A Generalization of Descent Polynomials

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Abstract

The notion of a *descent polynomial*, a function in enumerative combinatorics, that counts permutations with specific properties, enjoys a revived recent research interest due to its connection with other important notions in combinatorics, viz. *peak polynomials* and *symmetric functions*. We define the function $\mathfrak{d}^m(I, n)$ as a generalization of the descent polynomial (for which $m = 1$) and we prove that this function is a polynomial in n for sufficiently large n (similarly to the descent polynomial). We look at the coefficients of $\mathfrak{d}^m(I, n)$ in falling factorial bases. We prove positivity of the coefficients and discover a combinatorial interpretation for them. This result generalizes the positivity result of Diaz-Lopez et al. for the descent polynomial. We obtain an explicit formula for $\mathfrak{d}^m(I, n)$ in some special cases.

Summary

Permutations are a central topic in combinatorics, which correspond to mixing of objects in a straight line. The original problem under consideration is about finding the number of permutations with some specific properties. We make a generalization of it, in which we allow the occurrence same object in the permutation more than once. We prove a number of analogs of the traditional results for the generalization we are considering.

1 Introduction

The *descent polynomial* is a function in enumerative combinatorics that counts permutations with specific properties. It originates from 1915, when MacMahon[1] introduced it in his book *Combinatory Analysis*. However, in the following years there was almost nothing published about the subject. It was only in 2019 when the topic was revisited again by Diaz-Lopez et al.[2], but since then it has gained much popularity. The close relation of the *descent polynomial* with symmetric functions, in particular Schur polynomials, makes it a fundamental object in mathematics. We further explore this function by giving one interesting generalization of it.

First, let us state the original problem. Let I be a finite set of positive integers. We consider permutations of the set $\{1, 2, \dots, n\}$. The *descent polynomial* $d(I, n)$ is equal to the number of permutations for which the set of all indices where the corresponding elements are bigger than the next ones is exactly I. We research the following generalization, which has not been studied yet: instead of the set $\{1, 2, \dots, n\}$, we are considering permutations of the multiset $\{\underbrace{1, \dots, 1}_{m}, \underbrace{2, \dots, 2}_{m}, \dots, \underbrace{n, \dots, n}_{m}\}$, where each number is of multiplicity m. In this case we use the notation $\mathfrak{d}^m(I, n)$. Recall that $d(I, n) = \mathfrak{d}^1(I, n)$.

The paper is organized in the following way. In Section 2, we introduce some useful notation. In Section 3, we show some general properties of $\mathfrak{d}^m(I, n)$. Then in Section 4, we prove that $\mathfrak{d}^m(I, n)$ is a polynomial in n for n sufficiently large. In Section 5 we introduce the *infinity descent polynomial* and we look at its coefficients. Lastly, in Section 6 we give an explicit formula for computing the *infinity descent polynomial* in some special cases.

2 Preliminaries

In this section we introduce some notation which is used throughout the paper.

For the rest of the paper, we assume that n and m are positive integers and I is a finite set of positive integers. If I is non-empty, we let $I = \{\alpha_1, \alpha_2, \dots, \alpha_t\}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_t$. By I^- we denote the set I without its biggest element α_t, so $I^- = \{\alpha_1, \alpha_2, \dots, \alpha_{t-1}\}$. Also, for a non-empty set I, we let L be the length of the longest sequence of consecutive numbers in I.

For the set $I = \{\alpha_1, \alpha_2, \dots, \alpha_t\}$, we let $\beta = (\beta_1, \beta_2, \dots, \beta_t)$ denote the sequence of the *first differences* of the sequence $(0, \alpha_1, \alpha_2, \dots, \alpha_t)$. So, $\beta_1 = \alpha_1 - 0$, $\beta_2 = \alpha_2 - \alpha_1, \dots,$ $\beta_t = \alpha_t - \alpha_{t-1}$. We say that the set I is *separated* if for all $i = 2, 3, \dots, t$, we have $\beta_i \geq 2$.

Let $\mathrm{Comp}(t)$ be the set of all compositions of t and let $\mathrm{Comp}(t, m)$ be the set of all compositions of t, which do not have elements bigger than m. Recall that a composition of a positive integer t is an ordered sequence of positive integers with sum t.

Definition 2.1. Let $v = (v_1, v_2, v_3, \dots, v_\ell)$ be a finite sequence of positive integers. We define

$$\mathrm{Des}(v) = \{i \in \{1, 2, 3, \dots, \ell - 1\} : v_i > v_{i+1}\}$$

to be the *descent set* of the sequence v.

Definition 2.2. For the set $I = \{\alpha_1, \alpha_2, \dots, \alpha_t\}$, we define $D^m(I, n)$ to equal the number of sequences $v = (v_1, v_2, \dots, v_{\alpha_t})$, which satisfy the following conditions:

- $\mathrm{Des}(v) = I^-$,
- $v_i \in \{1, 2, \dots, n\}$,
- each number from the set $\{1, 2, \dots, n\}$ appears at most m times in v.

Definition 2.3. Let $S_n^{(m)}$ be the set of all permutations of the multiset

$$\{\underbrace{1, 1, \dots, 1}_{m}, \underbrace{2, 2, \dots, 2}_{m}, \dots, \underbrace{n, n, \dots, n}_{m}\}.$$

We define the function $\mathfrak{d}^m(I, n)$ such that

$$\mathfrak{d}^m(I, n) = \#\{w \in S_n^{(m)} : \mathrm{Des}(w) = I\}.$$

3 General properties of $\mathfrak{d}^m(I,n)$

In this section, we prove some basic properties of the function $\mathfrak{d}^m(I,n)$. We see how the value of $\mathfrak{d}^m(I,n)$ changes as m changes.

Proposition 3.1. *For a positive integer $n > \alpha_t$, we have the following inequality*

$$\mathfrak{d}^1(I,n) \leq \mathfrak{d}^2(I,n) \leq \mathfrak{d}^3(I,n) \leq \mathfrak{d}^4(I,n) \leq \cdots$$

Proof. Recall that $\max(I) = \alpha_t$. By determining the first α_t elements of the permutation, we determine the entire permutation, because the rest of the elements are in increasing order. We juxtapose each permutation $w \in S_n^{(m)}$ with a permutation $w' \in S_n^{(m+1)}$, by using the first α_t elements of w in w'. Thus, we preserve the descent set I everywhere except possibly at position α_t. We want to compare w'_{α_t} and w'_{α_t+1} (α_t-th and (α_t+1)-th element in w'). In the first α_t elements of w' the number 1 appears at most m times, therefore there is at least one 1 in the rest of the elements. The rest of the elements are in increasing order, so $w'_{\alpha_t+1} = 1$. This leads us to:

$$w'_{\alpha_t} = w_{\alpha_t} > w_{\alpha_t+1} \geq 1 = w'_{\alpha_t+1}.$$

Because we can juxtapose each permutation in $S_n^{(m)}$ with a different permutation in $S_n^{(m+1)}$, it follows that $\mathfrak{d}^m(I,n) \leq \mathfrak{d}^{m+1}(I,n)$. $\square$

Proposition 3.2. *For any positive integer n, the function $\mathfrak{d}^m(I,n)$ stabilizes for $m \geq \alpha_t$.*

Proof. By determining the first α_t elements of the permutation, we determine the entire permutation, because the rest of the elements are in increasing order. Let $m \geq \alpha_t$. The number 1 appears at most $(\alpha_t - 1)$ times in the first α_t elements ($w_{\alpha_t} > w_{\alpha_t+1} \geq 1$), so $w_{\alpha_t+1} = 1$. Also we can use each number from 1 to n as many times as we want in the first α_t elements. Therefore, when $m \geq \alpha_t$, $\mathfrak{d}^m(I,n)$ is equal to the number of sequences $v = (v_1, v_2, \ldots, v_{\alpha_t})$, which satisfy the following conditions:

- $\mathrm{Des}(v) = I^-$,

- $v_i \in \{1, 2, \ldots, n\}$,
- $v_{\alpha_t} \neq 1$.

So, for $m \geq \alpha_t$, the value of $\mathfrak{d}^m(I, n)$ does not depend on m. Therefore, for $m \geq \alpha_t$, the function $\mathfrak{d}^m(I, n)$ stabilizes.

$\square$

4 Polynomiality

In this section, we first prove that $D^m(I, n)$ is a polynomial in n for sufficiently large n. Then, we establish a relation between $D^m(I, n)$ and $\mathfrak{d}^m(I, n)$. Using this relation, we deduce that $\mathfrak{d}^m(I, n)$ is also a polynomial in n for sufficiently large n.

Definition 4.1. Let the set $I = \{\alpha_1, \alpha_2, \ldots, \alpha_t\}$ be non-empty. Let $A = (a_1, a_2, \ldots, a_r) \in \mathrm{Comp}(\alpha_t, m)$ be a composition. By $C(A, I)$, we denote the number of sequences $v = (v_1, v_2, \ldots, v_{\alpha_t})$, which satisfy the following conditions:

- $\mathrm{Des}(v) = I^-$,
- $v_i \in \{1, 2, \ldots, r\}$,
- for each $j \in \{1, 2, \ldots, r\}$, the number j appears exactly a_j times in v.

Remark 4.1. Notice that the value of $C(A, I)$ does not change if we replace the set $\{1, 2, 3, \ldots, r\}$, from where we choose the elements of v, with another set of r elements as long as the smallest number in the set appears a_1 times, the next in size appears a_2 times and so on.

Lemma 4.1. *For $n \geq \alpha_t$, the function $D^m(I, n)$ is a polynomial in n.*

Proof. Using Definition 4.1 we compute that

$$D^m(I,n) = \sum_{A \in \mathrm{Comp}(\alpha_t, m)} C(A,I) \binom{n}{r},$$

where r is the length of the composition A and we sum over all compositions.

We multiply $C(A,I)$ by $\binom{n}{r}$, because as we noted in Remark 4.1, the set $\{1,2,3,\ldots,r\}$ can be replaced by any set with r elements. Because we have the numbers from 1 to n, there are $\binom{n}{r}$ ways to choose r of them to form a set.

Let $n \geq \alpha_t$. Because r is the length of the composition $A \in \mathrm{Comp}(\alpha_t, m)$, it follows that $\alpha_t \geq r$. We obtain $n \geq \alpha_t \geq r$, which means that $\binom{n}{r}$ is a polynomial in n. Also, $C(A,I)$ and the number of elements in the sum are both determined only by m and I. Therefore, $D^m(I,n)$ is polynomial in n for $n \geq \alpha_t$. $\square$

Theorem 4.2. *For $n \geq \alpha_t$, the function $\mathfrak{d}^m(I,n)$ is a polynomial in n.*

Proof. To prove this we use induction on t — the size of I.

When $t = 0$, we get that $I = \varnothing$ and $\mathfrak{d}^m(n, \varnothing) = 1$.

Let $w = w_1 w_2 \ldots w_{nm}$ be a permutation in $S_n^{(m)}$. By determining the first α_t elements of it, we determine the entire permutation because the rest of the elements are in increasing order. There are $D^m(I,n)$ ways to define the first α_t elements. However, by defining the first α_t elements we do not know what is happening between w_{α_t} and w_{α_t+1}. If $w_{\alpha_t} > w_{\alpha_t+1}$ we get that $\mathrm{Des}(w) = I$. If $w_{\alpha_t} \leq w_{\alpha_t+1}$ we get that $\mathrm{Des}(w) = I^-$. So, we obtain the equation

$$D^m(I,n) = \mathfrak{d}^m(I,n) + \mathfrak{d}^m(I^-,n).$$

From the induction hypothesis we get that $\mathfrak{d}^m(I^-,n)$ is a polynomial in n for $n \geq \alpha_{t-1}$. Using Lemma 4.1 we get that $D^m(I,n)$ is a polynomial in n for $n \geq \alpha_t$. Therefore, $\mathfrak{d}^m(I,n)$ is also a polynomial in n for $n \geq \alpha_t$, which finishes the induction. $\square$

5 Coefficients for $d^{\infty}(I, n)$

In Section 3, we proved that $\mathfrak{d}^m(I, n)$ stabilizes for $m \geq \alpha_t$ and in Section 4, we proved that $\mathfrak{d}^m(I, n)$ is a polynomial in n for $n \geq \alpha_t$. Therefore, it is reasonable to introduce a new notation for the stabilized polynomial when $n, m \geq \alpha_t$.

Definition 5.1. Let $d^{\infty}(I, n)$ be the polynomial that equals the function $\mathfrak{d}^m(I, n)$ for $n, m \geq \alpha_t$. We call this polynomial the *infinity descent polynomial.*

In this section, we look at the coefficients of $d^{\infty}(I, n)$ in bases of the type $\left(\binom{n+k}{i}\right)_{i=0}^{\infty}$. We determine when these coefficients are positive and derive the exact value for some of them.

Definition 5.2. For a non-empty set I and an integer i, we define $d_i^{\infty}(I, n)$ to equal the number of sequences $v = (v_1, v_2, \ldots, v_{\alpha_t})$, which satisfy the following conditions:

- $\mathrm{Des}(v) = I^-$,
- $v_j \in \{1, 2, \ldots n\}$,
- exactly i numbers from the set $\{2, 3, \ldots n\}$ appear in v,
- $v_{\alpha_t} \neq 1$.

Definition 5.3. Let $b_0(I), b_1(I), b_2(I), \ldots$ be the coefficients of $d^{\infty}(I, n)$ in the base $\left(\binom{n-1}{i}\right)_{i=0}^{\infty}$:

$$d^{\infty}(I, n) = b_0\binom{n-1}{0} + b_1\binom{n-1}{1} + b_2\binom{n-1}{2} + \cdots$$

In the next theorem, we prove some interesting properties of the coefficients $b_i(I)$. We use ideas similar to the ideas that Diaz-Lopez et al.[2] used in the proof of Theorem 3.1. in their paper *Descent polynomials.*

Theorem 5.1. *Let the set $I = \{\alpha_1, \ldots, \alpha_t\}$ be non-empty and L be the length of the longest sequence of consecutive numbers in I. For each i such that $L \leq i \leq \alpha_t$, the coefficient $b_i(I)$ is a positive integer and for i such that $i < L$ or $i > \alpha_t$, the coefficient $b_i(I) = 0$.*

Proof. To prove this statement, we first want to find the combinatorial meaning of the coefficients: $b_0(I), b_1(I), b_2(I), \ldots$

In the proof of Proposition 3.2 we conclude that the value of $d^{\infty}(I,n)$ is equal to the number of sequences $v = (v_1, v_2, \ldots, v_{\alpha_t})$, which satisfy the following conditions:

- $\mathrm{Des}(v) = I^-$,
- $v_j \in \{1, 2, \ldots n\}$,
- $v_{\alpha_t} \neq 1$.

Therefore, from Definition 5.2, we obtain that

$$d^{\infty}(I,n) = d_0^{\infty}(I,n) + d_1^{\infty}(I,n) + d_2^{\infty}(I,n) + d_3^{\infty}(I,n) + \cdots \, . \tag{1}$$

Let v be a sequence that contains exactly i of the elements from the set $\{2, 3, 4, \ldots, n\}$. Let us replace these i elements with other i elements from the set $\{2, 3, 4, \ldots, n\}$, such that the j-th number in size from the old i elements is replaced with the j-th number in size from the new i elements. By doing that, we keep the descent set I^- and the last element remains bigger than 1. So, $d_i^{\infty}(I,n)$ is equal to $\binom{n-1}{i}$ times the number of sequences $v = (v_1, v_2, \ldots, v_{\alpha_t})$, which satisfy the following conditions:

- $\mathrm{Des}(v) = I^-$,
- $v_j \in \{1, 2, \ldots i+1\}$,
- for every $\ell \in \{2, 3, \ldots i+1\}$, exists $j : \ v_j = \ell$,
- $v_{\alpha_t} \neq 1$.

If we replace $d_i^{\infty}(I,n)$ in Equation (1), we get a representation of $d^{\infty}(I,n)$ in the base $\left(\binom{n-1}{i}\right)_{i=0}^{\infty}$. Therefore, the coefficient $b_i(I)$ represents the number of sequences that satisfy the four conditions above.

Let v be one such sequence. Because the length of v is α_t, we get that there at most α_t different numbers in v. Also, from the first condition we get that there are at least L different numbers in v bigger than 1. However, from the third condition we get that there are exactly i different numbers in v bigger than 1, which leads us to the inequality:

$$L \leq \ i \leq \alpha_t.$$

From this inequality, we conclude that:

$$0 = b_0(I) = b_1(I) = \cdots = b_{L-1}(I) = b_{\alpha_t+1}(I) = b_{\alpha_t+2}(I) = \cdots$$

Now, we must prove that $b_L(I), b_{L+1}(I), \ldots, b_{\alpha_t}(I)$ are positive integers. For each i such that $L \leq i \leq \alpha_t$, we need to show at least one sequence counted by $b_i(I)$. Such sequences can be expressed by semistandard Young tableaux of ribbon shape. Recall that a semistandard Young tableau is a Young diagram filled with positive integers such that the numbers are weakly increasing from left to right and are strongly decreasing from bottom to top. Recall also that a ribbon is a connected Young diagram without any 2×2 squares in it. The shape of the ribbon is determined by the descent set and its length is equal to the length of the sequence.

The example on Figure 1 shows a sequence for $i = L$, while the one on Figure 2 shows a sequence for $i = \alpha_t$.

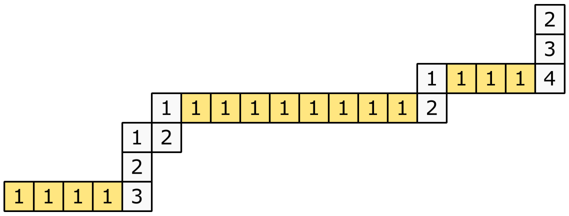

Figure 1: Sequence for $i = L$

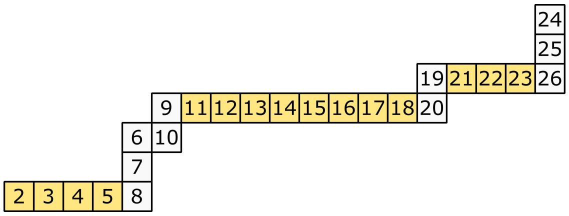

Figure 2: Sequence for $i = \alpha_t$

We want to prove that there exists such example for every other i between L and α_t. Let us replace the numbers in Figure 2 one-by-one with the numbers in Figure 1 starting from the smallest and going in an increasing order. By doing this we preserve the semistandard Young tableau in all median positions. Because we change one number at a time the value of i is changing by at most 1, which assures us that we go through all values of i from α_t to L. $\square$

Corollary 1. *For every integer k, the coefficients of $d^\infty(I,n)$ in the base $\left(\binom{n+k}{i}\right)_{i=0}^\infty$ are integers.*

Proof. To prove this we use induction on k in both directions.

From Theorem 5.1, we know that the coefficients of $d^\infty(I,n)$ are integers in the base $\left(\binom{n-1}{i}\right)_{i=0}^\infty$.

Let us suppose that the coefficients of $d^\infty(I,n)$ are integers in the base $\left(\binom{n+k}{i}\right)_{i=0}^\infty$. We want to prove that they are integers in both the bases $\left(\binom{n+k-1}{i}\right)_{i=0}^\infty$ and $\left(\binom{n+k+1}{i}\right)_{i=0}^\infty$. From Theorem 5.1, we know that the degree of $d^\infty(I,n)$ is α_t, so we get:

$$d^\infty(I,n) = a_{\alpha_t}\binom{n+k}{\alpha_t} + a_{\alpha_t-1}\binom{n+k}{\alpha_t-1} + \cdots + a_0\binom{n+k}{0}$$

for some integers $a_0, a_1, \ldots, a_{\alpha_t}$ ($a_{\alpha_t} \neq 0$). Using the identities:

$$\binom{n+k}{i} = \binom{n+k-1}{i} + \binom{n+k-1}{i-1}$$

$$\binom{n+k}{i} = \sum_{j=0}^{i}(-1)^{i-j}\binom{n+k+1}{j}$$

we can express $d^\infty(I,n)$ in both the bases $\left(\binom{n+k-1}{i}\right)_{i=0}^\infty$ and $\left(\binom{n+k+1}{i}\right)_{i=0}^\infty$ and conclude that the coefficients in both bases are integers. $\square$

Definition 5.4. For the non-empty set $I = \{\alpha_1, \ldots, \alpha_t\}$, let $x_i(I)$ equal the number of sequences $v = (v_1, v_2, \ldots, v_{\alpha_t})$, which satisfy the following conditions:

- $\mathrm{Des}(v) = I^-$,
- $v_j \in \{2, 3, \ldots i+1\}$,
- for every $\ell \in \{2, 3, \ldots i+1\}$, exists $j : \ v_j = \ell$.

Definition 5.5. For a non-empty set $I = \{\alpha_1, \ldots, \alpha_t\}$, let $y_i(I)$ equal the number of sequences $v = (v_1, v_2, \ldots, v_{\alpha_t})$, which satisfy the following conditions:

- $\mathrm{Des}(v) = I^-$,
- $v_j \in \{1, 2, \dots i+1\}$,
- for every $\ell \in \{1, 2, \dots i+1\}$, exists $j:\ v_j = \ell$,
- $v_{\alpha_t} \neq 1$.

Definition 5.6. Let $c_0(I), c_1(I), c_2(I), \dots$ be the coefficients of $d^\infty(I, n)$ in the base $\left(\binom{n}{i}\right)_{i=0}^{\infty}$:

$$d^\infty(I, n) = c_0\binom{n}{0} + c_1\binom{n}{1} + c_2\binom{n}{2} + \cdots$$

In the next theorem, we look at the coefficients $c_i(I)$. We derive a relation between $c_i(I)$ and $b_i(I)$ and find the exact values of $c_i(I)$ for some i.

Theorem 5.2. *Let $t = |I|$ and let L be the length of the longest sequence of consecutive numbers in I. Then $c_i(I) = (-1)^{i+t}$ for all i such that $0 \le i \le L$.*

Proof. Because we do not have much information about the coefficients $c_0(I), c_1(I), c_2(I), \dots$ we want to express them in terms of $b_0(I), b_1(I), b_2(I), \dots$. To do this we use the identity $\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$.

$$\begin{aligned}
d^\infty(I, n) &= b_{\alpha_t}(I)\binom{n-1}{\alpha_t} + b_{\alpha_t - 1}(I)\binom{n-1}{\alpha_t - 1} + \cdots + b_L(I)\binom{n-1}{0} \\
&= b_{\alpha_t}(I)\left(\binom{n-1}{\alpha_t} + \binom{n-1}{\alpha_t - 1}\right) + (b_{\alpha_t - 1}(I) - b_{\alpha_t}(I))\left(\binom{n-1}{\alpha_t - 1} + \binom{n-1}{\alpha_t - 2}\right) + \cdots \\
&\quad + \sum_{i=1}^{\alpha_t}(-1)^{i+1} b_i(I)\left(\binom{n-1}{1} + \binom{n-1}{0}\right) + \sum_{i=0}^{\alpha_t}(-1)^i b_i(I)\binom{n-1}{0} \\
&= b_{\alpha_t}(I)\binom{n}{\alpha_t} + \cdots + \sum_{i=1}^{\alpha_t}(-1)^{i+1} b_i(I)\binom{n}{1} + \sum_{i=0}^{\alpha_t}(-1)^i b_i(I)\binom{n}{0}.
\end{aligned}$$

Therefore, we obtain that:

$$c_k(I) = \sum_{i=k}^{\alpha_t}(-1)^{i+k} b_i(I).$$

Because the coefficients $0 = b_0(I) = b_1(I) = \cdots = b_{L-1}(I)$, we get that

$$c_0(I) = -c_1(I) = c_2(I) = -c_3(I) = \cdots = (-1)^L c_L(I). \tag{2}$$

Our goal is to compute $c_0(I)$.

$$c_0(I) = \sum_{i=0}^{\alpha_t} (-1)^i b_i(I).$$

From Definitions 5.4 and 5.5 we get

$$b_i(I) = x_i(I) + y_i(I). \tag{3}$$

Let us consider sequences with length α_t and non-empty descent set I^-. The number of such sequences whose elements are numbers from the set $\{2, 3, \dots, i+2\}$ is equal to $x_{i+1}(I)$. The value of $x_{i+1}(I)$ is equal to the number of such sequences whose elements are taken from the set $\{1, 2, \dots, i+1\}$. The last number can be divided into two parts: the number of sequences with last element bigger than 1, which is $y_i(I)$, and the number of sequences with last element exactly one 1, which is $b_i(I^-)$. Therefore, we get the equation

$$x_{i+1}(I) = y_i(I) + b_i(I^-).$$

Combining it with Equation (3) we obtain the relation:

$$b_i(I) = x_{i+1}(I) + x_i(I) - b_i(I^-).$$

Also, it is worth mentioning that:

$$b_0(I) = x_1(I) - b_0(I^-),$$
$$b_{\alpha_t}(I) = x_{\alpha_t}(I) - b_{\alpha_t}(I^-),$$

because $x_0(I) = x_{\alpha_t+1}(I) = 0$. Using this information, let us compute $c_0(I)$.

$$c_0(I) = \sum_{i=0}^{\alpha_t} (-1)^i b_i(I) = \sum_{i=0}^{\alpha_t} (-1)^i \left(x_{i+1}(I) + x_i(I) - b_i(I^-)\right) = -\sum_{i=0}^{\alpha_t} (-1)^i b_i(I^-) = -c_0(I^-).$$

We can continue this process until we get to $c_0(\{\alpha_1\})$. So, we get that:

$$c_0(I) = (-1)^{t-1} c_0(\{\alpha_1\}). \tag{4}$$

Therefore, our main goal is to calculate $c_0(\{\alpha_1\})$. We start by computing $d^\infty(\{\alpha_1\}, n)$. As we know from Proposition 3.2, $d^\infty(\{\alpha_1\}, n)$ is equal to the number of sequences $v = (v_1, v_2, \dots, v_{\alpha_1})$, whose elements are numbers from the set $\{1, 2, \dots, n\}$, which have last element bigger than 1, and which have a descent set $\{\varnothing\}$. So, the sequence v look like

$$1 \leq v_1 \leq v_2 \leq \ldots \leq v_{\alpha_1} \leq n.$$

We have $\binom{n+\alpha_t-1}{\alpha_1} - 1$ such sequences. We remove one because of the sequence $(1, 1, \dots, 1)$, which is the only sequence with last element equal to 1. Using the following identity, we find the coefficients $c_i(\{\alpha_1\})$.

$$-1+\binom{n+\alpha_t-1}{\alpha_1}=-\binom{n}{0}+\sum_{i=1}^{\alpha_1}\binom{\alpha_1-1}{\alpha_1-i}\binom{n}{i}.$$

We get that $c_0(\{\alpha_1\})=-1$. After replacing it in Equation (4) we obtain that $c_0(I)=(-1)^t$. Finally, if we replace this in Equation (2) we derive:

$$c_i(I)=(-1)^{i+t} \text{ for } 0\leq i\leq L.$$

□

The next theorem is a continuation of Theorems 5.1 and 5.2. We are considering the sign of the coefficients of d^∞(I,n) in the base $\left(\binom{n+k}{i}\right)_{i=0}^{\infty}$ for different values of k.

Theorem 5.3. *Let k be a positive number and the set I be non-empty. We consider the coefficients of $d^\infty(I,n)$ in the base $\left(\binom{n+k}{i}\right)_{i=0}^{\infty}$. If $k\leq -1$ all coefficients are non-negative and if $k\geq 0$, there exists at least one negative coefficient.*

Proof. From the identity $\binom{n+k}{i}=\binom{n+k-1}{i}+\binom{n+k-1}{i-1}$, it follows that if the coefficients of $d^\infty(I,n)$ are non-negative in the base $\left(\binom{n+k}{i}\right)_{i=0}^{\infty}$, they should also be non-negative in the base $\left(\binom{n+k-1}{i}\right)_{i=0}^{\infty}$. From Theorems 5.1 and 5.2, we know that the coefficients in the base $\left(\binom{n-1}{i}\right)_{i=0}^{\infty}$ are non-negative and there exists at least one negative coefficient in the base $\left(\binom{n}{i}\right)_{i=0}^{\infty}$. Therefore, for $k\leq -1$ the coefficients of $d^\infty(I,n)$ in the base $\left(\binom{n+k}{i}\right)_{i=0}^{\infty}$ are non-negative integers and for $k\geq 0$ there exists at least one negative coefficient. □

For $k\geq 0$, there exist negative coefficients in the representation of $d^\infty(I,n)$ in the base $\left(\binom{n+k}{i}\right)_{i=0}^{\infty}$. This means that we cannot give a combinatorial meaning to these coefficients. Therefore, the base $\left(\binom{n-1}{i}\right)_{i=0}^{\infty}$ is optimal for assigning a combinatorial meaning to the coefficients.

6 Formula for $d^{\infty}(I,n)$ when I is separated.

In this section, we derive an explicit formula for $d^{\infty}(I,n)$, when the set I is separated. Recall that a separated set is a set of integers, which does not contain consecutive numbers.

When we talk about compositions we imagine putting separators between balls in a line. For example the division of the balls below correspond to the composition $(3,1,2,2)$.

$$. \; . \; . \mid . \mid . \; . \mid . \; .$$

As we can see the numbers of balls between the separators give us the elements of the composition. Let us put a weight on each ball and instead of taking the number of balls, we take the total weight of the balls between the separators. Let us look at the previous example, but this time put weight on the balls. We take the weights to be $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8$. Now the composition $(3,1,2,2)$ corresponds to

$$\beta_1 \; \beta_2 \; \beta_3 \mid \beta_4 \mid \beta_5 \; \beta_6 \mid \beta_7 \; \beta_8.$$

We can write this in short as $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, where $\sigma_1 = \beta_1 + \beta_2 + \beta_3,\ \sigma_2 = \beta_4,\ \sigma_3 = \beta_5 + \beta_6$ and $\sigma_4 = \beta_7 + \beta_8$.

Definition 6.1. Let $A \in \mathrm{Comp}(t)$ be a composition of t and $A = (a_1, a_2, \ldots, a_s)$. For a finite sequence of positive integers $\beta = (\beta_1, \beta_2, \ldots, \beta_t)$, we define the function f_β, such that $f_\beta(A) = (\sigma_1, \sigma_2, \ldots, \sigma_s)$, where

$$\sigma_1 = \beta_1 + \beta_2 + \cdots + \beta_{a_1}$$

$$\sigma_i = \beta_{a_1+a_2+\cdots+a_{i-1}} + \beta_{a_1+a_2+\cdots+a_{i-1}+1} + \cdots + \beta_{a_1+a_2+\cdots+a_i} \quad \text{for all } i:\ 2 \le i \le s.$$

Lemma 6.1. *Let j, n, and t be positive integers such that $j \le n$. Let the set $I = \{\alpha_1, \alpha_2, \ldots, \alpha_t\}$ be a separated. Define $\beta = (\beta_1, \beta_2, \ldots, \beta_t)$ as in Section 2. Then, the number of sequences $v = (v_1, v_2, \ldots, v_{\alpha_t})$, which satisfy the following conditions:*

- $v_i \in \{1, 2, \ldots, n\}$

- $v_{\alpha_t} = j$,
- $\mathrm{Des}(v) = I^-$.

is

$$\sum_{A \in \mathrm{Comp}(t)} (-1)^{t-s} \binom{n-1+\sigma_1}{\sigma_1} \cdots \binom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \binom{j-1+\sigma_s-1}{\sigma_s-1} \tag{5}$$

where, in the summation above, we sum over all compositions $A \in \mathrm{Comp}(t)$. Recall that from Definition 6.1 the variables $\sigma_1, \sigma_2, \ldots, \sigma_s$ depend on A by the relation $f_\beta(A) = (\sigma_1, \sigma_2, \ldots, \sigma_s)$.

Proof. To prove this statement we make an induction on t — the number of elements in I.

When $t = 1$, we have that $I = \alpha_1$ and $I^- = \varnothing$. Therefore the sequence v should look like

$$1 \leq v_1 \leq v_2 \leq v_3 \leq \cdots \leq v_{\alpha_1} = j.$$

There are $\binom{j-1+\alpha_1-1}{\alpha_1-1}$ such sequences. Let us see what result we get by applying formula (5). When $t = 1$, we get that the sequence of the first differences is $\beta = (\alpha_1)$ and the set $\mathrm{Comp}(1) = \{(1)\}$. Therefore, $f_\beta((1)) = \alpha_1$ and $\sigma_1 = \alpha_1$. Substituting this in formula (5), we get

$$\sum_{A \in \mathrm{Comp}(t)} (-1)^{t-s} \binom{n-1+\sigma_1}{\sigma_1} \cdots \binom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \binom{j-1+\sigma_s-1}{\sigma_s-1} =$$

$$= (-1)^{1-1} \binom{j-1+\sigma_1-1}{\sigma_1-1} = \binom{j-1+\alpha_1-1}{\alpha_1-1}.$$

Proposition (5) is true for $t = 1$. Let us suppose it is true for $t-1$. We want to prove that it is true for t.

We fix $v_{\alpha_{t-1}} = i$. From the induction hypothesis we know that there are

$$\sum_{A \in \mathrm{Comp}(t-1)} (-1)^{t-s-1} \binom{n-1+\sigma_1}{\sigma_1} \cdots \binom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \binom{i-1+\sigma_s-1}{\sigma_s-1}$$

ways to choose the first α_{t-1} elements of the sequence.

We look at the last $\alpha_t - \alpha_{t-1}$ elements

$$i > v_{\alpha_{t-1}+1} \leq v_{\alpha_{t-1}+2} \leq \cdots \leq v_{\alpha_t - 1} \leq v_{\alpha_t} = j.$$

If $i + 1 \geq j$, there are

$$\binom{j-1+\alpha_{t+1}-\alpha_t-1}{\alpha_{t+1}-\alpha_t-1} = \binom{j-1+\beta_t-1}{\beta_t-1}$$

ways to choose the last $\alpha_t - \alpha_{t-1}$ elements of the sequence.

If $i \leq j$, there are

$$\binom{j-1+\beta_{t+1}-1}{\beta_{t+1}-1} - \binom{j-i+\beta_{t+1}-1}{\beta_{t+1}-1}$$

ways to choose the last $\alpha_t - \alpha_{t-1}$ elements of the sequence.

Therefore, the number of sequences v with a descent set $I^- = \{\alpha_1, \alpha_2, \ldots, \alpha_{t-1}\}$ and last element $v_{\alpha_t} = j$ is

$$\begin{aligned}
&\sum_{i=2}^{j} \left(\tbinom{j-1+\beta_t-1}{\beta_t-1} - \tbinom{j-i+\beta_t-1}{\beta_t-1}\right) \sum_{A\in\mathrm{Comp}(t-1)} (-1)^{t-s-1} \tbinom{n-1+\sigma_1}{\sigma_1} \ldots \tbinom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \tbinom{i-1+\sigma_s-1}{\sigma_s-1} \\
&+ \sum_{i=j+1}^{n} \tbinom{j-1+\beta_t-1}{\beta_t-1} \sum_{A\in\mathrm{Comp}(t-1)} (-1)^{t-s-1} \tbinom{n-1+\sigma_1}{\sigma_1} \ldots \tbinom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \tbinom{i-1+\sigma_s-1}{\sigma_s-1} \\
&= \sum_{i=2}^{n} \tbinom{j-1+\beta_t-1}{\beta_t-1} \sum_{A\in\mathrm{Comp}(t-1)} (-1)^{t-s-1} \tbinom{n-1+\sigma_1}{\sigma_1} \ldots \tbinom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \tbinom{i-1+\sigma_s-1}{\sigma_s-1} \\
&+ \sum_{i=2}^{j} \tbinom{j-i+\beta_t-1}{\beta_t-1} \sum_{A\in\mathrm{Comp}(t-1)} (-1)^{t-s} \tbinom{n-1+\sigma_1}{\sigma_1} \ldots \tbinom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \tbinom{i-1+\sigma_s-1}{\sigma_s-1} \\
&= \tbinom{j-1+\beta_t-1}{\beta_t-1} \sum_{A\in\mathrm{Comp}(t-1)} (-1)^{t-s-1} \tbinom{n-1+\sigma_1}{\sigma_1} \ldots \tbinom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \left(\tbinom{n-1+\sigma_s}{\sigma_s} - 1\right) \\
&+ \sum_{A\in\mathrm{Comp}(t-1)} (-1)^{t-s} \tbinom{n-1+\sigma_1}{\sigma_1} \ldots \tbinom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \left(\tbinom{j-1+\sigma_s+\beta_t-1}{\sigma_s+\beta_t-1} - \tbinom{j-1+\beta_t-1}{\beta_t-1}\right) \\
&= \sum_{A\in\mathrm{Comp}(t-1)} (-1)^{t-(s+1)} \tbinom{n-1+\sigma_1}{\sigma_1} \ldots \tbinom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \tbinom{n-1+\sigma_s}{\sigma_s} \tbinom{j-1+\beta_t-1}{\beta_t-1} \\
&+ \sum_{A\in\mathrm{Comp}(t-1)} (-1)^{t-s} \tbinom{n-1+\sigma_1}{\sigma_1} \ldots \tbinom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \tbinom{j-1+\sigma_s+\beta_t-1}{\sigma_s+\beta_t-1} \\
&= \sum_{A\in\mathrm{Comp}(t)} (-1)^{t-s} \tbinom{n-1+\sigma_1}{\sigma_1} \ldots \tbinom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \tbinom{j-1+\sigma_s-1}{\sigma_s-1},
\end{aligned}$$

which finishes the induction. □

Theorem 6.2. *Let the set $I = \{\alpha_1, \alpha_2, \ldots, \alpha_t\}$ be separated and $\beta = (\beta_1, \beta_2, \ldots, \beta_t)$ be the sequence of the first differences for I. Then:*

$$d^{\infty}(I, n) = \sum_{A\in\mathrm{Comp}(t)} (-1)^{t-s} \binom{n-1+\sigma_1}{\sigma_1} \ldots \binom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \left(\binom{n-1+\sigma_s}{\sigma_s} - 1\right),$$

where, in the summation above, we sum over all compositions $A \in \mathrm{Comp}(t)$. Recall that from Definition 6.1 the variables $\sigma_1, \sigma_2, \ldots, \sigma_s$ depend on A by the relation $f_\beta(A) = (\sigma_1, \sigma_2, \ldots, \sigma_s)$.

Proof. From the proof of Proposition 3.2, we know that the *infinity descent polynomial*

$d^\infty(I,n)$ is equal to the number of sequences $v = (v_1, v_2, v_3, \ldots, v_{\alpha_t})$, which satisfy the following conditions:

- $\mathrm{Des}(v) = I^-$,
- $v_i \in \{1, 2, \ldots, n\}$,
- $v_{\alpha_t} \neq 1$.

We can compute the number of sequences v by summing over all j from 2 to n in Lemma 6.1. Therefore, we obtain that

$$\begin{aligned} d^\infty(I,n) &= \sum_{j=2}^{n} \sum_{A\in\mathrm{Comp}(t)} (-1)^{t-s}\binom{n-1+\sigma_1}{\sigma_1}\cdots\binom{n-1+\sigma_{s-1}}{\sigma_{s-1}}\binom{j-1+\sigma_s-1}{\sigma_s-1} \\ &= \sum_{A\in\mathrm{Comp}(t)} (-1)^{t-s}\binom{n-1+\sigma_1}{\sigma_1}\cdots\binom{n-1+\sigma_{s-1}}{\sigma_{s-1}}\left(\binom{n-1+\sigma_s}{\sigma_s}-1\right). \end{aligned}$$

$\square$

7 Future Work

In the future we want to find an explicit formula for $\mathfrak{d}^m(I,n)$ in more general cases. Also, we want to further explore the connection between the descent polynomials and symmetric functions. Representing permutations by semistandard Young tableaux of ribbon shape as we do in Theorem 5.1, gives us a new prospective of the problem. We can use strong results for Young tableaux, for example the Jacobi-Trudi identity and Naruse hook length formula, in exploring the descent polynomials. Another venue for research is to look at a similar generalization for the peak polynomial, a brother of the descent polynomial, with the hope to prove some analogs of the traditional results for the generalization we are considering.

8 Conclusion

We look at the function $\mathfrak{d}^m(I,n)$, which is a generalization of the descent polynomial. We prove that as m is increasing the function $\mathfrak{d}^m(I,n)$ is weakly increasing and from some point onward it stabilizes. We prove that $\mathfrak{d}^m(I,n)$ is a polynomial in n for sufficiently large n similarly to the descent polynomial. We introduce the stabilized polynomial $d^\infty(I,n)$, which equals $\mathfrak{d}^m(I,n)$, when both n and m are large enough. We express $d^\infty(I,n)$ in falling factorial bases of the type $\left(\binom{n+k}{i}\right)_{i=0}^{\infty}$. Our main motivation to do this is the previous results[2, 3] about the coefficients of the descent polynomial in these bases. We prove that the coefficients in these bases are integers and we determine when they are non-negative. We also find a combinatorial meaning for these coefficients and compute the exact value of some coefficients. Lastly, we give an explicit formula for $d^\infty(I,n)$, when the set I does not contain consecutive numbers.

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