computing the knot mosaic number of reduced projections of certain knots and links without completing an exhaustive search of small mosaic boards. We also answer a question asked by Lee and others by proving that the 8_3 knot does not fit in reduced form on a 6×6 board.

Summary

Take a piece of string, tie it in a knot, and glue the ends together. The result is something that cannot be untied. This is a mathematical knot. Much of knot theory is based in trying to distinguish two given knots. To do so, mathematicians look at knot invariants, which are properties of a knot that do not vary when the knot is stretched or deformed. We look at a particular invariant called the knot mosaic number. We improve our ability to compute the knot mosaic number of certain knots and links. In particular, we show that a well known knot with 8 crossings, the 8_3 knot, cannot be represented on a 6×6 board with only 8 crossing tiles.

1 Introduction

In 2008, Lomonaco and Kauffman [1] introduced the concept of knot mosaics in their paper on quantum knots. They define a *knot mosaic* as a representation of a knot on an $n \times n$ grid. For example, the mosaic representation of the figure eight knot is given in Figure 1.

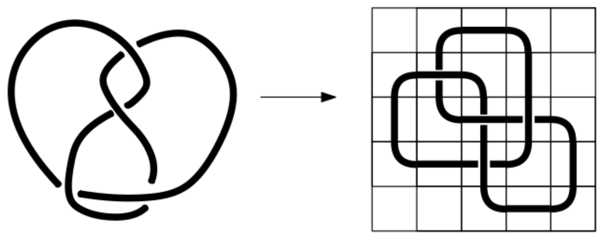


Figure 1: The figure eight knot and its mosaic representation.

The mosaic grids are composed of the 11 tiles defined in Figure 2, denoted T_0 to T_{10}. Every knot can be represented using only these 11 tiles.

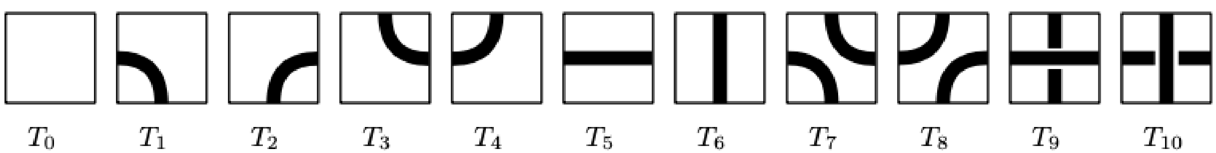


Figure 2: Knot mosaic tiles T_0 to T_{10}.

Lomonaco and Kauffman defined Reidemeister-like moves for knot mosaic representations and conjectured that tame knot theory was equivalent to knot mosaic theory. In other words, two knots are equivalent if and only if there exists a series of mosaic Reidemeister moves relating their two mosaic representations. This conjecture was later proved by Kuriya and Shehab [2]. A knot invariant one can look at is the mosaic number.

Definition 1.1 (Mosaic number)**.** For a knot K, we define the *mosaic number* $m(K)$ as the smallest integer n such that there exists a projection of K that can be represented on $n \times n$

mosaic grid.

A natural question one can ask is how the knot mosaic number relates to other knot invariants. In particular, the mosaic number can be bounded by the crossing number of a knot K, that is the minimum number of crossings in any projection of K, denoted by $c(K)$. Lee et al. [3], proved that $m(K) \geq \left\lceil \sqrt{c(K)} \right\rceil + 2$. Intuitively, it seems that the mosaic number of a knot should be realized when the crossing number is realized. If the projection of the knot uses more crossing tiles (tiles T_9 and T_{10}), this may increase the total number of tiles in the knot mosaic. Yet, this was disproved by Ludwig, Evans, and Paat [4], who produced an infinite family of knots whose mosaic number is realized in non-reduced projections.

Computing the mosaic number for a knot can be difficult because we need to show that it is impossible to represent a projection of a knot K on any board of smaller size. Previous papers have relied on completing an exhaustive search of knot mosaics for small boards with Knotscape or using crossing number arguments. Apart from these techniques, it is not obvious how to show that no projections of K fit on a smaller grid. We introduce other elementary tools to help compute knot mosaic number.

We begin in Section 3 by examining the impact of flypes on the different n-gons of a knot. We use those results in Sections 4 and 5 to compute the mosaic number of the 8_3 and 6_1 knots in reduced projections, answering a conjecture made by Lee et al. [3]. We then construct an infinite family of knots for which we compute the mosaic number in Section 6. Finally, in Section 7, we introduce crossing tile and shared tile bounds to help compute the mosaic number of certain split links. Section A.1 of the Appendix contains proofs omitted from the main text.

2 Preliminaries

We define some terminology related to knot theory as given by Adams [5].

Definition 2.1 (Knot)**.** A *knot* is a closed curve in 3-space that does not intersect itself anywhere. We do not distinguish between the original closed knotted curve and the deformations of that curve through space that do not allow the curve to pass through itself. The different pictures of the knot that result from these deformations are called *projections* of the knot.

Invariants are used to classify knots. One of the most natural invariants to look at is the crossing number.

Definition 2.2 (Crossing number)**.** The *crossing number* of a knot K is the minimal number of crossings of any projection of K. It is denoted $c(K)$.

We can classify knots on the basis of particular properties of their crossings.

Definition 2.3 (Alternating knot)**.** An *alternating knot* is a knot with a projection that has crossings that alternate between over and under as one travels around the knot in a fixed direction.

Given two knots, we can define a new knot by composing the two.

Definition 2.4 (Composition of two knots)**.** The *composition* of two knots J and K is the knot obtained by removing a small arc from each knot projection and then connecting the four endpoints by two new arcs. It is denoted by $J\#K$.

An example of the composition of a trefoil and a figure eight knot is given in Figure 3.

Definition 2.5 (Prime knot)**.** If a knot is not the composition of two non-trivial knots, we call it a *prime* knot.

We can also look at a collection of knots.

Definition 2.6 (Link)**.** A *link* is a set of knots that do not intersect each other but can be tangled together. Each knot that makes up the link is called a *component*.

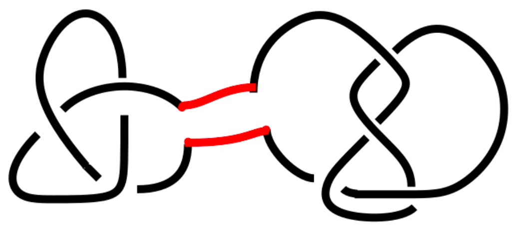


Figure 3: The composition of a trefoil and a figure eight knot.

Definition 2.7 (Split link)**.** A *split link* is a link whose components can be deformed so that they lie on different sides of a plane in 3-space.

We now set up definitions specific to knot mosaics that we use in the remainder of the paper.

Definition 2.8 (Connection point)**.** We call the midpoint of an edge of a tile a *connection point* if it is also the endpoint of a curve drawn on that tile, see Figure 4.

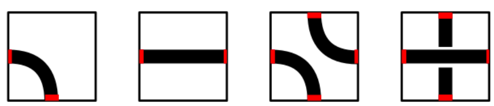
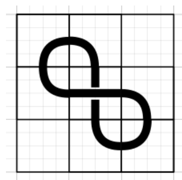


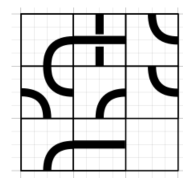
Figure 4: In red, connection points of tiles T_1, T_5, T_7, and T_9.

Definition 2.9 (Suitably connected)**.** A tile in a mosaic is said to be *suitably connected* if all of its connection points touch a connection point on another tile.

An example of a board in which all tiles are suitably connected is given in Figure 5a and a board in which some tiles are not suitably connected is given in Figure 5b. Boards in which all tiles are suitably connected define knots or links.



(a) A mosaic in which every tile is suitably connected.



(b) A mosaic in which not all tiles are suitably connected.

Figure 5: Examples of 3×3 mosaic boards in which some or all tiles are suitably connected.

We define a set of coordinates on our $n \times n$ grid where (i, j) denotes the tile in the i-th row and the j-th column. We set the upper left corner as $(1, 1)$ and the lower right corner as (n, n).

Definition 2.10 (Inner boundary)**.** The *inner boundary* is the set of tiles exactly one tile away from the border of the mosaic. More formally, on an $n \times n$ mosaic board, the inner boundary is the set of tile positions

$$\{(i, j) : 1 < i < n, 2 < j < n\} \setminus \{(i, j) : 2 < i < n - 1, 2 < j < n - 1\},$$

see Figure 6.

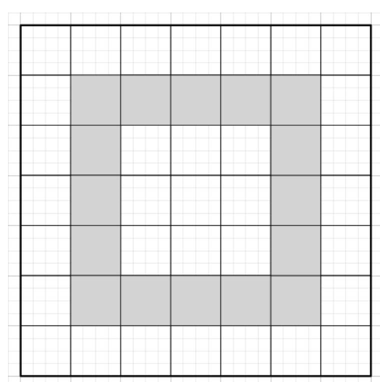


Figure 6: The inner boundary of a 7×7 mosaic board.

A mosaic representation of a knot delimits several shapes as defined by Ludwig et al. [4]. We can look both at the outer boundary of the knot, which we call the perimeter, and the n-gons that appear on the inside of the knot.

Definition 2.11 (Perimeter of a knot mosaic)**.** The *perimeter* of a knot mosaic is the set of boundary tiles of the knot. To find the perimeter of a knot, take a horizontal bar starting above the knot and drag it down until a mosaic tile is hit. We know the tile hit is not a crossing tile because crossing tiles cannot be placed on the boundary of a mosaic board. Pick any one of the mosaic tiles hit and start traveling counterclockwise. Follow the arc of that tile until a crossing tile is reached. At each crossing tile reached, rotate 90° clockwise and follow that strand. Repeat this process until you return to the original tile. An example is given in Figure 7.

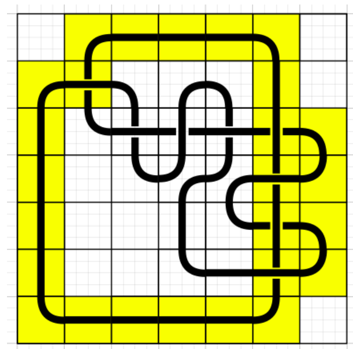


Figure 7: The perimeter tiles of a knot mosaic of 8_3.

Definition 2.12 (n-gon of a knot)**.** Similarly to the perimeter of knot, an *n-gon* is a shape found by starting at any crossing tile of the knot, traveling counterclockwise along the arc of that tile and rotating 90° clockwise at every crossing tile reached. If n crossing tiles were reached in the making of this shape, it is an n-gon.

An example of the n-gons in a trefoil is given in Figure 8.

Finally, we introduce a proposition that is essential to linking reduced alternating projections of knots to their crossing number.

Proposition 2.1 (Kauffman [6], Murasagi [7], and Thistlethwaite [8])**.** *The crossing number of an alternating knot occurs in a reduced alternating projection.*

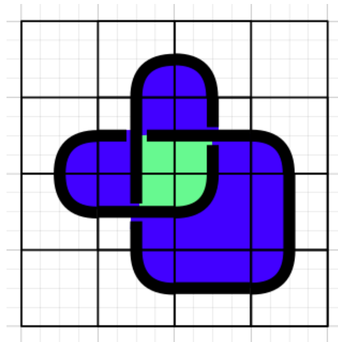


Figure 8: A knot mosaic of the trefoil with 3-gons in green and 2-gons in blue.

3 Flypes and n-gons

The Tait Flyping Conjecture, proved by Menasco and Thistlethwaite [9] tells us that given any two reduced alternating diagrams D_1 and D_2 of an oriented, prime alternating knot, D_1 may be transformed to D_2 by a sequence of flypes. A flype is a move in which a portion of a knot, represented by a shaded disc labeled T, is rotated 180° so that the crossing to its left is removed by untwisting and a new crossing is created to its right, see Figure 9.

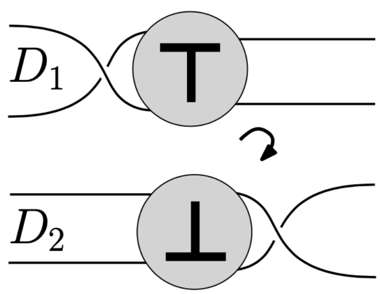


Figure 9: A flype of a portion of a knot.

When looking at the n-gons of a knot, it is important to look at the different projections of the knot obtained by a sequence of flypes because the number of n-gons in a knot can change after a flype, as in Figure 10. In this example, the portion of a knot being flyped is framed in red. As a result of the flype, the figure loses a 4-gon and 2-gon and gains two 3-gons.

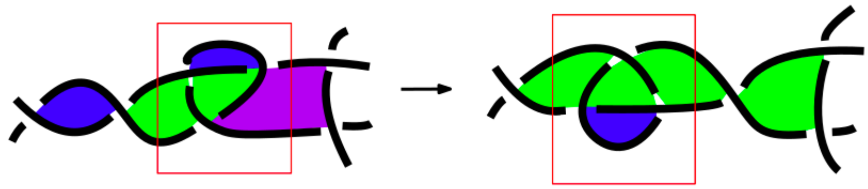


Figure 10: A flype in which the number of n-gons is not preserved, with 4-gons in purple, 3-gons in green, and 2-gons in blue.

Some knots have very few flypes. Ersnt et al. [10] proved that some knots have no flypes at all, meaning that two reduced projections of the knot are related by a series of planar isotopies only. Planar isotopy does not modify the number of n-gons in projections of a knot.

Lemma 3.1. *Planar isotopy preserves the number of n-gons in a projection of a knot.*

Proof. We want to show that planar isotopy does not change the number of n-gons in a projection of a knot. To do so, we refer to the 11 planar isotopy moves on knot mosaics given by Lomonaco and Kauffman [1]. Moves P_1 through P_7 do not change any crossing tiles so therefore do not alter the n-gons in a projection of the knot. Thus, we only need to consider moves P_8, P_9, P_{10}, and P_{11}, see Figure 11. For each of the four polygons bounded by the two strands in the knot diagrams of the isotopy moves, the number of crossing tiles on the perimeter of each n-gon does not vary. Therefore, the n-gons are preserved. □

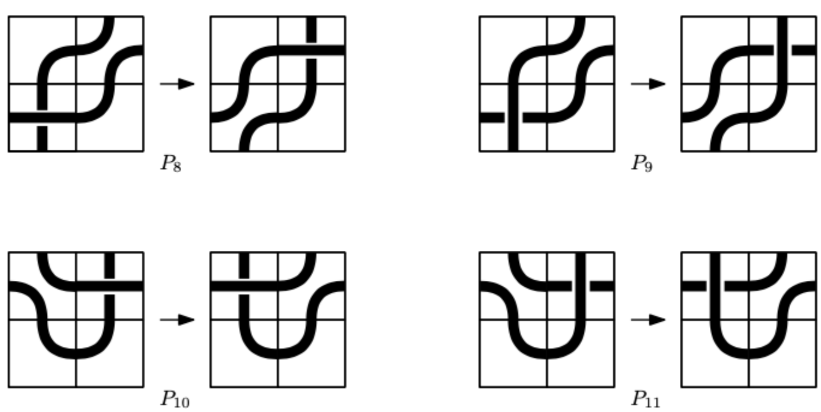


Figure 11: Planar isotopy moves P_8 through P_{10} that preserve n-gons.

4 Mosaic number of a reduced projection of 8_3

Certain knots realize their mosaic number only in non-reduced projections. Yet, it is hard to prove this result for any given knot without enumerating all the possible boards of a certain size. For example, the result that a reduced projection of 6_1 does not fit on a 5×5 board was proved only by exhaustive computation. Lee et al. [3] asked whether there exists a representation of 8_3, 8_6, 8_9, or 8_{11} on a 6×6 board with only eight crossing tiles. We prove that 8_3 does not fit in reduced form on a 6×6 board. A reduced projection of 8_3 fits on a 7×7 board, as shown in Figure 12.

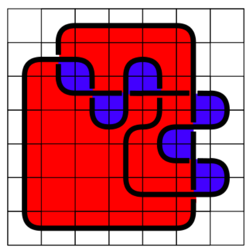


Figure 12: Reduced projection of 8_3 on a 7×7 board with 5-gons in red and 2-gons in blue.

Ernst et al. [10] showed that 8_3 had no possible flypes. Therefore, the different projections of a reduced diagram of the 8_3 vary only up to planar isotopy.

Lemma 4.1. *All reduced alternating projections of* 8_3 *only contain exactly four 5-gons (including the perimeter) and six 2-gons.*

Proof. The projection of 8_3 in Figure 12 satisfies this property. Since there are no flypes possible on 8_3, two reduced alternating projections of the knot are related by a series of planar isotopies. By Lemma 3.1, we know that these planar isotopy moves do not change the n-gons that make up 8_3. □

Lemma 4.2. *A reduced alternating projection of* 8_3 *on a* 6×6 *board has exactly 5 crossing tiles on the inner boundary.*

We call the configuration of tiles in Figure 13 an *almost-square.*

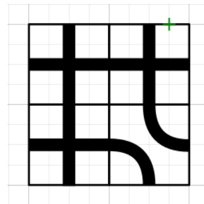


Figure 13: An almost-square configuration on a 2×2 mosaic board.

Lemma 4.3. *If a reduced projection of* 8_3 *is represented on a 6×6 mosaic board, the inner 2×2 board looks like an almost-square up to rotation.*

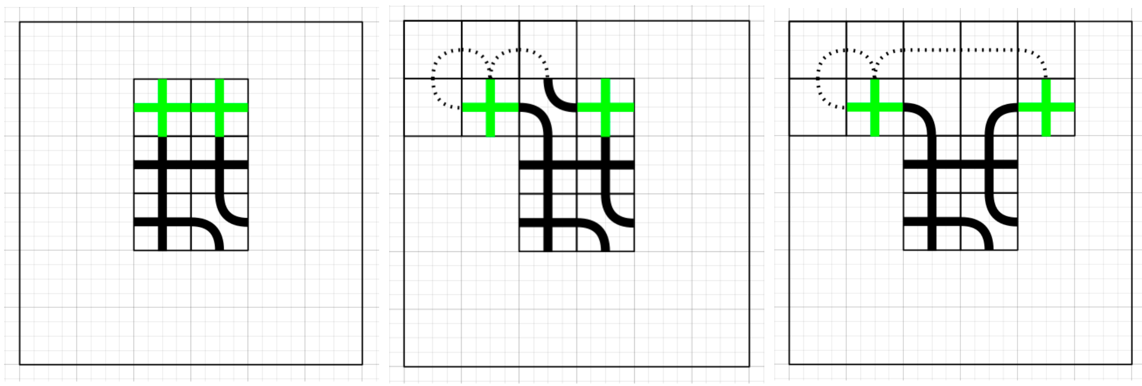
We use Lemmas 4.1 through 4.3 to prove that a reduced projection of 8_3 does not fit on a board smaller than 7×7.

Theorem 4.4. *There is no reduced alternating projection of* 8_3 *that fits onto a 6×6 board.*

Proof. By Lemmas 4.2 and 4.3, we know the inner 2×2 board is equivalent to an almost-square up to rotation and that there are 5 crossing tiles on the inner boundary. We then separate our work into two cases based on the positions on the inner boundary on which these 5 tiles are placed.

Case 1: There are two crossing tiles in $\{(2, m) : 2 \leq m \leq 5\}$ or there are two crossing tiles in $\{(m, 2) : 2 \leq m \leq 5\}$.

We can suppose without loss of generality that the two crossing tiles are in $\{(2, m) : 2 \leq m \leq 5\}$. There are three configurations of these two green crossing tiles that we need to look at, shown in Figure 14.



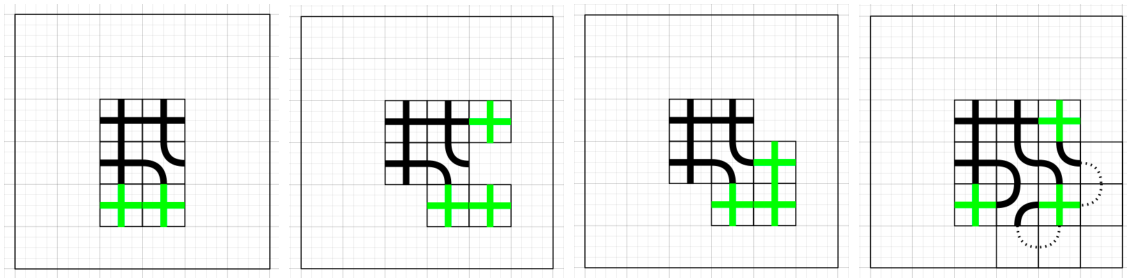
(a) Config. 1 (b) Config. 2 (c) Config. 3

Figure 14: Configurations of two crossing tiles in Case 1.

- In Config. 1, there is a 4-gon, which contradicts Lemma 4.1.
- In the Config. 2, the tile in position (2,3) has at least three connection points. To avoid creating a 3-gon or a 4-gon, the tile must be T_7. Looking at the connection point at the top of tile (2,2), the two possible paths coming out of it represented by dotted lines either create a 4-gon or create a loop easily removable by a Reidemeister Type I move, meaning our knot is not reduced.
- In the Config. 3, the tile in positions (2,3) needs to be T_1 or T_7 and the tile in (2,4) needs to be T_2 or T_8 in order to avoid 3-gons and 4-gons. Looking at the connection point at the top of tile (2,2), the two possible paths either create an easily removable loop or a 4-gon.

Case 2: There are at least 3 crossing tiles in $\{(5, m) : 3 \leq m \leq 5\} \bigcup \{(m, 5) : 3 \leq m \leq 5\}$.

There are four configurations of tiles that we need to examine, shown in Figure 15. Up to symmetry, these configurations eliminate all possibilities for the placement of three crossing tiles within that area.



(a) Config. 1 (b) Config. 2 (c) Config. 3 (d) Config. 4

Figure 15: Configurations of two crossing tiles in Case 12.

- In Config. 1, there is a 3-gon no matter where the last of the green crossing tiles is placed.

- In Config. 2, if the tile in position (4,5) is T_8 or a crossing tile, then the n-gon created by the three center crossings and the two bottom crossings is greater than a 5-gon. If the tile is T_7, the resulting projection will either contain a 3-gon or have a loop that will make it non-reduced.

- In the Config. 3, there is a 6-gon.

- In Config. 4, if a crossing tile is placed in either position (5,4) or (4,5), a 3-gon is created. If both (5,4) and (4,5) are T_7 or both are T_8, the center n-gon is either a 4-gon a 6-gon respectively. Yet, if one of (5,4) and (4,5) is T_7 and the other is T_8, the resulting knot will either be non reduced or will contain a 3-gon.

□

Remark. This proof relies heavily on the fact that 8_3 has no flypes and only has 5-gons and 2-gons. The methods used are hard to extend to the other knots mentioned by Lee et al. [3] (8_6, 8_9, and 8_{11}) because these knots have multiple projections up to flypes and have many different types of n-gons.

5 Mosaic number of a reduced projection of 6_1

Similarly to Theorem 5.5, we want to prove that 6_1 cannot fit in reduced form on a 5×5 board. Ludwig et al. [4] claim that Jacob Shapiro of Purdue University used an exhaustive search to prove this result, but we offer an elementary proof. Ludwig et al. proved that 6_1 only has one possible flype. Up to planar isotopy, the two projections of 6_1 are given in Figure 16.

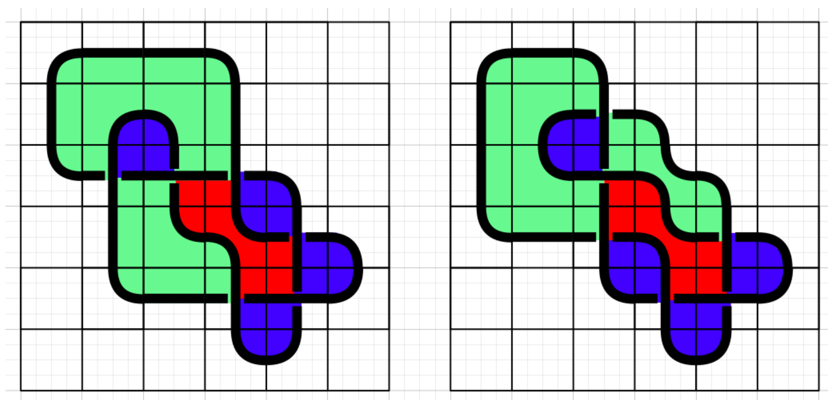


Figure 16: Both reduced projections of 6_1 with 5-gons in red, 3-gons in green, and 2-gons in blue.

Definition 5.1 (Consecutive corners)**.** Two corners of the inner boundary of an $n \times n$ board are said to be *consecutive corners* if they lie on the same row or the same column.

Lemma 5.1. *In a reduced projection of* 6_1 *on a* 5×5 *board, there are either three consecutive crossing tiles along the inner boundary or two consecutive corners contain crossing tiles.*

Theorem 5.2. *No reduced projection of* 6_1 *fits on a board of size* 5×5*.*

Proof. Because both projections have two 5-gons (including the perimeter), two 3-gons, and four 2-gons, we conclude that all reduced alternating projections of 6_1 are composed of exactly these n-gons. We show that, in the cases when there are three consecutive crossing tiles along the inner boundary or two consecutive corners with crossing tiles, the resulting knot is not reduced.

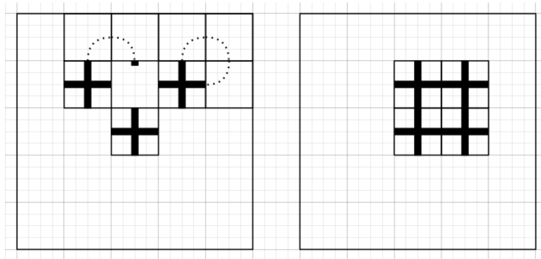


Figure 17: Crossing tile configuration in a reduced projection of 6_1 on a 5×5 board.

We look at the two configurations in Figure 17. In the first configuration, the tile in position $(2, 3)$ has three connection points, so it must have a fourth. There is no way to connect the top row without creating a loop easily removable by a Type I Reidemeister move. In the second configuration, there is a 4-gon which contradicts the fact that 6_1 is soleley composed of 5-gons, 3-gons, and 2-gons. □

6 Construction of an infinite family of knots for which we can compute the mosaic number

Computing the mosaic number of the composition of knots can be difficult because the number of crossings in the composition of knots grows linearly whereas the crossing bound given by the size of the grid grows quadratically. To overcome this problem, we take the composition of a given knot with an L-shape we define below.

Definition 6.1 ($L_{2n,2m+1}$)**.** We define $L_{2n,2m+1}$ as the alternating knot which contains crossing tiles in positions

$$\{(i,j) : 1 < i < 2n+2m+1, 1 < jM2n+2m+1\} \setminus \{(i,j) : 1 < i \leq 2n+1, 1 < j \leq 2n+1\}$$
$$\setminus \{(2n+2m,2),(2n+2m,2)\}.$$

In other words, the knot is creating an L-shape on the board, as in Figure 18.

The parity of the different portions of the L-shape ensure that it is always possible to create such a knot. Given that $L_{2n,2m+1}$ is an alternating knot and that it has densely dispersed crossing tiles, fixing one of the crossing tiles as either T_9 or T_{10} forces the type of the remaining crossing tiles. We can conclude that $L_{2n,2m+1}$ is unique. By Proposition 2.1, because there are no obvious reductions, $L_{2n,2m+1}$ is reduced.

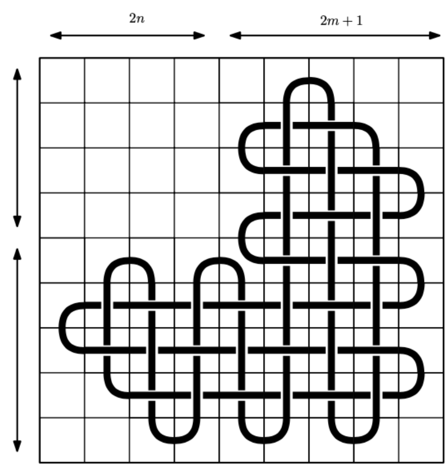


Figure 18: Example of $L_{4,5}$.

For any reduced alternating projection of a knot K with mosaic number less than or equal to $2n$, we can take the composition of K with any $L_{2n,2m+1}$ by placing K in the upper left-hand corner of a board of size $(2n+2m+1) \times (2n+2m+1)$, (Figure 19). We now would like to determine the mosaic number of this composition. We first introduce a lemma relating crossing number and knot composition.

Lemma 6.1 (Grandy [11]). *Crossing number is additive with respect to the composition of two alternating knots.*

Theorem 6.2. *For any reduced, alternating projection of a knot K such that $m(K) \leq 2n$, if $m > n^2 - n + 1$, $K\#L_{2n,2m+1}$ has mosaic number $2n + 2m + 1$.*

Proof. By Lemma 6.1, $c\left(K\#L_{2n,2m+1}\right) = c\left(L_{2n,2m+1}\right) + c(K)$. Because $L_{2n,2m+1}$ is reduced and alternating by construction, we determine that $c\left(L_{2n,2m+1}\right) = 4m^2 - 4m + 8nm - 4n - 1$.

We know that $K\#L_{2n,2m+1}$ fits on a board of size $(2n+2m+1) \times (2n+2m+1)$. A board of size $(2n+2m) \times (2n+2m)$ only has space for $4n^2-8n+4m^2-8m+8nm+4$ crossing tiles. Yet, when $m > n^2-n+1$,

$$c\left(K\#L_{2n,2m+1}\right) > c\left(L_{2n,2m+1}\right) > 4n^2-8n+4m^2-8m+8nm+4.$$

Therefore, $K\#L_{2n,2m+1}$ cannot fit on a smaller board. For all $m > n^2-n+1$, we can compute the mosaic number of the knot $K\#L_{2n,2m+1}$. □

For example, the composition of the trefoil and $L_{4,9}$ has mosaic number exactly $2n+2m+1=13$, as shown in Figure 19.

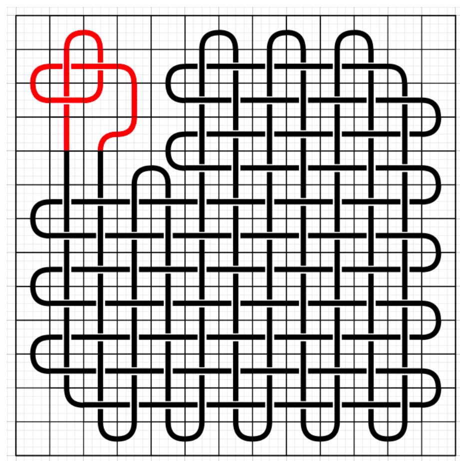


Figure 19: Composition of a trefoil in red and $L_{4,9}$.

7 Computing the mosaic number of split links

The mosaic number of split links is often much smaller than the sum of the mosaic number of the components because the link fills the space of a square grid better. Interesting problems arise when attempting to compute the mosaic number of certain split links. We try and generalize information about a single knot to a split link of multiple copies of that knot. Apart from computing the mosaic number of a knot, the smallest grid on which a knot fits, we can also compute the minimum number of tiles needed to create a knot mosaic for a certain knot, the tile number. We introduce definitions as given by Heap and Knowles [12].

Definition 7.1 (Tile number)**.** The *tile number* of a knot K, denoted $t(K)$, is the minimum number of tiles needed to represent K on a knot mosaic.

Definition 7.2 (Space-efficient)**.** A mosaic representation of a knot K is said to be *space-efficient* if it is reduced and the number of tiles used is minimised.

We introduce notation useful for quantifying the number of tiles used in a split link.

Definition 7.3 ($t_\lambda(K)$)**.** For a knot or link K in a given knot mosaic λ, we let $t_\lambda(K)$ denote the number of tiles used by K.

Definition 7.4 ($s_\lambda(K)$)**.** For a split link K in a knot mosaic λ, we let $s_\lambda(K)$ denote the number of tiles shared by two distinct components of the link.

7.1 Mosaic number of a reduced projection of the split link of two knots that form hammer shapes in space-efficient form

We make arguments which leverage space-efficiency and tile numbers. This is the first example of using tile number to compute mosaic number. To do so, we introduce a lemma bounding the maximum number of tiles that can be used in a reduced knot mosaic.

Lemma 7.1 (Heap, Knowles [12])**.** *For any reduced projection of a knot K with mosaic number m, if m is even, $t(K) \leq m^2 - 4$ and if m is odd then $t(K) \leq m^2 - 8$.*

Heap and Knowles [12] enumerated the possible layouts for space-efficient knots on a 6×6 mosaic board. In particular, up to rotation, we call the layout in Figure 20 a *hammer*.

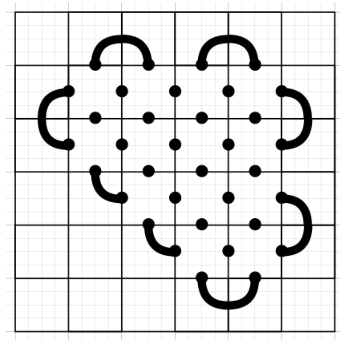


Figure 20: The hammer layout for a space-efficient knot on a 6×6 board.

We would like to compute the mosaic number for the split link of two knots that form hammer shapes in space-efficient form.

Theorem 7.2. *The mosaic number of a reduced projection of the split link of two knots that form hammers in space-efficient form is 8.*

Proof. In Figure 21, we have a knot mosaic of the split link of two hammers on an 8×8 board.

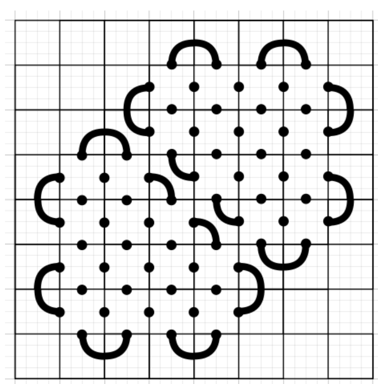


Figure 21: Knot mosaic of the split link of two hammers on an 8×8 board.

We want to show that this link does not fit on a 7×7 board. We call the two knots that are hammers in space-efficient form K_1 and K_2 and denote by L the split link of K_1 and K_2. We suppose for contradiction that L fits on a 7×7 board in some reduced knot mosaic λ. By Lemma 7.1, if a knot or link has a mosaic representation on a 7×7 board, it has tile number less than or equal to $7^2 - 8 = 41$. Therefore, $t_\lambda(L) \leq 41$. Since both K_1 and K_2 form hammer shapes in space-efficient form, we know that $t(K_1) = t(K_2) = 27$. By Definitions 7.3 and 7.4, we know that $t_\lambda(L) = t_\lambda(K_1) + t_\lambda(K_2) - s_\lambda(L) \geq t(K_1) + t(K_2) - s_\lambda(L) = 54 - s_\lambda(L)$. Putting together the two inequalities, we get that $s_\lambda(L) \geq 13$. Heap and Knowles [13] showed that space-efficient hammers have 9, 10, or 11 crossings, so $c(L) = c(K_1) + c(K_2) \geq 18$. Therefore, $c(L) + s_\lambda(L) \geq 18 + 13 = 31$. Tiles T_7 through T_{10} cannot be placed on the boundary of a suitably connected mosaic, so a 7×7 board only has space for 25 tiles with four connection points. Yet, both the shared tiles and the crossing tiles of L must have four connection points, so $c(L) + s_\lambda(L) \leq 25$. Thus, by contradiction, L does not fit on a 7×7 board. □

7.2 Mosaic number of a reduced projection of the split link of four knots that form gummy bears in space-efficient form

We use similar crossing tile and shared tile arguments for knots that form different shapes in space-efficient form. In particular, we want to look at knots that have tile number 17 and are space-efficient on a 5×5 mosaic board. Heap and Knowles [12] showed that any space-efficient mosaic on a 5×5 board has a *gummy bear* layout, as shown in Figure 22.

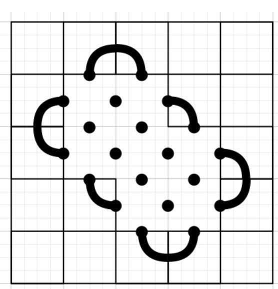


Figure 22: Gummy bear layout for a space-efficient knot on a 5×5 board.

Similarly to the proof of Theorem 7.2, we look to bound the crossing and shared tiles. To come to a contradiction, we need the crossing number of the knots to be at least 7. Heap and Knowles [13] showed that the only prime knot with 7 crossings and tile number 17 is 7_4.

Theorem 7.3. *The mosaic number of a reduced projection of the split link of four 7_4 knots is 9.*

Proof. Similarly to the proof of Theorem 8.2, we bound the number of crossing tiles and shared tiles of the link. We call the link L and the four components K_1 through K_4. We have a knot mosaic of L that fits on a 9×9 board, see Figure 23.

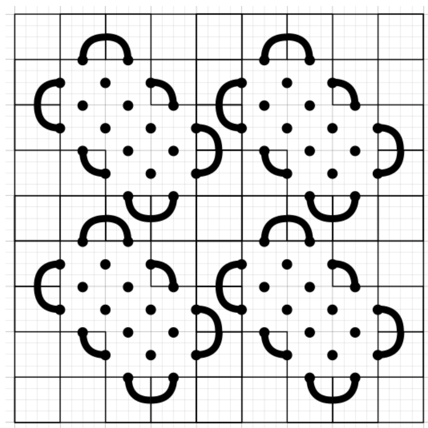


Figure 23: Knot mosaic of the split link of four gummy bears on a 9×9 board.

Suppose that L fits on an 8×8 board. We are looking at the split link formed by four 7_4 knots, so $t(K_1) = t(K_2) = t(K_3) = t(K_4) = 17$ and $c(K_1) = c(K_2) = c(K_3) = c(K_4) = 7$. We have that $t_\lambda(L) = \sum_{i=1}^{4} t_\lambda(K_i) - s_\lambda(L) \geq 68 - s_\lambda(L)$. Since $t_\lambda(L) \leq 8^2 - 4 = 60$, we know that $s_\lambda \geq 8$, so $\sum_{i=1}^{4} c(K_i) + s_\lambda(L) \geq 28 + 8 = 36$. There are 36 spaces to place tiles with four connection points, so $\sum_{i=1}^{4} c(K_i) + s_\lambda(L) \leq 36$. We therefore know that $\sum_{i=1}^{4} c(K_i) + s_\lambda(L) = 36$. This means that if L fits on an 8×8 board, then all of its components must use exactly 17 tiles and every inner tile is either a crossing tile or a shared tile. Our bound also tells us that every position on the board except for the corners must contain a tile. In particular, positions (1,2), (2,1), (2,2) and (1,7), (2,7), (2,8) must contain tiles. All four components use exactly 17 tiles, so they must form gummy bears. Yet, the only way to place gummy bears to fill tiles (1,2), (2,1), (2,2) and (1,7), (2,7), (2,8) creates overlap, as shown in Figure 24, because it is impossible for a single gummy bear to fill all those tiles.

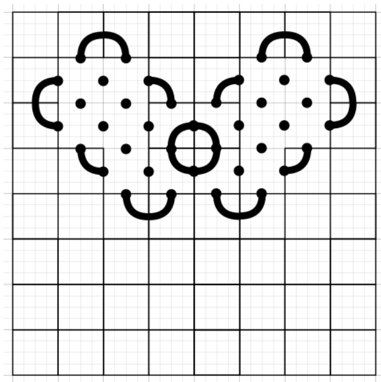


Figure 24: Overlap created when trying to fit four gummy bears on a 7×7 board.

Not only is the overlap impossible to create with tiles T_0 through T_{10}, it also would not result in a reduced link. Thus, by contradiction, the mosaic number of a reduced projection of the split link of four 7_4 knots is 9. □

Remark. We can extend these bounds to knots that form other shapes in space-efficient form. For small cases, even if the bound is not sufficient to get a contradiction, as in Theorem 7.3, we can split the problem into several cases to prove the desired result. Further work could be done on computing the mosaic number for reduced projections of split links of different knots by bounding the crossing tiles and shared tiles.

8 Practical Takeaways

Knot invariants are crucial for distinguishing knots. It is always useful to have a knot invariant that is simple enough to compute but strong enough to distinguish between two knots. Finding ways to compute the knot mosaic number for specific knots and links is therefore an effective way to gain more intuition on how to compute this invariant in more general cases. In particular, finding elementary arguments that do not rely on an exhaustive search is a helpful step in generalizing results about knot mosaic number. The proofs provided in this paper yield insight which can be applied to future computations of the knot mosaic number. More generally, work on knot theory has applications ranging from biology and chemistry to physics, such as using knots to describe the structure of tangles in DNA.

9 Future Work

We develop tools to help compute the mosaic number of certain reduced projections of knots and links. Further work could be done on trying to improve and generalize these tools to compute the mosaic number for a broader set of knots and links.

In particular, work could be done to gain a better understanding of the relationship between flypes and n-gons of a knot. For example, we could ask what the minimum number of tiles or minimum size $x \times y$ board that can contain an n-gon is. We could also investigate how the number of n-gons vary in different reduced projections of a given knot and ask how can we compute the number of flypes that a certain reduced projection of a knot has.

Further work could be done to compute the mosaic number of reduced and non-reduced projections of knots. For instance, can we prove that there does not exist a representation of 8_6, 8_9, and 8_{11} in reduced form on a 6×6 mosaic board? We can also ask whether or not there are knots with mosaic number n that can only be represented in reduced form on an $(n+2) \times (n+2)$ grid.

Finally, we could improve our understanding of the mosaic number of split links. We could ask whether the mosaic number of the split link of 6_1 and $L_{4,2m+1}$ is only realized in a non-reduced projection. We could also ask if there are other knots with different space-efficient shapes for which we can compute the mosaic number of the split link of multiple of these knots.

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References

[1] S. J. Lomonaco and L. H. Kauffman. Quantum knots and mosaics. *Quantum Information Processing*, 7(2):85–115, 2008.

[2] T. Kuriya and O. Shehab. The lomonaco–kauffman conjecture. *Journal of Knot Theory and Its Ramifications*, 23(01):1450003, 2014.

[3] H. J. Lee, L. Ludwig, J. Paat, and A. Peiffer. Knot mosaic tabulation. *Involve, a Journal of Mathematics*, 11(1):13–26, 2017.

[4] L. D. Ludwig, E. L. Evans, and J. S. Paat. An infinite family of knots whose mosaic number is realized in non-reduced projections. *Journal of Knot Theory and Its Ramifications*, 22(07):1350036, 2013.

[5] C. C. Adams. *The Knot Book*. American Mathematical Soc., 1994.

[6] L. H. Kauffman. New invariants in the theory of knots. *The American Mathematical Monthly*, 95(3):195–242, 1988.

[7] K. Murasugi. Jones polynomials and classical conjectures in knot theory. *Topology*, 26(2):187–194, 1987.

[8] M. B. Thistlethwaite. A spanning tree expansion of the jones polynomial. *Topology*, 26(3):297–309, 1987.

[9] W. W. Menasco and M. B. Thistlethwaite. The tait flyping conjecture. *Bulletin (New Series) of the American Mathematical Society*, 25(2):403–412, 1991.

[10] C. Ernst, C. Hart, T. Menezes, and D. Price. A complete list of minimal diagrams of an oriented alternating knot. *Journal of Knot Theory and Its Ramifications*, 30(08):2150063, 2021.

[11] J. Grandy. The additivity of crossing number with respect to the composition of knots. 2010.

[12] A. Heap and D. Knowles. Tile number and space-efficient knot mosaics. *Journal of Knot Theory and Its Ramifications*, 27(06):1850041, 2018.

[13] A. Heap and D. Knowles. Space-efficient knot mosaics for prime knots with mosaic number 6. *Involve, a Journal of Mathematics*, 12(5):767–789, 2019.

[14] H. J. Lee, K. Hong, H. Lee, and S. Oh. Mosaic number of knots. *Journal of Knot Theory and Its Ramifications*, 23(13):1450069, 2014.

A Appendix

A.1 Omitted Proofs

Proof of Lemma 4.2.

Proof. From Lemma 4.1, we know that the perimeter of any reduced alternating projection of 8_3 is a 5-gon. Because each crossing tile on the inner boundary contributes an edge to the perimeter, if there are more than 5 crossings on the inner boundary, then the perimeter is not determined by a 5-gon. We represent a projection of 8_3 with exactly 8 crossings. Because crossing tiles cannot be placed on the outer boundary of the mosaic, if there are fewer than 4 crossing tiles on the inner boundary, there will not be enough space remaining on the board to place the rest of the crossing tiles. Finally, if there are exactly 4 crossings on the inner boundary, then the mosaic contains a 2×2 board filled with crossing tiles. Yet, this creates a 4-gon, which, by Lemma 4.1, does not appear in any reduced alternating projection of 8_3. □

Proof of Lemma 4.3.

Proof. We want to show that the inner 2×2 board is equivalent to Figure 13 up to rotation. By Lemma 4.2, we know that there are 5 crossing tiles on the inner boundary. Since our knot has exactly 8 crossings, there must be one non-crossing tile in position (3,3), (3,4), (4,3), or (4,4). We can suppose without loss of generality that the non-crossing tile is in position (4,4) because of rotational symmetry. Given the connection points, the only possibilities for the tile in position (4,4) are T_4, T_7, T_8, T_9, or T_{10}. If the tile is T_4 or T_8, a 3-gon is created, and if the tile is T_9 or T_{10}, a 4-gon is created, which both contradict Lemma 4.1. Therefore, the tile must be T_7. □

Proof of Lemma 5.1.

Proof. Similar to the proof of Lemma 4.2, we know that the perimeter of 6_1 is determined by a 5-gon so there are at most 5 crossing tiles on the inner boundary. There cannot be fewer than 5 crossings on the inner boundary because there is not enough space on the inner 1×1 board to fit more than one crossing tile. We want to show that there are either three consecutive crossing tiles along the inner boundary or that two consecutive corners contain crossing tiles. We suppose for contradiction that this is not the case. Then, there are 8 tiles on the inner boundary and exactly 5 of them are crossing tiles. We can therefore look at the positions of the 3 non-crossing tiles. If there are two adjacent non-crossing tiles, then there will be three consecutive crossing tiles. If there are no two adjacent non-crossing tiles, then every non-crossing tile is surrounded by two crossing tiles. If a non-crossing tile is in position $(2,3), (3,2), (3,4)$, or $(4,3)$, there will be two consecutive corners with crossing tiles, which contradicts the original assumption. The three non-crossing tiles must be in the corners of the inner boundary. Yet, this results in three adjacent crossing tiles. □

A.2 Links K that satisfy $m(K) = \left\lceil \sqrt{c(K)} \right\rceil + 2$

To make use of crossing number bounds, the number of crossing tiles in the knot or link must be very close to the size of the grid. In contrast with the rest of the paper, where naive crossing number bounds are not sufficient to compute mosaic number, we now look at an example in which the crossing number is maximized; that is, $m(K) = \left\lceil \sqrt{c(K)} \right\rceil + 2$.

All the inner tiles of the mosaic must be crossing tiles. For the resulting link to be reduced, Lee et al. [3] showed that the mosaic number $m(K)$ must be even. Lee et al. [14] also showed that there are only two ways to complete the board to get a suitably connected mosaic. Firstly, a Reidemeister Type I move can be applied to any corner of the board, meaning the knot or link is not reduced. Otherwise, if the resulting knot or link is alternating, it will be reduced. By Proposition 2.1, the crossing number is realized in this link and $m(K) = \left\lceil \sqrt{c(K)} \right\rceil + 2$. An example of a link that realizes the maximal number of crossings on a 4×4 board is given in Figure 25.

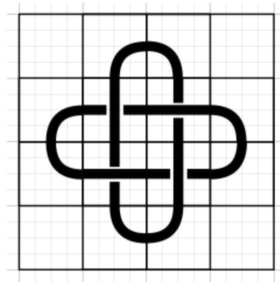

Figure 25: Solomon's knot, a link that has maximal crossings on a 4×4 board.