

# Mod $p$ Homology of Unordered Configuration Spaces of Surfaces

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## Abstract

We prove that dimensions of the mod  $p$  homology groups of the unordered configuration space  $B_k(T)$  of  $k$  points in a torus are equivalent to its Betti numbers for  $p > 2$  and  $k \leq p$ . Hence, the integral homology has no  $p$ -power torsion. I use the same argument to explicitly compute the dimensions of the mod 2 homology groups of the unordered configuration space  $B_2(\mathbb{R}P^{2n+1})$  of two points in any odd dimensional real projective space. The same argument works for the punctured genus  $g$  surface with  $g > 0$ , thereby recovering a result of Brantner-Hahn-Knudsen via Lubin-Tate theory.

## Summary

Given two surfaces, it is a classical interest of topologists to determine whether you can continuously deform one surface into the other, as if the two surfaces were like play-doh. One approach is to look for algebraic structures that remain unchanged regardless of how you deform either surface. Then, if the two surfaces have different such structures, we can conclude they cannot be deformed into one another. In this project, we compute a special algebraic structure for two classes of topological objects by breaking down the structure into different dimensional components. By computing the sizes of these different dimensional components, we can also better understand how complex these topological objects are—the larger the size, the more complex.

# 1 Introduction

Configuration spaces are important mathematical objects with various applications, such as motion planning robotics in Farber [1]. For a closed manifold  $M$ , the configuration space  $\text{Conf}_k(M)$  is defined as the space of  $k$ -tuples of distinct points of  $M$ . The  $k$ th symmetric group acts freely on  $\text{Conf}_k(M)$  by permuting the distinct points, so the quotient of the action can be used to obtain  $B_k(M)$  as the unordered configuration space of  $k$  points in  $M$ .

One can associate a topological space with its homology, which is a collection of algebraic structures called homology groups. Having larger and more complicated homology groups indicates a more complex space. Moreover, the homology of a space is a homotopy invariant; that is, if two topological spaces have different homology groups, there exists no continuous deformation transforming one space to the other, allowing topologists to better classify and distinguish spaces.

It has been a classical challenge to compute the homology of configuration spaces over various coefficient rings for different possible manifolds. Research on configuration spaces dates back to the 1960s, with generalizations of  $B_k(M)$  to labeled configuration spaces beginning in 1973 with Segal [2] and McDuff [3]. A recent development in the computation of the homology of configuration spaces by Knudsen [4] showed that when  $M$  is a framed manifold, the homology with coefficients in a ring  $R$  of labeled configuration spaces can be computed using a series of homological approximations called a spectral sequence. When  $R = \mathbb{Q}$ , Knudsen [5] showed that the spectral sequence could be computed from only its first approximation. Building on the work of Knudsen, Drummond-Cole and Knudsen [6] computed the dimensions of the rational homology groups of unordered configuration spaces of surfaces.

This paper is primarily interested in mod  $p$  homology; that is, homology over finite field  $\mathbb{F}_p$ . However, the mod  $p$  homology of unordered configuration spaces are difficult to compute. The only cases known classically are  $M = \mathbb{R}^\infty$  by May [7] in 1972,  $M = \mathbb{R}^n$  by Cohen [8] in 1986, and any odd-dimensional  $M$  by Bödiger et al. [9] in 1988. Using Knudsen's spectral sequence for the homology of labeled configuration spaces with coefficients in Lubin-Tate  $E$ -theory, Brantner et al. [10] computed the mod  $p$  homology of  $B_p(\Sigma_{g,1})$  in 2019, where  $\Sigma_{g,1}$  is the genus  $g$  surface with a point removed and  $p > 2$ . However, their methods do not extend easily to genus  $g$  surfaces. In 2021, Zhang [11] showed that for  $k = 2$  or  $3$ , Knudsen's spectral sequence for the mod  $p$  homology of  $B_k(M)$  remains relatively computable for any framed manifold  $M$ , giving an algorithm to compute an explicit basis for the mod  $p$  homology of  $B_k(M)$ .

We build off the spectral sequence from Zhang [11] to compute the mod  $p$  homology of  $B_k(T)$  for  $p \geq 5$  and larger  $k$ , where  $T$  denotes the torus, the only closed and parallelizable genus  $g$  surface. In Section 2, I define the homology and introduce preliminaries. We outline the general method in Section 3, which will be used for all later sections for given  $M$  and  $k$ . Specifically, in Section 4, we prove that when  $M = T$  and  $p \geq 5$ , the dimensions of the mod  $p$  homology groups of  $B_k(T)$  agree with the Betti numbers obtained by Drummond-Cole and Knudsen [6] for  $k \leq p$ . In Section 5, I

compute the mod 2 homology when  $M = \mathbb{R}\mathbb{P}^{2n+1}$  for  $k = 2$  and  $n \geq 0$ .

## 2 Preliminaries

A *chain complex*  $C_\bullet$  over a ring  $R$  is a sequence of  $R$ -modules

$$\dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

equipped with  $R$ -module homomorphisms  $\partial_n : C_n \rightarrow C_{n-1}$  such that  $\partial_n \partial_{n+1} = 0$  for all  $n$ . For a given topological space  $X$ , one can associate a chain complex  $C_\bullet(X; R)$  with  $C_n(X; R)$  generated by the  $n$ -dimensional simplices of  $X$ . The *homology* of  $X$  with coefficients in  $R$ , denoted by  $H_*(X; R)$ , is a sequence of  $R$ -modules  $H_0(X; R), H_1(X; R), H_2(X; R), \dots$ , where the  $n$ th homology group is the quotient

$$H_n(X; R) := \ker \partial_n / \text{im } \partial_{n+1}.$$

The intuition behind this definition is that, over a rational coefficient ring, the rank of the basis of  $H_n$  is the number of  $n$ -dimensional holes of  $X$ . Indeed,  $\ker \partial_n$  is the set of  $n$ -dimensional cycles, and  $\text{im } \partial_{n+1}$  is the set of  $n$ -dimensional boundaries of  $(n+1)$ -dimensional elements, so the quotient leaves non-boundary cycles—or in other words, holes. We are interested in the dimension of the homology groups over  $\mathbb{F}_p$ , namely  $H_i(X; \mathbb{F}_p) \cong \mathbb{F}_p^{\dim H_i(X; \mathbb{F}_p)}$ .

**Example 2.1** (Hatcher [12]). For an  $n$ -sphere  $S^n$  and any ring  $R$ , the 0th homology group is  $H_0(S^n; R) \cong R$ , the  $n$ th homology group is  $H_n(S^n; R) \cong R$ , and 0 for all other groups.

**Example 2.2** (Hatcher [12]). For any genus  $g$  surface  $\Sigma_g$  and ring  $R$ , the 0th homology group is  $H_0(\Sigma_g; R) \cong R$ , the 1st homology group is  $H_1(\Sigma_g; R) \cong R^{2g}$ , the 2nd homology group is  $H_2(\Sigma_g; R) \cong R$ , and 0 for all other groups.

Henceforth, denote the generators of  $H_*(\Sigma_g; R)$  with  $c$  as the generator of  $H_2(\Sigma_g; R)$ , generators  $a_1, \dots, a_g, b_1, \dots, b_g$  for  $H_1(\Sigma_g; R)$ , and  $d$  as the generator of  $H_0(\Sigma_g; R)$ .

**Example 2.3.** For odd dimensional real projective space  $\mathbb{R}\mathbb{P}^{2n+1}$ , the  $i$ th mod 2 homology group is  $H_i(\mathbb{R}\mathbb{P}^{2n+1}; \mathbb{F}_2) \cong \mathbb{F}_2$  for  $0 \leq i \leq 2n+1$ . Equivalently, one can let  $x^k$  denote the generator for  $H_k$ , so the homology is

$$H_*(\mathbb{R}\mathbb{P}^{2n+1}; \mathbb{F}_2) = \mathbb{F}_2[x]/x^{2n+2}.$$

For closed and oriented manifolds  $M^n$ , there is a dual notion to homology called *cohomology*. The  $k$ th cohomology group,  $H^k(M^n)$ , was shown by Poincaré duality to satisfy

$$H^k(M^n) \cong H_{n-k}(M^n).$$

For Examples 2.1, 2.2, and 2.3,  $H^* \cong H_*$ . There also exists a bilinear operation

$$\cup : H^p(M^n; R) \times H^q(M^n; R) \longrightarrow H^{p+q}(M^n; R)$$

on the cohomology called the *cup product*. For classes  $a \in H^k(M^n; R)$  and  $b \in H^\ell(M^n; R)$ ,  $a \cup b$  is a class in  $H^{k+\ell}(M^n; R)$ . Although there exists a formula for the cup product in Appendix A, the cup products we compute lie in cohomology groups with  $\dim H^{k+\ell} = 0$  or  $1$  for the surfaces we encounter, making it trivial to compute  $a \cup b$ .

The chain complexes we work with are defined in terms of free shifted Lie algebras, free  $\bar{R}$  algebras, or as free  $\text{sLie}_{\bar{R}}$  algebras. We define all three algebras.

A *shifted Lie algebra*  $L$  over a ring  $R$  is a graded  $R$ -module equipped with a *shifted Lie bracket*  $[-, -] : L_i \otimes L_j \rightarrow L_{i+j-1}$ , where  $L_k$  denotes the elements of  $L$  with *internal degree*  $k$ .  $L$  is graded by internal degree, and for  $x \in L$ , we denote the internal degree of  $x$  as  $|x|$ . For brevity, we refer to the shifted Lie bracket as simply a Lie bracket henceforth. The Lie bracket obeys the following properties.

- Bilinearity:  $[ax + by, z] = a[x, z] + b[y, z]$  and  $[z, ax + by] = a[z, x] + b[z, y]$ ,
- Graded anti-commutativity:  $[x, y] = (-1)^{|x||y|}[y, x]$ ,
- Jacobi Identity:  $(-1)^{|z||x|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$ .

The free shifted Lie algebra on an  $R$ -module  $M$  also admits a *weight grading*, making it a bigraded module. Explicitly, every element of  $M$  has weight 1, and the Lie bracket  $[x, y]$  adds the weights of  $x, y$ . For a chain complex  $\mathcal{C}$ , let  $\text{wt}_k(\mathcal{C})$  denote the weight  $k$  part of  $\mathcal{C}$ .

There is a free functor  $\text{Free}^{\text{sLie}}$  sending an  $R$ -module  $M$  to the free shifted Lie algebra on  $M$ , which can be thought of as applying iterations of Lie brackets on elements in  $M$  but not evaluating the Lie bracket. One can apply  $\text{Free}^{\text{sLie}}$  to the underlying  $R$ -module of a shifted Lie algebra  $L$  over  $R$ ; note that this is not the same as the Lie bracket being in  $L$ . This gives rise to the shifted Lie algebra structure map  $d : \text{Free}^{\text{sLie}}(L) \rightarrow L$  of  $L$ . For instance, suppose  $x$  and  $y$  are elements of  $L$ . Then

$$[x, y] \xrightarrow{d} [x, y],$$

where the different colored Lie brackets indicate that the free functor is applying the red Lie bracket to act on  $x$  and  $y$ , while  $d$  maps to the element  $[x, y]$  in  $L$ . The free functor and structure maps can naturally extend into a longer chain complex. For example, let  $x, y, z$  be generators of  $L$  and  $d_1 : \text{Free}^{\text{sLie}} \circ \text{Free}^{\text{sLie}}(L) \rightarrow \text{Free}^{\text{sLie}}(L)$  and  $d_2 : \text{Free}^{\text{sLie}}(L) \rightarrow L$ . The color of the functor corresponds to which color Lie bracket it is applying. Then

$$[[x, y], z] \xrightarrow{d_1} [[x, y], z] \xrightarrow{d_2} [[x, y], z].$$

In short, each of the Lie brackets decrease in level after each structure map, represented by a change to the next color.

On the other hand, the *free  $\bar{R}$ -algebra* is bigraded by internal degree and weight and generated by a set of unary operations  $\{\beta^\epsilon \bar{Q}^i \mid i \in \mathbb{Z}, \epsilon = 0, 1\}$ . For a free module  $M$  over  $\bar{R}$ , these unary operations satisfy the following properties.

**Definition 2.1** (Kjaer [13]). Let  $p > 2$  be a prime. Suppose that  $x$  is an element in a module  $M$  over the algebra  $\bar{R}$ . Then the following hold:

- $\beta^\epsilon \bar{Q}^i x \neq 0$  for  $\epsilon = 0, 1$  if and only if  $i \geq |x|/2$ ,
- The internal degree of  $\beta^\epsilon \bar{Q}^i x$  is  $|\beta^\epsilon \bar{Q}^i x| = 2i(p-1) + |x| - 1 - \epsilon$ ,
- The weight of  $\beta^\epsilon \bar{Q}^i x$  is  $p$  times the weight of  $x$ .

There is a free functor  $\text{Free}^{\bar{R}}$  sending an  $\mathbb{F}_p$ -module  $M$  to the free  $\bar{R}$ -module over  $M$ , behaving analogously to  $\text{Free}^{\text{sLie}}$  but applying  $\beta^\epsilon \bar{Q}^i$  as opposed to a Lie bracket.

The *free sLie $_{\bar{R}}$  algebra* is equipped with both unary operations and the Lie bracket. The two structures interact in the following way: for any module  $M$  over  $\text{sLie}_{\bar{R}}$ , Kjaer [13] proved that for all  $x$  and  $y$  in  $M$ ,  $[x, \beta^\epsilon \bar{Q}^i y] = 0$ . There also exists a free functor for the  $\text{sLie}_{\bar{R}}$ -algebra, with underlying  $\mathbb{F}_p$ -module given by

$$\text{Free}^{\text{sLie}_{\bar{R}}}(M) = \text{Free}^{\bar{R}} \circ \text{Free}^{\text{sLie}}(M).$$

For a shifted Lie algebra  $L$ , it is known that there exists an isomorphism between the *shifted Lie algebra homology* of  $L$  and the homology of another chain complex called the *Chevalley-Eilenberg complex* [14] of  $L$ . We define both homologies.

**Definition 2.2.** The *shifted Lie algebra homology* of  $L$  is the homology of the chain complex  $D_\bullet$  with  $D_i = (\text{Free}^{\text{sLie}})^{\circ i}(L)$  and boundary maps  $\partial_i : D_i \rightarrow D_{i-1}$  given by

$$\partial_i = \sum_{j=0}^i (-1)^j d_j,$$

where  $d_j$  is induced by the structure map  $\text{Free}^{\text{sLie}} \circ (\text{Free}^{\text{sLie}})^{\circ(i-j)}(L) \rightarrow (\text{Free}^{\text{sLie}})^{\circ(i-j)}(L)$ .

Computing the homology of  $D_\bullet$  is equivalent to computing the homology of the Chevalley-Eilenberg complex over rationals. The Chevalley-Eilenberg complex is expressed in terms of exterior and divided power algebras. The divided power algebra  $\Gamma$  is a commutative ring  $A$  equipped with ideal  $I$  and divided power operations  $\gamma_n$  for  $n \geq 1$  satisfying a set of conditions available in Appendix A. Of most relevance to us is if  $px \neq 0$  for all  $x \in A$  and  $x \neq 0$ . In such instance, we say  $A$  is  $p$ -torsion free, and  $n!\gamma_n(x) = x^n$  for all  $x \in I$ ,  $n \geq 0$ . Over an ideal  $I$  where  $x^n$  is an  $n!$ -multiple, such as in  $\mathbb{Q}$ , then  $\gamma_n(x) = x^n/n!$ .

For a vector space  $V$  over a field, the exterior algebra  $\Lambda V$  is an algebra satisfying associativity and unitality. It is equipped with the binary operation wedge product  $\wedge$  with the identity that  $x \wedge x = 0$  for all  $x \in V$ . For generators  $x_1, x_2, \dots, x_n$ , denote for brevity's sake,

$$x_1 \wedge x_2 \wedge \dots \wedge x_n = \langle x_1, x_2, \dots, x_n \rangle.$$

**Definition 2.3** (Chevalley-Eilenberg [14]). For a shifted Lie algebra  $L$  over a field, let  $L_{\text{even}}$  and  $L_{\text{odd}}$  denote the elements in  $L$  with even and odd degree, respectively. The *Chevalley-Eilenberg* complex of  $L$  over any coefficient ring  $R$  is the chain complex

$$CE(L) = (\Gamma(L_{\text{even}}) \otimes \Lambda(L_{\text{odd}}), \partial),$$

where the value of differential  $\partial$  on a general element

$$\gamma_{k_1}(x_1)\gamma_{k_2}(x_2)\cdots\gamma_{k_m}(x_m)\langle y_1, y_2, \dots, y_n \rangle \in \Gamma(L_{\text{even}}) \otimes \Lambda(L_{\text{odd}})$$

is the expression

$$\begin{aligned} & \sum_{1 \leq i < j \leq m} \gamma_{k_1}(x_1) \cdots \gamma_{k_{i-1}}(x_{i-1}) \cdots \gamma_{k_{j-1}}(x_{j-1}) \cdots \gamma_{k_m}(x_m) \langle [x_i, x_j], y_1, \dots, y_n \rangle \\ & + \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} \gamma_{k_1}(x_1) \cdots \gamma_{k_m}(x_m) \langle [y_i, y_j], y_1, \dots, \widehat{y}_i, \dots, \widehat{y}_j, \dots, y_n \rangle \\ & + \frac{1}{2} \sum_{i=1}^m \gamma_{k_1}(x_1) \cdots \gamma_{k_{i-2}}(x_{i-2}) \cdots \gamma_{k_m}(x_m) \langle [x_i, x_i], y_1, \dots, y_n \rangle \\ & + \sum_{i=1}^m \sum_{j=1}^n (-1)^{j-1} \gamma_1([x_i, y_j]) \gamma_{k_1}(x_1) \cdots \gamma_{k_{i-1}}(x_{i-1}) \cdots \gamma_{k_m}(x_m) \langle y_1, \dots, \widehat{y}_j, \dots, y_n \rangle. \end{aligned}$$

where  $\widehat{y}$  denotes the omission of  $y$  from the expression.

All elements in the chain complex are bigraded by simplicial degree and internal degree; the simplicial degree is the number of Lie brackets in an element and determines the chain group the element is in. For a chain complex of  $\text{sLie}_{\bar{R}}$  algebras, applying a unary operation also adds one to the simplicial degree. The simplicial degree is commonly denoted by  $s$  and internal degree by  $t$ . Together,  $s + t$  is the *total degree* of the element.

The dimension of the  $i$ th homology group over  $\mathbb{Q}$  is the  $i$ th *Betti number*  $\beta_i$ . Drummond-Cole and Knudsen [6] computed the Betti numbers of  $B_k(\Sigma_g)$  for  $g > 0$ .

**Theorem 2.1** (Drummond-Cole, Knudsen [6]). *The Betti numbers of  $B_k(\Sigma_g)$  are equivalent to the Betti numbers of  $\text{wt}_k(CE(\mathfrak{g}'))$ , where  $\mathfrak{g}'$  is the shifted Lie algebra*

$$\mathfrak{g}' = H^*(\Sigma_g; \mathbb{Q}) \otimes \text{Free}^{\text{sLie}_{\mathbb{Q}}}(\mathbb{Q}\{x_2\}).$$

The basis elements of  $\mathfrak{g}'$  with weight at most  $p$  are

$$\begin{cases} \{c \otimes x_2, a_i \otimes x_2, b_i \otimes x_2, d \otimes x_2\} & 1 \leq i \leq g, \\ \{c \otimes [x_2, x_2], a_i \otimes [x_2, x_2], b_i \otimes [x_2, x_2], d \otimes [x_2, x_2]\} & 1 \leq i \leq g. \end{cases}$$

Drummond-Cole and Knudsen [6] gave explicit formulas for the Betti numbers of  $B_k(\Sigma_g)$  for all  $k$  and  $g$ . These formulas and tables for small values can be found in Appendix A.

There is the following well established inequality in mathematical folklore regarding mod  $p$  and rational homology groups of a finite type topological space  $X$ .

**Theorem 2.2.** *For any finite type topological space  $X$ , the dimension of  $H_n(X; \mathbb{F}_p)$  is at most the  $n$ th Betti number of  $X$  for all  $n$ .*

### 3 General Method of Computing the $\mathbb{F}_p$ Homology

We present a general outline of the method we use to compute the mod  $p$  homology of an arbitrary framed closed manifold  $M^n$  of dimension  $n$ , building on Sections 6 and 7 in Zhang [11]. In subsequent sections, we implement this general algorithm for specific values of  $k$  with respect to  $p$  and framed manifolds  $M^n$ .

We compute the  $\mathbb{F}_p$  homology of unordered configuration spaces using a spectral sequence. In general, a *spectral sequence* consists of bigraded objects  $E_{s,t}^r$ , called the  $E^r$ -pages, for  $r \geq 2$  and differentials  $d_r : E_{s,t}^r \rightarrow E_{s-r,t+r-1}^r$  such that  $E_{s,t}^{r+1} = H_*(E_{s,t}^r, d_r)$ . For certain sequences, the spectral sequence converges to the  $E^\infty$ -page

$$E_{s,t}^2 \rightarrow E_{s,t}^3 \rightarrow \cdots \rightarrow E^\infty.$$

If  $d_r = 0$ , then  $E_{s,t}^{r+1} \cong E_{s,t}^r$ . We say that the spectral sequence *collapses* on the  $E^r$ -page when  $d_i = 0$  for all  $i \geq r$ .

Denote by  $\mathfrak{g}$  the  $\text{sLie}_{\bar{R}}$ -algebra

$$\mathfrak{g} = H^*(M^n; \mathbb{F}_p) \otimes \text{Free}^{\text{sLie}_{\bar{R}}}(\mathbb{F}_p\{x_n\}),$$

where the  $\mathbb{F}_p$ -module generator  $x_n$  has internal degree  $n$  and tensoring with an element of  $H^i(M^n; \mathbb{F}_p)$  decreases the internal degree by  $i$ . Define a chain complex  $C_\bullet(\mathfrak{g})$  as follows:  $C_\bullet$  consists of chain groups  $C_s = (\text{Free}^{\text{sLie}_{\bar{R}}})^{\circ s}(\mathfrak{g})$  and boundary maps  $\partial_s : C_s \rightarrow C_{s-1}$  such that

$$\partial_s = \sum_{i=0}^s (-1)^i d_i,$$

where  $d_i$  is the structure map

$$d_i : \text{Free}^{\text{sLie}_{\bar{R}}} \circ (\text{Free}^{\text{sLie}_{\bar{R}}})^{\circ(s-i)}(\mathfrak{g}) \rightarrow (\text{Free}^{\text{sLie}_{\bar{R}}})^{\circ(s-i)}(\mathfrak{g}).$$

In particular, the differentials in the chain complex preserve the weight grading. Let  $C_\bullet^k$  denote the weight  $k \geq 1$  part of  $C_\bullet$  and  $D_\bullet^k$  the weight  $k$  part of the shifted Lie algebra chain complex  $D_\bullet$  defined in Definition 2.2. Then

$$H_*(C_\bullet) = \bigoplus_{k \geq 1} \text{wt}_k H_*(C_\bullet) = \bigoplus_{k \geq 1} H_*(C^k).$$

The  $E^2$ -page of the spectral sequence is  $H_*(C_\bullet)$ , and it converges to  $\bigoplus_{k \geq 1} H_*(B_k(M^n); \mathbb{F}_p)$ . More specifically, because  $E^2$  and  $E^\infty$  are both graded by weight, then we consider the spectral sequence in weight  $k$  to compute  $H_*(B_k(M^n); \mathbb{F}_p)$ .

This gives a general approach for computing the mod  $p$  homology for the configuration space of a genus  $g$  surface: compute the homology of the weight  $k$  part of  $C_\bullet$  and find the nonzero differentials

in the spectral sequence to compute the  $E^\infty$ -page.

## 4 Mod $p$ Homology of Configuration Spaces of $M^n = T$

In this section, we compute the mod  $p$  homology of the configuration spaces of  $k \leq p$  points in a torus. Depending on the size of  $k$  with respect to  $p$ , the operations in the weight  $k$  part of  $C_\bullet$  differ, and we compute these cases separately. For this section,  $M^n = T$  is a 2-dimensional manifold of genus 1, so in the chain complex  $C_\bullet(\mathfrak{g})$ ,

$$\mathfrak{g} = H^*(T; \mathbb{F}_p) \otimes \text{Free}^{\text{sLie}_{\bar{R}}}(\mathbb{F}_p\{x_2\}).$$

The basis elements of  $\mathfrak{g}$  with weight at most  $p$  are the following:

$$\begin{cases} \{c \otimes x_2, a_1 \otimes x_2, b_1 \otimes x_2, d \otimes x_2\} \\ \{c \otimes [x_2, x_2], a_1 \otimes [x_2, x_2], b_1 \otimes [x_2, x_2], d \otimes [x_2, x_2]\} \\ \{d \otimes \beta^\epsilon \bar{Q}^j x_2, a_1 \otimes \beta^\epsilon \bar{Q}^j x_2, b_1 \otimes \beta^\epsilon \bar{Q}^j x_2\} & j \geq 1, \epsilon \in \{0, 1\}, \\ \{c \otimes \beta^\epsilon \bar{Q}^j x_2\} & j \geq 0, \epsilon \in \{0, 1\}. \end{cases}$$

The Lie bracket  $[p \otimes x_2, q \otimes x_2]$  in  $\mathfrak{g}$  for  $p, q \in H^*(T; \mathbb{F}_p)$  is given by

$$[p \otimes x_2, q \otimes x_2] = (p \cup q)[x_2, x_2].$$

A unary operation in the  $\bar{R}$  algebra acts on  $\mathfrak{g}$  via

$$\beta^\epsilon \bar{Q}^i(p \otimes x_2) = p \otimes \beta^\epsilon \bar{Q}^i x_2.$$

### 4.1 Homology for $k < p$ and $p \geq 3$

For  $k < p$ , all elements of weight  $k$  must be the iteration of Lie brackets, for applying a unary operation via the functor  $\text{Free}^{\text{sLie}_{\bar{R}}}$  to any element in  $C_\bullet$  results in weight  $p > k$ . As a result,  $C^k \cong \text{wt}_k(\text{CE}(\mathfrak{g}; \mathbb{F}_p)) = D^k$ , implying  $E_{s,t}^2 = H_*(C^k) \cong H_*(D^k)$ . We show that the spectral sequence collapses on the  $E^2$ -page and compute the homology of the Chevalley-Eilenberg complex over  $\mathbb{F}_p$ .

**Theorem 4.1.** *For  $k < p$ , the dimensions of the mod  $p$  homology groups of the Chevalley-Eilenberg complex of  $\mathfrak{g}$  as  $\mathbb{F}_p$ -modules equal the Betti numbers of the Chevalley-Eilenberg complex of  $\mathfrak{g}'$  in Theorem 2.1.*

*Proof.* For a given basis element  $e$  with weight  $k$  of the form  $x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} \langle y_1, y_2, \dots, y_n \rangle$  in  $\text{CE}(\mathfrak{g}'; \mathbb{Q})$ , define the map  $\phi$  mapping the basis of the weight  $k < p$  part of  $\text{CE}(\mathfrak{g}'; \mathbb{Q})$  to the basis of the weight  $k < p$  part of  $\text{CE}(\mathfrak{g}; \mathbb{F}_p)$  such that

$$\phi(e) = \phi(x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} \langle y_1, y_2, \dots, y_n \rangle) = \frac{x_1^{k_1}}{k_1!} \frac{x_2^{k_2}}{k_2!} \cdots \frac{x_m^{k_m}}{k_m!} \langle y_1, y_2, \dots, y_n \rangle. \quad (\star)$$

Because  $k < p$ , then  $k_i < p$  for all  $i$ , so the weight  $k$  part of  $CE(\mathfrak{g}'; \mathbb{F}_p)$  is  $p$ -torsion free and

$$\phi(e) = \gamma_{k_1}(x_1)\gamma_{k_2}(x_2)\cdots\gamma_{k_m}(x_m)\langle y_1, y_2, \dots, y_n \rangle.$$

The basis elements only differ by a unit, so  $\phi$  is a surjection mapping a basis in  $CE(\mathfrak{g}'; \mathbb{Q})$  to a basis in  $CE(\mathfrak{g}; \mathbb{F}_p)$ . Moreover, two distinct basis elements cannot be mapped to the same element in  $CE(\mathfrak{g}; \mathbb{F}_p)$  lest  $x_i^{k_i}$  is the same for all  $i$ , so  $\phi$  is also injective and bijective. Because  $e$  and  $\phi(e)$  differ only by a unit, then this bijection is also preserved under the differential. Therefore, when  $e$  is in the rational homology of  $CE(\mathfrak{g}'; \mathbb{Q})$ ,  $\phi(e)$  is in the mod  $p$  homology of  $CE(\mathfrak{g}; \mathbb{F}_p)$ .

We claim the spectral sequence over  $\mathbb{F}_p$  collapses on the  $E^2$ -page. Suppose for the sake of contradiction there existed a nonzero differential  $d_r$  in the spectral sequence with  $r \geq 2$ . Then, in the  $E^{r+1}$ -page, some elements vanish from the  $\ker d_r$  and by extension from the  $E^r$ -page.

On the other hand, no elements vanish from the  $E^r$ -page in the rational homology from Knudsen [5]. Therefore,  $\dim H_*(E^r, d_r)$  over  $\mathbb{F}_p$  is strictly less than  $\dim H_*(E^r, d_r)$  over  $\mathbb{Q}$ , contradicting Theorem 2.2. Instead, it must be that the  $E^2$ -page over  $\mathbb{F}_p$  collapses as well, so the dimensions of the mod  $p$  homology of  $CE(\mathfrak{g}; \mathbb{F}_p)$  agree with the Betti numbers of  $CE(\mathfrak{g}'; \mathbb{Q})$ .  $\square$

Using the formula from Theorem 2.1, we can compute the Betti numbers of the rational homology case and subsequently the dimensions of the mod  $p$  homology groups of  $B_k(T)$  for  $k < p$  by Theorem 4.1.

## 4.2 Homology for $k = p$ and $p \geq 5$

For  $k = p$ , all elements of weight  $p$  must be either iterations of Lie brackets or a single unary operation applied to a weight one element  $x \in \mathfrak{g}$ . In other words,

$$C^p = D^p \oplus \{\beta^\epsilon \bar{Q}^i x \mid x \in \text{wt}_1(\mathfrak{g})\}.$$

Hence, the homology of  $C^p$  is

$$H_*(C^p) \cong H_*(D^p) \oplus \{\beta^\epsilon \bar{Q}^i x \in \ker \partial_1\},$$

where  $\partial_1 : C_1 \rightarrow C_0$  is the boundary map. We first show that  $H_*(D^p)$  is computable. Then, we find all  $\beta^\epsilon \bar{Q}^i x \in \ker \partial_1$  and all nonzero differentials in the spectral sequences.

**Lemma 4.2.** *In weight  $p$ , the dimensions of the mod  $p$  homology groups of the Chevalley-Eilenberg complex of  $\mathfrak{g}$  as  $\mathbb{F}_p$ -modules equal the Betti numbers of the Chevalley Eilenberg complex of  $\mathfrak{g}'$ .*

*Proof.* For all basis elements with  $k_i < p$  for all  $i$ , the same mapping as  $(\star)$  from the proof of Theorem 4.1 produces a valid bijection. For a weight one generator  $x$ , we map the remaining element  $x^p$  over  $\mathbb{Q}$  to the remaining basis element  $\gamma_p(x)$  over  $\mathbb{F}_p$ . The basis element  $x^p$  in  $CE(\mathfrak{g}'; \mathbb{Q})$  corresponds to  $p!\gamma_p(x) = 0$  in  $CE(\mathfrak{g}; \mathbb{F}_p)$ , so  $CE(\mathfrak{g}; \mathbb{F}_p)$  is no longer  $p$ -torsion free.

It suffices to show that the differentials, which we denote by  $\partial_{\mathbb{Q}}$  and  $\partial_{\mathbb{F}_p}$ , in each complex are equivalent for their respective elements. Because we are interested in the homology, it suffices to

show that  $x^p \in \ker \partial_{\mathbb{Q}}$  if and only if  $\gamma_p(x) \in \ker \partial_{\mathbb{F}_p}$ . Indeed, applying the formula for the differential from Definition 2.3,

$$\partial_{\mathbb{Q}}(x^p) = \frac{1}{2}x^{p-2}[x, x],$$

while over  $\mathbb{F}_p$ ,

$$\partial_{\mathbb{F}_p}(\gamma_p(x)) = \frac{1}{2}\gamma_{p-2}[x, x] = \frac{1}{2}\frac{x^{p-2}}{(p-2)!}[x, x].$$

Because  $(p-1)! \equiv -1 \pmod{p}$  from Wilson's Theorem,

$$\partial_{\mathbb{F}_p}(\gamma_p(x)) = \frac{1}{2}\frac{x^{p-2}}{(p-2)!}[x, x] = \frac{1}{2}(1-p)x^{p-2}[x, x] = \frac{1}{2}x^{p-2}[x, x].$$

Because the differentials are equivalent on  $x^p$  in  $CE(\mathfrak{g}'; \mathbb{Q})$  and  $\gamma_p(x)$  in  $CE(\mathfrak{g}; \mathbb{F}_p)$ , we conclude the bijection is indeed preserved under the differential. Consequently, the dimensions of the mod  $p$  homology groups of  $CE(\mathfrak{g}; \mathbb{F}_p)$  agree with the Betti numbers of  $CE(\mathfrak{g}; \mathbb{Q})$ .  $\square$

We can thus compute  $H_*(D^p)$  using the Betti number formulas in Theorem 2.1. To compute elements in the kernel of  $\partial_1$  with a unary operation, we claim that  $\beta^\epsilon \bar{Q}^i(q \otimes x_2) \in \ker \partial_1$  if and only if  $|q \otimes x_2|/2 \leq i < 1$  for generator  $q \in H^*(T)$ . The lower bound is necessary for  $\beta^\epsilon \bar{Q}^i(a \otimes x_2)$  to be nonzero. For the upper bound, if  $i < 1 = |x_2|/2$ , then

$$\beta^\epsilon \bar{Q}^i(a \otimes x_2) = a \otimes \beta^\epsilon \bar{Q}^i x_2 = a \otimes 0 = 0.$$

If  $i \geq |x_2|/2$ , then  $\beta^\epsilon \bar{Q}^i x_2 \neq 0$  and  $\beta^\epsilon \bar{Q}^i(a \otimes x_2)$  would not be in the kernel.

Therefore, the only elements in  $\{\beta^\epsilon \bar{Q}^i x \in \ker \partial_1\}$  are  $\bar{Q}^0(c \otimes x_2)$  and  $\beta \bar{Q}^0(c \otimes x_2)$ . We have computed  $H_*(C^p)$  and, equivalently, the weight  $p$  part of the  $E_{s,t}^2$ -page. It remains to find the nonzero differentials in the spectral sequence in order to compute  $E^\infty$ . We begin with the following lemma proving there exists a unique element with minimal total degree.

**Lemma 4.3.** *In the homology of the weight  $p$  part of the Chevalley-Eilenberg complex,  $\gamma_p(c \otimes x_2)$  is the unique element with minimal dimension  $s + t = 0$ .*

*Proof.* From Definition 2.1, given an element  $e = \gamma_{k_1}(x_1) \cdots \gamma_{k_i}(x_i) \langle y_1, \dots, y_j \rangle$  in  $D^p$  with simplicial degree  $s$ , the total degree of  $e$  is

$$\begin{aligned} |e| + s &= ((k_1|x_1| + \cdots + k_i|x_i|) + (|y_1| + \cdots + |y_j|) - s) + s, \\ &= (k_1|x_1| + \cdots + k_i|x_i|) + (|y_1| + \cdots + |y_j|), \\ &\geq (k_1 + \cdots + k_i) \cdot \min_{1 \leq t \leq i} |x_t| + j \cdot \min_{1 \leq t \leq j} |y_t|, \\ &\geq (k_1 + \cdots + k_i + j) \cdot \min_{x \in \mathfrak{g}} |x|, \\ &\geq p \cdot \min_{x \in \mathfrak{g}} |x|. \end{aligned}$$

because  $k_1 + \dots + k_i + j = p$  in weight  $p$ . Because the unique minimal degree element in  $\mathfrak{g}$  is  $c \otimes x_2$ , then  $\gamma_p(c \otimes x_2)$  is the unique minimal degree element with total degree 0. Moreover, under the differential in the Chevalley-Eilenberg complex from Definition 2.3, it is evident that

$$\begin{aligned} \partial_{\mathbb{F}_p}(\gamma_p(c \otimes x_2)) &= \frac{1}{2}\gamma_{p-2}(c \otimes x_2)[c \otimes x_2, c \otimes x_2], \\ &= \frac{1}{2}\gamma_{p-2}(c \otimes x_2) \cdot (c \cup c)[x_2, x_2], \\ &= 0. \end{aligned}$$

because  $c \cup c = 0$ . Therefore  $\gamma_p(c \otimes x_2)$  is in the kernel of  $\partial_{\mathbb{F}_p}$ , so it is in the homology of the Chevalley Eilenberg complex.  $\square$

Equipped with Theorem 2.2 and Lemmas 4.2 and 4.3, we are prepared to state the result for the  $k = p$  case.

**Theorem 4.4.** *The dimensions of the mod  $p$  homology  $H_*(B_p(T); \mathbb{F}_p)$  agree with the Betti numbers of  $B_p(T)$ .*

*Proof.* Using Definition 2.1,  $\bar{Q}^0(c \otimes x_2)$  and  $\beta\bar{Q}^0(c \otimes x_2)$  have internal degree  $-1$  and  $-2$ , respectively. Under a nonzero differential  $d_r : E_{s,t}^r \rightarrow E_{s-r,t+r-1}^r$ , the total degree of a given element in the  $E^r$  page decreases by 1. In particular,  $\dim \gamma_p(c \otimes x_2) = 0$ , so  $\dim d_r(\gamma_p(c \otimes x_2)) = -1$ .

We claim that  $d_r(\gamma_p(c \otimes x_2)) = \beta\bar{Q}^0(c \otimes x_2)$ . By Lemma 4.3, all elements apart from  $\gamma_p(c \otimes x_2)$  have dimension strictly greater than 0 while  $\dim \beta\bar{Q}^0(c \otimes x_2) = -1$ . It follows that only  $d_r(\gamma_p(c \otimes x_2)) = \beta\bar{Q}^0(c \otimes x_2)$ . If not, then  $\beta\bar{Q}^0(c \otimes x_2)$  survives to the  $E^\infty$ -page and becomes a generator in the  $H_{-1}$  homology group, contradicting the fact that  $B_p(T)$  is a space and only has nonnegative dimension homology groups.

The simplicial degrees of  $\gamma_p(c \otimes x_2)$  and  $\beta\bar{Q}^0(c \otimes x_2)$  are  $p - 1$  and 1, respectively, so the only possible nonzero differential is  $d_{p-2}$ . As a result, because  $d_r = 0$  for  $2 \leq r < p - 2$ , then

$$E_{s,t}^2 \cong E_{s,t}^3 \cong \dots \cong E_{s,t}^{p-2}.$$

It remains to show that one can uniquely determine the  $E_{s,t}^{p-1}$ -page from  $d_{p-2}$ . Because  $\bar{Q}^0(c \otimes x_2)$  already has lowest simplicial degree, it cannot be that  $\bar{Q}^0(c \otimes x_2)$  maps to a nonzero element. It therefore suffices to verify that no elements in  $H_*(D^p)$  map to  $\bar{Q}^0(c \otimes x_2)$ .

Suppose for the sake of contradiction there exists  $x \in H_*(D^p)$  such that  $d_{p-2}(x) = \bar{Q}^0(c \otimes x_2)$ . Then,  $\bar{Q}^0(c \otimes x_2)$  is quotiented out by the image of  $d_{p-2}$ , so it vanishes from the homology of  $E^{p-2}$ . Likewise,  $\gamma_p(c \otimes x_2)$  is no longer in the kernel of  $d_{p-2}$ , so it too vanishes from the homology of  $E^{p-2}$ . Then  $E^{p-1} \cong E^\infty$  is isomorphic to  $H_*(D^p)$  excluding  $\gamma_p(c \otimes x_2)$ , implying

$$\dim H_*(B_p(T); \mathbb{F}_p) < \dim H_*(D^p) = \dim H_*(B_p(T); \mathbb{Q}),$$

contradicting Theorem 2.2. We thus conclude the only elements which are removed from the  $E_{s,t}^{p-2}$ -page after  $d_{p-2}$  are  $\gamma_p(c \otimes x_2)$  and  $\beta\bar{Q}^0(c \otimes x_2)$ .

Hence, we have determined that  $d_{p-2}$  is the only nonzero differential, the two elements removed from the  $E^2 \cong E^{p-2}$ -page are  $\beta\bar{Q}^0(c \otimes x_2)$  and  $\gamma_p(c \otimes x_2)$ , and  $\bar{Q}^0(c \otimes x_2)$  is the only element added. Moreover, because  $\dim \gamma_p(c \otimes x_2) = \dim \bar{Q}^0(c \otimes x_2)$ , the dimensions of the mod  $p$  homology groups of  $B_p(T)$  remain unchanged from  $CE(\mathfrak{g}; \mathbb{F}_p)$ , which agree with the Betti numbers of  $B_p(T)$  by Lemma 4.2.  $\square$

**Remark 4.5.** When  $p = 3$ ,  $\gamma_3(c \otimes x_2)$  has internal degree  $-2$  and simplicial degree 2, whereas  $\beta\bar{Q}^0(c \otimes x_2)$  has internal degree  $-2$  and simplicial degree 1. For all differentials  $d_r$  with  $r \geq 2$  in the spectral sequence, the simplicial degree of  $d_r(c \otimes x_2)$  is less than the simplicial degree of  $\beta\bar{Q}^0(c \otimes x_2)$ . This implies there must exist a generator in the  $H_{-1}$  homology group, contradiction. Instead, we suspect there exists a missing equivalence relation

$$\beta\bar{Q}^0(c \otimes x_2) \sim \gamma_3(c \otimes x_2)$$

in Definition 2.1 undiscovered by Kjaer [13]. For this reason, we restrict our results to  $p \geq 5$  for now.

**Remark 4.6.** Because they are framed, the exact same argument also works for punctured genus  $g$  surfaces  $\Sigma_{g,1}$  by comparing

$$\mathfrak{g} = H^*(\Sigma_{g,1}; \mathbb{F}_p) \otimes \text{Free}^{\text{sLie}_{\bar{R}}}(\mathbb{F}_p\{x_2\}) \quad \text{and} \quad \mathfrak{g}' = H^*(\Sigma_{g,1}; \mathbb{Q}) \otimes \text{Free}^{\text{sLie}_{\mathbb{Q}}}(\mathbb{Q}\{x_2\})$$

to show that the dimensions of the  $\mathbb{F}_p$  homology groups equal the Betti numbers of  $B_k(\Sigma_{g,1})$  for  $k \leq p$ . Consequently, this provides an elementary proof for Corollary 1.11 of Brantner-Hahn-Knudsen [10].

## 5 Mod 2 Homology of Configuration Spaces of $\mathbb{RP}^{2n+1}$

The method in Section 3 applies to other framed manifolds, so long as the Chevalley-Eilenberg complex can be computed over  $\mathbb{F}_p$ . It is therefore natural to compute the mod  $p$  homology of configuration spaces for other well known framed manifolds, not only the torus.

When the manifold is an odd-dimensional real projective space  $\mathbb{RP}^{2n+1}$ , its mod  $p$  homology groups are trivially 0 for all  $p > 2$ . Therefore, I compute the nontrivial mod 2 homology of the configuration space for 2 points on  $\mathbb{RP}^{2n+1}$ . When  $p = 2$ , the Lie bracket and unary operation are different from those in the odd primary cases.

**Definition 5.1** (Antolín-Camarena [15]). Suppose that  $M$  is an  $\mathbb{F}_2$ -module over the  $\text{sLie}_{\bar{R}}$  algebra. The action of the Lie bracket and the unary operations  $\bar{Q}^i$  on  $M$  satisfy the following.

- Applying  $\bar{Q}^i$  to  $x \in M$  multiplies the weight of  $x$  by 2 instead of by  $p$ ,
- The condition  $\bar{Q}^i x \neq 0$  occurs when  $i \geq |x|$  instead of  $i \geq |x|/2$ ,

- There is an identity  $\bar{Q}^{|x|}x = [x, x]$ ,
- The Lie bracket is now commutative, implying  $[y_{2n+1}, y_{2n+1}] \neq 0$ .

To compute  $H_*(B_k(\mathbb{RP}^{2n+1}); \mathbb{F}_2)$ , it still suffices to compute the homology of the weight  $k$  part of  $C_\bullet(\mathfrak{g})$ , where for  $\mathbb{RP}^{2n+1}$ ,

$$\mathfrak{g} = H^*(\mathbb{RP}^{2n+1}) \otimes \text{Free}^{\text{Lie}_{\bar{R}}}(\mathbb{F}_2\{y_{2n+1}\}).$$

**Theorem 5.1.** *The dimension of the  $m$ th homology group of  $H_*(B_2(\mathbb{RP}^{2n+1}); \mathbb{F}_2)$  for  $n \geq 1$  is*

$$\begin{cases} \left\lfloor \frac{\min(m, 4n-m)}{2} \right\rfloor + \frac{1-(-1)^m}{2} + \left\lceil \frac{m+1}{2} \right\rceil & 0 \leq m \leq 2n, \\ \left\lfloor \frac{\min(m, 4n-m)}{2} \right\rfloor + 1 + \binom{\lceil \frac{4n+3-m}{2} \rceil}{2} & 2n+1 \leq m \leq 4n. \end{cases}$$

*Proof.* For a weight 1 generator  $x$  in  $\mathfrak{g}$ , any weight 2 element must be either of the form  $\bar{Q}^i x$  or  $[x, x]$ , both of which have simplicial degree 1. As a result, there are no nonzero differentials in the spectral sequence, meaning the  $E^\infty$ -page is the kernel of  $\partial_1$ . There are three possible cases for basis elements in  $\ker \partial_1$ :

1.  $[x^a \otimes y_{2n+1}, x^b \otimes y_{2n+1}] = x^{a+b} \otimes [y_{2n+1}, y_{2n+1}]$  for unordered pairs  $(a, b)$  where  $a + b \geq 2n + 2$  and  $a, b \leq 2n + 1$ ,
2.  $\bar{Q}^i(x^k \otimes y_{2n+1})$  with  $2n + 1 - k < i < 2n + 1$  for  $k \geq n + 1$ , where the lower bound is strict because  $i = 2n + 1 - k$  is counted in case 1 from the identity  $\bar{Q}^{|x|}x = [x, x]$ , and  $2n + 1 - k \leq i < 2n + 1$  for  $k \leq n$ .
3.  $[x^a \otimes y_{2n+1}, x^b \otimes y_{2n+1}] - [x^c \otimes y_{2n+1}, x^d \otimes y_{2n+1}] = (x^{a+b} - x^{c+d}) \otimes [y_{2n+1}, y_{2n+1}]$  for unordered quadruples  $(a, b, c, d)$  where  $a + b = c + d < 2n + 2$  and  $(a, b) \neq (c, d)$ .

*Case 1:* For a given  $2n + 2 \leq a + b \leq 4n + 2$ , the total degree of  $x^{a+b} \otimes [y_{2n+1}, y_{2n+1}]$  is  $4n + 2 - (a + b)$ . Therefore, for  $0 \leq m \leq 2n$ , the ordered pairs  $(a, b)$  with total degree  $m$  are

$$(2n + 1 - m, 2n + 1), (2n + 2 - m, 2n + 1), \dots, (2n + 1, 2n + 1 - m),$$

giving  $\left\lceil \frac{m+1}{2} \right\rceil$  many unordered pairs and, accordingly, kernel basis elements with dimension  $m$  in  $\ker \partial_1$ .

*Case 2:* The total degree of  $\bar{Q}^i(x^k \otimes y_{2n+1})$  for given pair  $(i, k)$  is  $m = 2n + 1 + i - k$ . Note that the number of elements in  $H_m$  are the number of pairs  $(i, k)$  with  $i - k$  constant for  $1 \leq k \leq 2n + 1$  and  $2n + 1 - k \leq i \leq 2n$ . Plotting all such pairs  $(i, k)$  in Table 1 with their total degrees, pairs lying on the same diagonal have equal total degrees. Note that for  $k \geq n + 1$ , if  $i + k = 2n + 1$ , then from the identity  $\bar{Q}^{|x|}x = [x, x]$ ,

$$\bar{Q}^{2n+1-k}(x^k \otimes y_{2n+1}) = [x^k \otimes y_{2n+1}, x^k \otimes y_{2n+1}] = x^{2k} \otimes [y_{2n+1}, y_{2n+1}] = 0.$$

However, elements of this form were counted in Case 1, so I exclude  $\bar{Q}^{2n+1-k}(x^k \otimes y_{2n+1})$  for  $n+1 \leq k \leq 2n+1$ . In Table 1, I denote excluded elements with a  $\emptyset$ . From here, it is evident that in general, the number of  $m$ 's in the table for  $0 \leq m \leq 2n$  is

$$\left\lfloor \frac{\min(m, 4n-m)}{2} \right\rfloor + \frac{1 - (-1)^m}{2},$$

and for  $2n+1 \leq m \leq 4n$ , there are

$$\left\lfloor \frac{\min(m, 4n-m)}{2} \right\rfloor + 1.$$

$k \setminus i$	0	1	2	$\dots$	$n-1$	$n$	$n+1$	$\dots$	$2n-2$	$2n-1$	$2n$
1	$\emptyset$	$\emptyset$	$\emptyset$	$\dots$	$\emptyset$	$\emptyset$	$\emptyset$	$\dots$	$\emptyset$	$\emptyset$	$4n$
2	$\emptyset$	$\emptyset$	$\emptyset$	$\dots$	$\emptyset$	$\emptyset$	$\emptyset$	$\dots$	$\emptyset$	$4n-2$	$4n-1$
3	$\emptyset$	$\emptyset$	$\emptyset$	$\dots$	$\emptyset$	$\emptyset$	$\emptyset$	$\dots$	$4n-4$	$4n-3$	$4n-2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$n$	$\emptyset$	$\emptyset$	$\emptyset$	$\dots$	$\emptyset$	$\emptyset$	$2n+2$	$\dots$	$3n-1$	$3n$	$3n+1$
$n+1$	$\emptyset$	$\emptyset$	$\emptyset$	$\dots$	$\emptyset$	$\emptyset$	$2n+1$	$\dots$	$3n-2$	$3n-1$	$3n$
$n+2$	$\emptyset$	$\emptyset$	$\emptyset$	$\dots$	$\emptyset$	$2n-1$	$2n$	$\dots$	$3n-3$	$3n-2$	$3n-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
$2n-1$	$\emptyset$	$\emptyset$	$\emptyset$	$\dots$	$n+1$	$n+2$	$n+3$	$\dots$	$2n$	$2n+1$	$2n+2$
$2n$	$\emptyset$	$\emptyset$	3	$\dots$	$n$	$n+1$	$n+2$	$\dots$	$2n-1$	$2n$	$2n+1$
$2n+1$	$\emptyset$	1	2	$\dots$	$n-1$	$n$	$n+1$	$\dots$	$2n-2$	$2n-1$	$2n$

Table 1: Total degrees for all unary elements in the mod 2 homology of  $B_2(\mathbb{RP}^{2n+1})$ , where  $\emptyset$  denotes an excluded unary element equivalent to a previously counted Lie bracket from the identity  $\bar{Q}^{|x|}x = [x, x]$ . Note that all pairs  $i, k$  with total degree  $m$  lie on the same diagonals.

*Case 3:* Suppose  $0 \leq a+b = c+d = k \leq 2n+1$ . For a given  $k$ , the total degree of  $(x^{a+b} - x^{c+d}) \otimes [y_{2n+1}, y_{2n+1}]$  is  $4n+2-k$ , implying that for  $2n+1 \leq m = 4n+2-k \leq 4n+2$ , the possible ordered pairs with total degree  $m$  are  $(0, k), (1, k-1), \dots, (k, 0)$ , giving  $\lceil \frac{k+1}{2} \rceil = \lceil \frac{4n+3-m}{2} \rceil$  unordered pairs with total degree  $m$ . Therefore, there are  $\binom{\lceil \frac{4n+3-m}{2} \rceil}{2}$  distinct kernel basis elements of the form  $[x^a \otimes y_{2n+1}, x^b \otimes y_{2n+1}] - [x^c \otimes y_{2n+1}, x^d \otimes y_{2n+1}]$ . Note that there are 0 such elements for  $m = 4n+1, 4n+2$ , so  $2n+1 \leq m \leq 4n$ .

Summing over all three cases then gives the desired formula.  $\square$

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# Appendix

## A Extra Preliminaries Definitions and Formulas

### Properties of $\gamma$ Operations in a Divided Power Algebra

The  $\gamma_n : I \rightarrow I$  operations over an ideal  $I$  in commutative ring  $A$  satisfy:

1. For any  $x \in I$ , define  $\gamma_0(x) = 1$  and  $\gamma_1(x) = x$ .
2. For  $x, y \in I$ ,  $\gamma_n(x + y) = \sum_{i=0}^n \gamma_{n-i}(x)\gamma_i(y)$ .
3. For any  $c \in A$  and  $x \in I$ ,  $\gamma_n(cx) = c^n \gamma_n(x)$ .
4. For every  $x \in I$ ,  $\gamma_{m+n}(x) = \frac{m! \gamma_m(x) \cdot n! \gamma_n(x)}{(m+n)!}$ .
5. For  $x \in I$ ,  $\gamma_m(\gamma_n(x)) = \frac{(nm)!}{(n!)^m m!} \gamma_{mn}(x)$ .

### Cup Product Formula

**Definition A.1** (Hatcher [12]). Denote  $X$  a simplicial complex with cohomology  $C^*(X; R)$  over coefficient ring  $R$ . For cochains  $a \in C^p(X; R)$  and  $b \in C^q(X; R)$ , the cup product  $a \smile b \in C^{p+q}(X; R)$  is the cochain whose value on a singular simplex  $\sigma : \Delta^{p+q} \rightarrow X$  is given by the formula

$$(a \smile b)(\sigma) = a(\sigma | [v_0, \dots, v_p])b(\sigma | [v_{p+1}, \dots, v_{p+q}]),$$

where  $\sigma | [v_i, \dots, v_j]$  denotes the simplex  $\sigma$  restricted to a face generated by vertices  $v_i, \dots, v_j$ .

### Betti Number Formulas for $B_k(\Sigma_g)$

Formulas for the Betti numbers  $\beta_i(B_k(\Sigma_g))$  were explicitly computed in general for all  $i \geq 0$  and  $g \geq 0$  by Drummond-Cole and Knudsen [6]. They showed that  $\beta_i(B_k(\Sigma_g)) = 0$  for all  $i > k + 1$  and  $\beta_i(B_k(\Sigma_g))$  is independent of  $k$  for all  $k > i$ . It therefore suffices to consider when  $i = k + 1$ ,  $i = k$ , and  $i < k$ .

**Theorem A.1** (Drummond-Cole, Knudsen [6]). *For  $i = k + 1 \geq 5$  and  $g \geq 0$ , the Betti number  $\beta_i(B_{i-1}(\Sigma_g))$  is*

$$\beta_i(B_{i-1}(\Sigma_g)) = \sum_{j=0}^{g-1} \sum_{m=0}^j (-1)^{g+j+1} \frac{2j - 2m + 2}{2j - m + 2} \binom{6j+2i+2g-2m-5-3(-1)^{i+j+g+m}}{m, 2j - m + 1}.$$

For  $i < 5$ , the Betti numbers are

$$\beta_i(B_{i-1}(\Sigma_g)) = \begin{cases} 0 & i = 1 \\ 1 & i = 2 \\ 0 & i = 3 \\ 2g & i = 4. \end{cases}$$

**Theorem A.2** (Drummond-Cole, Knudsen [6]). For  $i = k \geq 5$  and  $g \geq 0$ , the Betti number  $\beta_i(B_i(\Sigma_g))$  is

$$\beta_i(B_i(\Sigma_g)) = -\binom{2g+i-4}{2g-2} + \sum_{j=0}^{g-1} \sum_{m=0}^j (-1)^{g+j+1} \frac{2j-2m+2}{2j-m+2} \left[ \binom{6j+2i+2g-2m+1+3(-1)^{i+j+g+m}}{m, 2j-m+1} \right. \\ \left. + \binom{6j+2i+2g-2m-3+3(-1)^{i+j+g+m}}{m, 2j-m+1} + \binom{6j+2i+2g-2m-5-3(-1)^{i+j+g+m}}{m, 2j-m+1} \right].$$

For  $i < 5$ , the Betti numbers are

$$\beta_i(B_i(\Sigma_g)) = \begin{cases} 1 & i = 0 \\ 2g & i = 1 \\ 2g^2 - g & i = 2 \\ 4 & i = 3, g = 1 \\ (4g^3 - g + 3)/3 & i = 3, g \neq 1 \\ 0 & i = 4, g = 0 \\ 4 & i = 4, g = 1 \\ 24 & i = 4, g = 2 \\ (4g^4 + 4g^3 - g^2 + 11g)/6 & i = 4, g > 2. \end{cases}$$

**Theorem A.3** (Drummond-Cole, Knudsen [6]). For  $5 \leq i < k$  and  $g \geq 0$ , the Betti number is

$$\beta_i(B_k(\Sigma_g)) = -\binom{2g+i-1}{2g-2} - \binom{2g+i-4}{2g-2} + \sum_{j=0}^{g-1} \sum_{m=0}^j (-1)^{g+j+1} \frac{2j-2m+2}{2j-m+2} \left[ \binom{6j+2i+2g-2m+3-3(-1)^{i+j+g+m}}{m, 2j-m+1} + \binom{6j+2i+2g-2m+1+3(-1)^{i+j+g+m}}{m, 2j-m+1} \right. \\ \left. + \binom{6j+2i+2g-2m-3+3(-1)^{i+j+g+m}}{m, 2j-m+1} + \binom{6j+2i+2g-2m-5-3(-1)^{i+j+g+m}}{m, 2j-m+1} \right]$$

For  $i < 5$ , the Betti numbers are

$$\beta_i(B_k(\Sigma_g)) = \begin{cases} 1 & i = 0 \\ 2g & i = 1 \\ 0 & i = 2, g = 0 \\ 3 & i = 2, g = 1 \\ 2g^2 - g & i = 2, g > 1 \\ 1 & i = 3, g = 0 \\ 5 & i = 3, g = 1 \\ 16 & i = 3, g = 2 \\ (4g^3 - g + 3)/3 & i = 3, g > 2 \\ 0 & i = 4, g = 0 \\ 7 & i = 4, g = 1 \\ 28 & i = 4, g = 2 \\ 90 & i = 4, g = 3 \\ (4g^4 + 4g^3 - g^2 + 11g)/6 & i = 4, g > 3. \end{cases}$$

The dimensions of the first few Betti numbers are explicitly computed in Tables 2, 3, and 4 out of convenience for future reference. For  $k \leq p$ , by Theorem 4.4 the values in the column  $g = 1$  give the dimensions of the  $\mathbb{F}_p$  homology groups of  $B_k(T)$ , and all values listed in the tables correspond to the dimensions of the  $\mathbb{F}_p$  homology groups of  $B_k(\Sigma_{g,1})$ .

$k \setminus g$	0	1	2	3	4	5	6
1	0	0	0	0	0	0	0
2	1	1	1	1	1	1	1
3	0	0	0	0	0	0	0
4	0	2	4	6	8	10	12
5	0	1	6	15	28	45	66
6	0	3	10	35	84	165	286
7	0	2	20	70	210	495	1001
8	0	4	25	140	462	1287	3003
9	0	3	45	224	966	3003	8008
10	0	5	51	406	1764	6567	19448
11	0	4	84	574	3318	13035	44187
12	0	6	91	966	5370	25575	92950
13	0	5	140	1266	9372	45342	189761
14	0	7	148	2010	13959	82368	359359
15	0	6	216	2505	23001	134420	678678
16	0	8	225	3795	32263	230230	1186185
17	0	7	315	4565	50765	352781	2109536
18	0	9	325	6655	68068	576433	3461536
19	0	8	440	7799	103103	840762	5865782
20	0	10	451	11011	133497	1321398	9148074
21	0	9	594	12649	195832	1852630	14894516
22	0	11	606	17381	246610	2817750	22279180
23	0	10	780	19656	351988	3824898	35068110
24	0	12	793	26390	433356	5656002	50658674
25	0	11	1001	29470	604044	7474050	77452816
26	0	13	1015	38780	729912	10784970	108636528
27	0	12	1260	42860	996540	13932282	161938634
28	0	14	1275	55420	1185444	19676514	221476255
29	0	13	1560	60724	1589160	24931401	322850451
30	0	15	1576	77316	1865325	34545819	432010733
31	0	14	1904	84099	2460291	43046685	617367817
32	0	16	1921	105621	2854845	58640175	810508270
33	0	15	2295	114171	3711099	72014294	1137836128
34	0	17	2313	141645	4263448	96611086	1468946336
35	0	16	2736	152285	5470157	117137735	2029362335

Table 2: Chart of values for the  $k$ th Betti number of the  $(k - 1)$ th unordered configuration space of a genus  $g$  surface for  $1 \leq k \leq 35$  and  $0 \leq g \leq 6$ . By Theorem 4.4, if  $k \leq p + 1$ , these are also the dimensions of the  $k$ th  $\mathbb{F}_p$  homology group of  $B_{k-1}(\Sigma_{g,1})$ , and the  $g = 1$  column lists the dimensions of the  $k$ th  $\mathbb{F}_p$  homology group of  $B_{k-1}(T)$ .

$k \setminus g$	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	0	2	4	6	8	10	12
2	0	1	6	15	28	45	66
3	1	4	11	36	85	166	287
4	0	4	24	76	218	505	1013
5	0	7	31	155	490	1332	3069
6	0	7	60	259	1050	3168	8294
7	0	10	75	490	1974	7062	20449
8	0	10	126	728	3822	14322	47190
9	0	13	151	1266	6384	28710	100958
10	0	13	230	1751	11451	52074	209638
11	0	16	268	2835	17642	96668	404118
12	0	16	381	3736	29735	161579	776633
13	0	19	435	5706	43218	282358	1382589
14	0	19	588	7260	69161	443599	2500927
15	0	22	661	10575	96200	737593	4182333
16	0	22	860	13099	147371	1104003	7200247
17	0	25	955	18355	198066	1760298	11450386
18	0	25	1206	22258	292390	2534354	18910099
19	0	28	1326	30206	382263	3898335	28856751
20	0	28	1635	36001	546721	5436651	45987590
21	0	31	1783	47565	698698	8106307	67802360
22	0	31	2156	55881	972412	11008593	104755091
23	0	34	2335	72176	1219246	15973725	150011147
24	0	34	2778	83770	1657200	21209409	225537884
25	0	37	2991	106120	2044380	30046809	315004559
26	0	37	3510	121889	2721837	39130353	462297473
27	0	40	3760	151845	3311028	54270459	631846226
28	0	40	4361	172838	4328703	69498396	907507667
29	0	43	4651	212196	5201762	94581575	1217039318
30	0	43	5340	239626	6691811	119346458	1714480922
31	0	46	5673	290445	7955424	159689926	2261077027
32	0	46	6456	325701	10088309	198888712	3130046205
33	0	49	6835	390321	11879294	262088167	4066906701
34	0	49	7718	434980	14871584	322645059	5541326648
35	0	52	8146	516040	17362905	419338380	7104490887

Table 3: Chart of values for the  $k$ th Betti number of the  $k$ th unordered configuration space of a genus  $g$  surface for  $0 \leq k \leq 35$  and  $0 \leq g \leq 6$ . By Theorem 4.4, if  $k \leq p$ , these are also the dimensions of the  $k$ th  $\mathbb{F}_p$  homology group of  $B_k(\Sigma_{g,1})$ , and the  $g = 1$  column lists the dimensions of the  $k$ th  $\mathbb{F}_p$  homology group of  $B_k(T)$ .

$i \setminus g$	0	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	0	2	4	6	8	10	12
2	0	3	6	15	28	45	66
3	1	5	16	36	85	166	287
4	0	7	28	90	218	505	1013
5	0	9	48	169	532	1332	3069
6	0	11	75	335	1098	3300	8294
7	0	13	114	569	2289	7227	20878
8	0	15	162	979	4187	15587	47762
9	0	17	225	1531	7748	30294	105963
10	0	19	300	2396	13034	58860	216281
11	0	21	393	3520	22079	105118	436150
12	0	23	501	5151	34866	188319	818752
13	0	25	630	7211	55223	315369	1530869
14	0	27	777	10039	82965	529718	2693703
15	0	29	948	13529	124690	842884	4736380
16	0	31	1140	18125	179921	1343826	7912036
17	0	33	1359	23689	259302	2050653	13221792
18	0	35	1602	30784	361900	3132029	21159269
19	0	37	1875	39236	504021	4615128	33879846
20	0	39	2175	49741	684067	6800508	52294099
21	0	41	2508	62085	926002	9727432	80742936
22	0	43	2871	77111	1227304	13904838	120830579
23	0	45	3270	94561	1622011	19387707	180821641
24	0	47	3702	115439	2106363	27001767	263434743
25	0	49	4173	139439	2727348	36822006	383668154
26	0	51	4680	167740	3479594	50140352	545978070
27	0	53	5229	199984	4426415	67056804	776480287
28	0	55	5817	237539	5560388	89530551	1082270541
29	0	57	6450	279991	6965069	117692377	1507214918
30	0	59	7125	328911	8630475	154433796	2062327850
31	0	61	7848	383825	10664900	199922976	2818996389
32	0	63	8616	446521	13055217	258327002	3793935067
33	0	65	9435	516461	15939574	329858905	5100110873
34	0	67	10302	595664	19301036	420398901	6762379052
35	0	69	11223	683524	23313381	530213298	8955099361

Table 4: Chart of values for the  $i$ th Betti number of the  $k$ th unordered configuration space of a genus  $g$  surface for all  $i < k$ ,  $0 \leq i \leq 35$  and  $0 \leq g \leq 6$ . By Theorem 4.4, if  $k \leq p$ , these are also the dimensions of the  $i$ th  $\mathbb{F}_p$  homology group of  $B_k(\Sigma_{g,1})$ , and the  $g = 1$  column lists the dimensions of the  $i$ th  $\mathbb{F}_p$  homology group of  $B_k(T)$ .