Two Problems on Cantor Set Arithmetic

Lauren Chen

under the direction of

Yuqiu Fu
Department of Mathematics
Massachusetts Institute of Technology

Research Science Institute
July 30, 2019

Abstract

We find the cardinalities of the solution sets to the polynomial equations $c = a + b$ and $c = a - b$ on variants of the Cantor set. We also compute examples for the equation $c = ab$. A previous theorem states $f(C \times C) = [0, 1]$ for the Cantor set C where $f(x, y) = x^2 y$. Our second problem generalizes this to $f = x^\alpha y$ for α in the range $\frac{\log 2}{\log 3/2} \leq \alpha \leq 2$. We also explore the case when α is greater than 2. We consider the expansion of $f(C_n \times C_n)$ for a few small n, where C_n is the nth iteration of the Cantor set, to find intervals of $\alpha > 2$ such that $f(C \times C)$ does not cover the entire interval $[0, 1]$.

Summary

The numbers we typically use are in base 10, meaning they contain the digits 0 through 9. Similarly, numbers in base 3 only contain the digits 0, 1, and 2. The Cantor set is the set of all numbers from $[0, 1]$ that can be written in base 3 without 1. We study two problems. In the first problem, we consider three polynomial equations in base 3 where all numbers involved do not include one of the digits 0, 1, or 2. The second problem generalizes a theorem about the equation $u = x^2 y$ in the Cantor set to $u = x^\alpha y$.

1 Introduction

The Cantor set C is defined as all numbers that can be expressed in the form $\sum_{k=1}^{\infty} a_k 3^{-k}$, $a_k \in \{0, 2\}$. In other words, it is the set of points in the interval $[0, 1]$ that can be written in ternary without 1. It is produced by repeatedly removing the middle third of each of the remaining segments, starting with the interval $[0, 1]$. Denote these iterations C_n, as seen in Figure 1.

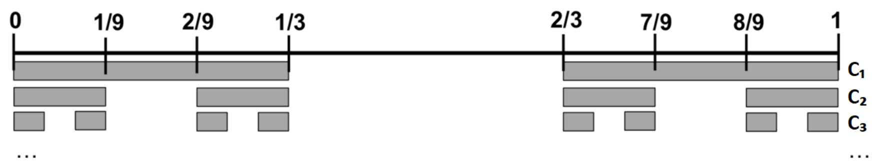

Figure 1: Iterations of the Cantor set C_n. The Cantor set satisfies $C = \lim_{n\to\infty} C_n$.

We define the set $\mathcal{C}(A)$ as $\{a \in \mathbb{Z}_{\geq 0} | a = \sum_{k=0}^{j} a_k 3^k$ for $a_k \in A$, $j \in \mathbb{Z}_{\geq 0}\}$, where A is a two-element subset of the set $\{0, 1, 2\}$. $\mathcal{C}(A)$ is an analog of the Cantor set. Numbers in both sets consist of only two distinct numbers from the set $\{0, 1, 2\}$ in their ternary representation. Because the Cantor set has been more extensively studied, the connection between the sets reveals some potential directions of exploration for our own problem.

We define the set $\mathcal{C}(A)_N$ to be $\{a \in \mathcal{C}(A) : a < 3^N\}$. We want to find the cardinality of the set $S_N := \{(a, b) \in \mathcal{C}(A)_N \times \mathcal{C}(A)_N : \exists\, c \in \mathcal{C}(A)$ s.t. $P(a, b) = c\}$ for some polynomial P. Note that S_N depends on the polynomial P. We will specify which polynomial S_N refers to in each section. We consider variations of this problem, such as changing A to be a different two-element subset of $\{0, 1, 2\}$. Our first problem is the discrete version of the problem of finding the Minkowski dimension of the solution set [1]. Motivation for the problem thus arises from this connection.

We also explore a second problem on the Cantor set itself. In 2017, Athreya, Reznick, and Tyson [2] proved that every element u of $[0, 1]$ can be written in the form $u = x^2 y$, where

x, y are elements of the Cantor set C. We investigate the range of possible values for α in $u = x^\alpha y$ for which the result still holds.

In Sections 2 and 3, we find formulas for the cardinalities of the solution sets for $P(a, b) = a + b$ and $P(a, b) = a - b$, respectively. In Section 4, we compute the cardinalities $|S_N|$ for some small N for $P(a, b) = ab$. In Section 5, we form a conjecture for the lower bound on α in $u = x^\alpha y$ based on the proof by Athreya, Reznick, and Tyson for $\alpha = 2$. Section 6 generalizes the result to a larger range for $\alpha < 2$, and Section 7 explores what happens for $\alpha > 2$.

2 $|S_N|$ for $P(a, b) = a + b$

For our first problem, we find formulas for the cardinalities of the solution sets to polynomial equations on the Cantor set variant $\mathcal{C}(A)$. We first consider the case where $P(a, b) = a+b$ and $\mathcal{C}(\{0, 1\})$.

Theorem 2.1. *When* $S_N := \{(a, b) \in \mathcal{C}(A)_N \times \mathcal{C}(A)_N : \exists\ c \in \mathcal{C}(A)$ *s.t.* $a + b = c\}$ *and* $\mathcal{C}(A) := \{a \in \mathbb{Z}_{\geq 0} : a = \sum_{k=0}^{j} a_k 3^k$ *for* $a_k \in \{0, 1\}$, $j \in \mathbb{Z}_{\geq 0}\}$, $|S_N| = 3^N$.

Proof. Let $a = \sum_{i=0}^{N-1} a_i 3^i$ and $b = \sum_{i=0}^{N-1} b_i 3^i$ with $a_i, b_i \in \{0, 1\}$, so $c = a + b = \sum_{i=0}^{N-1} c_i 3^i$ where $c_i \in \{0, 1, 2\}$. We want c_i to also be $0, 1$ for all i so $c \in \mathcal{C}(A)$. For each $i \in \{0, 1, ..., N-1\}$, there are four cases:

When $a_i = 0$ and $b_i = 0$, we have $a_i + b_i = 0$.

When $a_i = 0$ and $b_i = 1$, we have $a_i + b_i = 1$.

When $a_i = 1$ and $b_i = 0$, we have $a_i + b_i = 1$.

When $a_i = 1$ and $b_i = 1$, we have $a_i + b_i = 2$.

Thus, $a_i + b_i$ does not exceed 2 for any i, so $c_i = a_i + b_i$ and each of the N sums is independent from the others. Each sum can be any of the first three cases, so $|S_N| = 3^N$. $\square$

We now consider the case where $P(a, b) = a+b$ and A is the set $\{1, 2\}$, following a similar method of proof.

Theorem 2.2. *When* $S_N := \{(a,b) \in \mathcal{C}(A)_N \times \mathcal{C}(A)_N : \exists\ c \in \mathcal{C}(A)\ s.t.\ a+b=c\}$ *and* $\mathcal{C}(A) := \{a \in \mathbb{Z}_{\geq 0} : a = \sum_{k=0}^{j} a_k 3^k \text{ for } a_k \in \{1,2\},\ j \in \mathbb{Z}_{\geq 0}\}$, $|S_N| = \dfrac{3^N+1}{2}$.

Proof. As before, we let $a = \sum_{i=0}^{N-1} a_i 3^i$ and $b = \sum_{i=0}^{N-1} b_i 3^i$ with $a_i, b_i \in \{1,2\}$, so $c = a+b = \sum_{i=0}^{N-1} c_i 3^i$ where $c_i \in \{0,1,2\}$. We find a recursive formula for $|S_N|$. There are four cases:

When $a_i = 1$ and $b_i = 1$, we have $a_i + b_i = 2$.

When $a_i = 1$ and $b_i = 2$, we have $a_i + b_i = 3$.

When $a_i = 2$ and $b_i = 1$, we have $a_i + b_i = 3$.

When $a_i = 2$ and $b_i = 2$, we have $a_i + b_i = 4$.

When $x_0 = y_0 = 1$, there are $|S_{N-1}|$ such ordered pairs. The second and third cases are not possible when $i = 0$ because c_i would then equal 0, which is not part of our solution set. When the last case holds for $i = 0$, the sum $a_0 + b_0$ causes a regroup and increases z_1 by 1. If this happens, a_1 and b_1 have to be one of the last three cases. Then by the same reasoning, all following a_i and b_i must be one of the last three cases. Thus, there are 3^{N-1} ordered pairs with $i = 0$ as the last case. Thus, $|S_N| = 3^{N-1} + |S_{N-1}|$ with $S_0 = 1$. We can rewrite the recursive formula in closed form as $|S_N| = \dfrac{3^N+1}{2}$. □

3 $|S_N|$ for $P(a,b) = a - b$

For $P(a,b) = a-b$, we allow negative integers in $\mathcal{C}(A)$. By symmetry, $|S_N|$ is equal for $a > b$ and $a < b$. Thus, it suffices to consider when $a > b$. Define $S_{m,n}$ to be the solution set for a and b with exactly m and n digits, respectively. Note that $S_N = \bigcup_{m,n \leq N} S_{m,n} = \bigcup_{m=1}^{N} \bigcup_{n=1}^{N} S_{m,n}$.

Theorem 3.1. *When* $S := \{(a,b) \in \mathcal{C}(A)_N \times \mathcal{C}(A)_N : \exists\ c \in \mathcal{C}(A)\ s.t.\ a-b=c\}$ *and* $\mathcal{C}(A) := \{a \in \mathbb{Z}_{\geq 0} : a = \sum_{k=0}^{\jmath} a_k 3^k \text{ for } a_k \in \{0,1\},\ j \in \mathbb{Z}_{\geq 0}\}$, $|S_{m,n}| = 3^{n-1}2^{m-n}$.

Proof. As before, we let $a = \sum_{i=0}^{N-1} a_i 3^i$ and $b = \sum_{i=0}^{N-1} b_i 3^i$ with $a_i, b_i \in \{0,1\}$, so $c = a-b = \sum_{i=0}^{N-1} c_i 3^i$ where $c_i \in \{0,1,2\}$. We first consider the four cases:

When $a_i = 1$ and $b_i = 1$, we have $a_i - b_i = 0$.

When $a_i = 0$ and $b_i = 1$, we have $a_i - b_i = -1$.

When $a_i = 1$ and $b_i = 0$, we have $a_i - b_i = 1$.

When $a_i = 0$ and $b_i = 0$, we have $a_i - b_i = 0$.

If $m = n$, we assume $a_m = b_n = 1$. Then $i = 0, 1, ..., m-1$ has to be case 3 or 4, so there are 2^m possibilities.

Now let $m > n$. In case 2, the -1 is equivalent to a $c_i = 2$ and subtracting 1 from c_{i+1}. This cannot happen because we cannot have any $c_i = 2$. Thus, $i = 0, 1, ..., n-2$ can be cases 1, 3, or 4, so there are 3^{n-1} ways to assign a case to $i = 0, 1, ..., n-2$. We can assume that b_n is 1, so $i = n-1$ has to be case 1. Then for $i = n, ..., m-1$, a_i can equal 0 or 1 for each i, so there are 2^{m-n} ways to assign those. Thus, $S_{m,n} = 3^{n-1} \cdot 2^{m-n}$. □

We now consider the case where $P(a, b) = a - b$ and A is the set $\{1, 2\}$.

Theorem 3.2. *When* $S := \{(a, b) \in \mathcal{C}(A)_N \times \mathcal{C}(A)_N : \exists\ c \in \mathcal{C}(A)\ s.t.\ a - b = c\}$ *and* $\mathcal{C}(A) := \{a \in \mathbb{Z}_{\geq 0} : a = \sum_{k=0}^{j} a_k 3^k \text{ for } a_k \in \{1, 2\},\ j \in \mathbb{Z}_{\geq 0}\}$, $|S_{m,n}| = 2^{m-n}\left(\frac{3^n + 3}{4}\right)$.

Proof. If $m = n$, there are two ordered pairs in the solution set: $(\sum_{i=0}^{m} 2 \cdot 3^i, \sum_{i=0}^{m} 3^i)$ and $(\sum_{i=0}^{m} 3^i, \sum_{i=0}^{m} 2 \cdot 3^i)$.

Now let $m > n$. As before, we let $a = \sum_{i=0}^{N-1} a_i 3^i$ and $b = \sum_{i=0}^{N-1} b_i 3^i$ with $a_i, b_i \in \{1, 2\}$, so $c = a - b = \sum_{i=0}^{N-1} c_i 3^i$ where $c_i \in \{0, 1, 2\}$. We first consider the four cases:

When $a_i = 2$ and $b_i = 2$, we have $a_i - b_i = 0$.

When $a_i = 1$ and $b_i = 2$, we have $a_i - b_i = -1$.

When $a_i = 2$ and $b_i = 1$, we have $a_i - b_i = 1$.

When $a_i = 1$ and $b_i = 1$, we have $a_i - b_i = 0$.

By a similar recursive method as in the proof of Theorem 2.2, there are $\frac{3^n + 1}{2}$ possibilities for $i = 0, 1, ..., n-1$. There is exactly one way that there are no regroups in this range. For this, there are 2^{m-n} possibilities for $i = n, n+1, ..., m-1$. There are $\frac{3^n + 1}{2} - 1 = \frac{3^n - 1}{2}$ ways that there is a regroup in this range. For this, a_i has to be 2. There are 2^{m-n-1} possibilities for $i = n+1, ..., m-1$. Thus, $|S_{m,n}| = \frac{3^n - 1}{2} \cdot 2^{m-n-1} + 2^{m-n} = 2^{m-n}\left(\frac{3^n+3}{4}\right)$. $\square$

4 $|S_N|$ for $P(a,b) = ab$

We wrote a Python program to compute the cardinalities of S_N for $N = 1, 2, ..., 12$. The computed $|S_N|$ for $A = \{0, 1\}$ are 4, 15, 52, 161, 472, 1281, 3346, 8365, 20370, 47731, 109276, 243987.

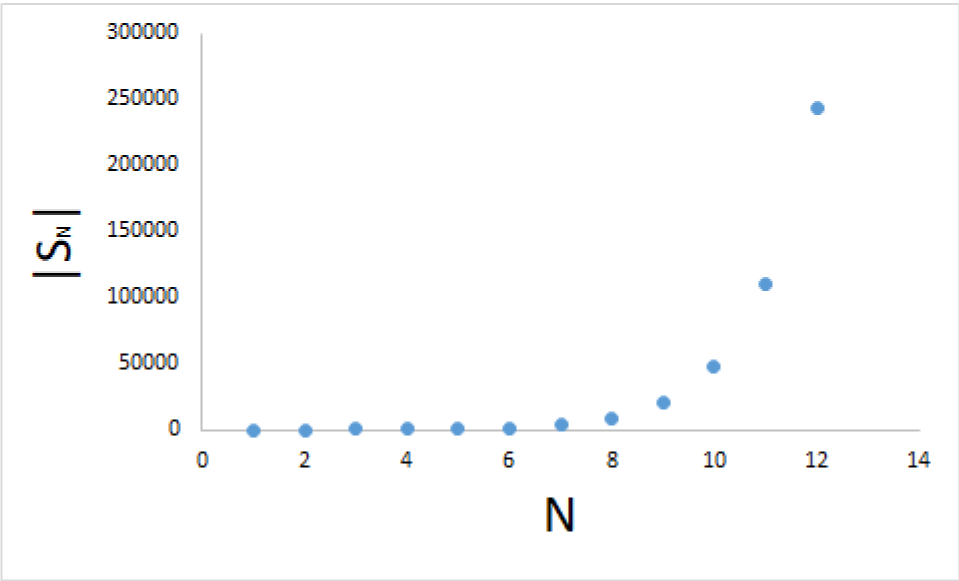

Figure 2: $|S_N|$ for $A = \{0, 1\}$

5 Necessary condition for good $\alpha < 2$ in $u = x^\alpha y$

The second problem is based on a theorem on products in the Cantor set. We define a value of α to be good if $f(C \times C) = [0,1]$ when $f(x,y) = x^\alpha y$. Thus, a value of α is bad if $f(C \times C)$ skips an interval in $[0,1]$.

Theorem 5.1 (Athreya, Reznick, Tyson [2]). *Every element u of $[0,1]$ can be written in the* *form $u = x^2y$, where x, y are elements of the Cantor set C.*

It was remarked in [2] that to generalize this, we consider $f(x, y) := x^a y^b$. Because $u = x^a y^b$ if and only if $u^{1/a} = xy^{b/a}$, for $u \in (0, 1)$, it suffices to consider $a = 1$. We first compute $C_1 \times C_1$ where $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. We get $\left[0, \frac{1}{3}\right] \cup \left[\left(\frac{2}{3}\right)^{1+\alpha}, 1\right]$. If $\alpha > 1$ and $\left(\frac{2}{3}\right)^{1+\alpha} > \frac{1}{3}$, we see that $f(C_1^2)$ misses an interval in $[0, 1]$. Thus, $\alpha > \frac{\log 2}{\log 3/2} \approx 1.0795$.

Using the same method, we compute $C_2 \times C_2$ to see if we can improve the lower bound on α. The resulting union of intervals appears to cover all of $[0, 1]$ for $\frac{\log 2}{\log 3/2} \leq \alpha \leq 2$. We thus conjecture that $\frac{\log 2}{\log 3/2}$ is the smallest good α.

6 Necessary and sufficient condition for good α when $\alpha < 2$

Lemma 6.1. *To represent the removal of the middle third, let $I = [a, a + 3t]$ and $\ddot{I} = [a, a+t] \cup [a+2t, a+3t]$. Suppose $F : \mathbb{R}^m \to \mathbb{R}$ is continuous, and suppose that for every choice of disjoint or identical subintervals $I_k \subset [a, b]$ of common length, $F(I_1, ..., I_m) = F(\ddot{I}_1, ..., \ddot{I}_m)$. Then $F(C_{a,b}^m) = F([a, b]^m)$.*

This lemma was proved by Athreya, Reznick, and Tyson in 2017 [2].

Theorem 6.2. *Let $\widetilde{C} = C \cap \left[\frac{2}{3}, 1\right]$. Then $f(\widetilde{C}^2) = \left[\left(\frac{2}{3}\right)^{\alpha+1}, 1\right]$ when $1.5 \leq \alpha \leq 2$.*

Proof. Let $I = [a, a + 3t]$ and $J = [b, b + 3t]$ be in $\left[\frac{2}{3}, 1\right]$. We first must show that $f(I, J) = f(\ddot{I}, \ddot{J})$ when $f(x, y) = x^\alpha y$. We begin by defining the intervals

$$[u_1, v_1] := [a^\alpha b, (a + t)^\alpha (b + t)];$$

$$[u_2, v_2] := [a^\alpha (b + 2t), (a + t)^\alpha (b + 3t)];$$

$$[u_3, v_3] := [(a + 2t)^\alpha b, (a + 3t)^\alpha (b + t)];$$

$$[u_4, v_4] := [(a+2t)^\alpha(b+2t), (a+3t)^\alpha(b+3t)].$$

We see that $f(I,J) = [u_1, v_4]$ and $f(\ddot{I}, \ddot{J}) = [u_1, v_1] \cup [u_2, v_2] \cup [u_3, v_3] \cup [u_4, v_4]$. Note that $u_1 < u_2$, $v_1 < v_2$, and $u_3 < u_4$, $v_3 < v_4$, so $[u_1, v_1] \cup [u_2, v_2] = [u_1, v_2]$ and $[u_3, v_3] \cup [u_4, v_4] = [u_3, v_4]$ if $v_1 > u_2$ and $v_3 > u_4$. And because $u_1 < u_3$ and $v_2 < v_4$, if $v_2 > u_3$, then $[u_1, v_2] \cup [u_3, v_4] = [u_1, v_4]$. Therefore, it suffices to show that $v_1 > u_2$, $v_3 > u_4$, and $v_2 > u_3$.

We first show that $v_3 > u_4$.

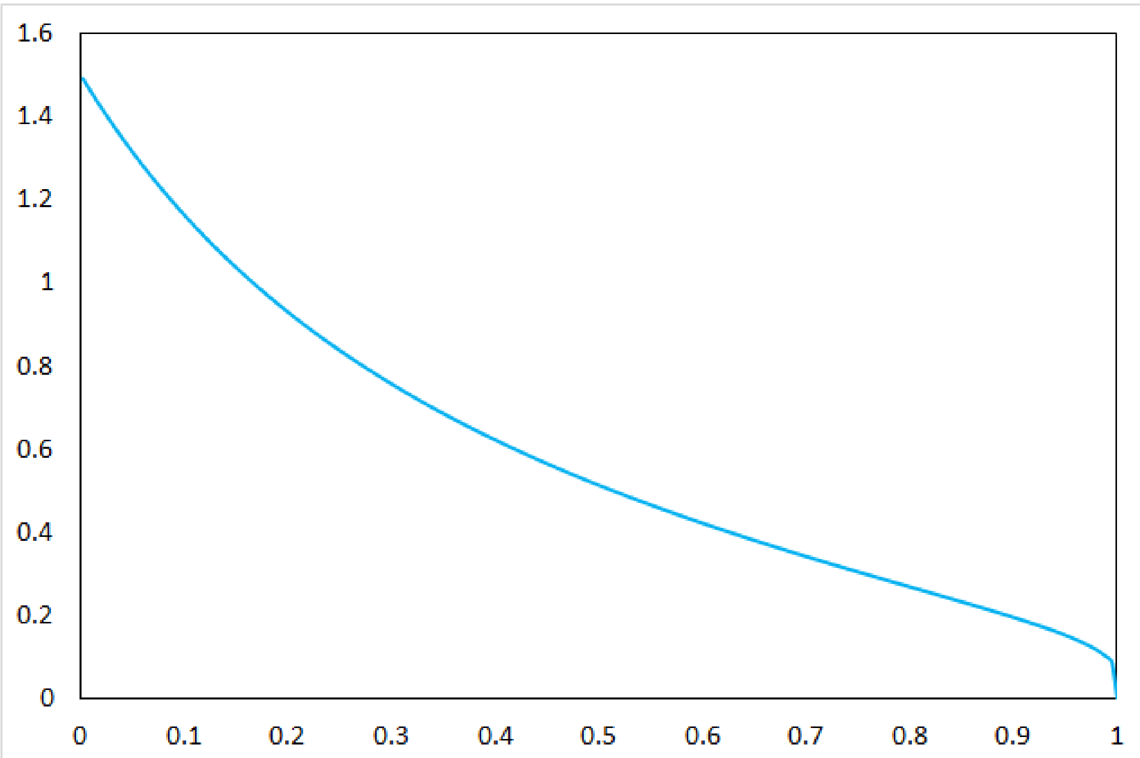

Figure 3: Graph of $y = \dfrac{\log\left(1 + \dfrac{t}{\frac{2}{3} + t}\right)}{\log\left(1 + \dfrac{t}{1-t}\right)}$

The maximum of the function $y = \dfrac{\log\left(1 + \dfrac{t}{\frac{2}{3} + t}\right)}{\log\left(1 + \dfrac{t}{1-t}\right)}$ in the range $0 \le t \le \dfrac{1}{9}$ is achieved as t approaches 0, so

$$\max_{0\le t\le \frac{1}{9}} \frac{\log\left(1+\dfrac{t}{\dfrac{2}{3}+t}\right)}{\log\left(1+\dfrac{t}{1-t}\right)} = \lim_{t\to 0} \frac{\log\left(1+\dfrac{t}{\dfrac{2}{3}+t}\right)}{\log\left(1+\dfrac{t}{1-t}\right)} = 1.5.$$

The limit equals 1.5, so α is always greater than or equal to the maximum of the function $y(x)$ over $0 \le t \le \dfrac{1}{9}$. Hence,

$$\left(1+\frac{t}{1-t}\right)^{\alpha} \ge 1+\frac{t}{\frac{2}{3}+t} \quad \forall\, 0\le t\le \frac{1}{9},$$

so

$$\left(1+\frac{1}{\dfrac{1-3t}{t}+2}\right)^{\alpha} \ge 1+\frac{1}{\frac{\frac{2}{3}}{t}+1}.$$

Because $[a, a+3t] \subset \left[\dfrac{2}{3}, 1\right]$, $a \ge \dfrac{2}{3}$ and $a+3t \le 1$, so $\dfrac{2}{3} \le a \le 1-3t$. Similarly, $\dfrac{2}{3} \le b \le 1-3t$. Because $a \le 1-3t$ and $b \ge \dfrac{2}{3}$,

$$\left(1+\frac{1}{\frac{a}{t}+2}\right)^{\alpha} \ge \left(1+\frac{1}{\frac{1-3t}{t}+2}\right)^{\alpha} \ge 1+\frac{1}{\frac{\frac{2}{3}}{t}+1} \ge 1+\frac{1}{\frac{b}{t}+1}.$$

Thus,

$$\left(1+\frac{1}{\frac{a}{t}+2}\right)^{\alpha} \ge 1+\frac{1}{\frac{b}{t}+1},$$

so

$$\left(\frac{a+3t}{a+2t}\right)^{\alpha} \ge \frac{b+2t}{b+t}.$$

Therefore,

$$(a+3t)^{\alpha}(b+t) \ge (a+2t)^{\alpha}(b+2t),$$

so $v_3 > u_4$.

We claim that $v_3 > u_4$ implies $v_1 > u_2$. By definition, $v_3 > u_4$ if and only if $\dfrac{t}{a+2t} > \sqrt[\alpha]{\dfrac{b+2t}{b+t}} - 1$, and $v_1 > u_2$ if and only if $\dfrac{t}{a} > \sqrt[\alpha]{\dfrac{b+2t}{b+t}} - 1$. Because a and t are both positive,

$\dfrac{t}{a}$ is always greater than $\dfrac{t}{a+2t}$, so $v_3 > u_4$ implies $v_1 > u_2$. It thus remains to show that $v_2 > u_3$.

We claim that if $\alpha \leq 2$, then $v_2 > u_3$. By definition, $v_2 > u_3$ if and only if $\dfrac{b+3t}{b} > \left(\dfrac{a+2t}{a+t}\right)^{\alpha}$. Athreya, Reznick, and Tyson proved that this is true for $\alpha = 2$. When $\alpha \leq 2$, $\left(\dfrac{a+2t}{a+t}\right)^{\alpha} \leq \left(\dfrac{a+2t}{a+t}\right)^{2}$ because $\dfrac{a+2t}{a+t} > 1$. Thus, $\dfrac{b+3t}{b} > \left(\dfrac{a+2t}{a+t}\right)^{\alpha}$ for $\alpha \leq 2$, so $v_2 > u_3$.

Therefore, $f(I,J) = f(\ddot{I},\ddot{J})$. By Lemma 6.1, $f(\widetilde{C}^2) = [(\frac{2}{3})^{\alpha+1}, 1]$. □

Theorem 6.3. *If* $\dfrac{\log 2}{\log 3/2} \leq \alpha \leq 2$, *then* $f(C^2) = [0,1]$.

Proof. Suppose $u \in [0,1]$. If $u = 0$, then $u = 0^2 \cdot 0$. If $u > 0$, then there exists a unique integer $r \geq 0$ so that $u = 3^{-r}v$, where $v \in (\frac{1}{3}, 1]$. When $\alpha = \dfrac{\log 2}{\log 3/2}$, $\left(\dfrac{2}{3}\right)^{\alpha+1} = \dfrac{1}{3}$, so when $\dfrac{\log 2}{\log 3/2} \leq \alpha \leq 2$, $\left(\dfrac{2}{3}\right)^{\alpha+1} \leq \dfrac{1}{3}$. By Theorem 6.2, $f(\widetilde{C}^2) = \left[\left(\dfrac{2}{3}\right)^{\alpha+1}, 1\right]$ if $1.5 \leq \alpha \leq 2$, so $f(\widetilde{C}^2) \supset \left[\dfrac{1}{3}, 1\right]$. Hence, $v = x^{\alpha}y$ for $x, y \in \widetilde{C} \subset C$, and since $x, 3^{-r}y \in C$, $u = x^{\alpha}(3^{-r}y)$ is the desired representation. □

7 Necessary condition for good α when $\alpha > 2$

Using a MATLAB program, we compute $f(C_2 \times C_2)$ directly for different values of $\alpha > 2$. We find that an interval in $[0,1]$ is skipped for α in the range from about 3.04 to 5.88. Because $f(C_n \times C_n) \supset f(C \times C)$ for all positive integers n, we know that $f(C \times C)$ also skips some interval in $[0,1]$ for that α. Examining the intervals, we determine the exact bounds of bad α from $f(C_2 \times C_2)$ to be $\dfrac{\log 3/2}{\log 8/7} < \alpha < \dfrac{\log 1/2}{\log 8/9}$.

Theorem 7.1. *If* $\dfrac{\log 3/2}{\log 8/7} < \alpha < \dfrac{\log 1/2}{\log 8/9}$, *then there exists some element* u *in* $[0,1]$ *that cannot be expressed in the form* $x^{\alpha}y$ *for some* x, y *in the Cantor set* C.

Proof. Expand $f(C_2 \times C_2)$ to get

$$\left[0, \left(\frac{1}{9}\right)^\alpha \left(\frac{1}{9}\right)\right] \cup \left[0, \left(\frac{1}{9}\right)^\alpha \left(\frac{1}{3}\right)\right] \cup \left[0, \left(\frac{1}{9}\right)^\alpha \left(\frac{7}{9}\right)\right] \cup \left[0, \left(\frac{1}{9}\right)^\alpha (1)\right]$$
$$\cup \left[0, \left(\frac{1}{3}\right)^\alpha \left(\frac{1}{9}\right)\right] \cup \left[\left(\frac{2}{9}\right)^\alpha \left(\frac{2}{9}\right), \left(\frac{1}{3}\right)^\alpha \left(\frac{1}{3}\right)\right] \cup \left[\left(\frac{2}{9}\right)^\alpha \left(\frac{2}{3}\right), \left(\frac{1}{3}\right)^\alpha \left(\frac{7}{9}\right)\right] \cup \left[\left(\frac{2}{9}\right)^\alpha \left(\frac{8}{9}\right), \left(\frac{1}{3}\right)^\alpha (1)\right]$$
$$\cup \left[0, \left(\frac{7}{9}\right)^\alpha \left(\frac{1}{9}\right)\right] \cup \left[\left(\frac{2}{3}\right)^\alpha \left(\frac{2}{9}\right), \left(\frac{7}{9}\right)^\alpha \left(\frac{1}{3}\right)\right] \cup \left[\left(\frac{2}{3}\right)^\alpha \left(\frac{2}{3}\right), \left(\frac{7}{9}\right)^\alpha \left(\frac{7}{9}\right)\right] \cup \left[\left(\frac{2}{3}\right)^\alpha \left(\frac{8}{9}\right), \left(\frac{7}{9}\right)^\alpha (1)\right]$$
$$\cup \left[0, \frac{1}{9}\right] \cup \left[\left(\frac{8}{9}\right)^\alpha \left(\frac{2}{9}\right), \frac{1}{3}\right] \cup \left[\left(\frac{8}{9}\right)^\alpha \left(\frac{2}{3}\right), \frac{7}{9}\right] \cup \left[\left(\frac{8}{9}\right)^\alpha \left(\frac{8}{9}\right), 1\right].$$

For $\alpha > 2$, this union can be condensed to

$$\left[0, \frac{1}{9}\right] \cup \left[\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{\alpha+1}, \left(\frac{3}{7}\right)\left(\frac{7}{9}\right)^{\alpha+1}\right] \cup \left[\left(\frac{2}{3}\right)^{\alpha+1}, \left(\frac{9}{7}\right)\left(\frac{7}{9}\right)^{\alpha+1}\right]$$
$$\cup \left[\left(\frac{1}{4}\right)\left(\frac{8}{9}\right)^{\alpha+1}, \frac{1}{3}\right] \cup \left[\left(\frac{3}{4}\right)\left(\frac{8}{9}\right)^{\alpha+1}, 1\right].$$

We must show that this union of intervals skips some element in $[0, 1]$ when α is in the range $\frac{\log 3/2}{\log 8/7} < \alpha < \frac{\log 1/2}{\log 8/9}$. To show this, we consider these four intervals:

If $3.04 \approx \frac{\log 3/2}{\log 8/7} < \alpha < \frac{\log 1/3}{\log 7/9} \approx 4.37$, we can combine the first two intervals into $\left[0, \left(\frac{3}{7}\right)\left(\frac{7}{9}\right)^{\alpha+1}\right]$.

If $4.37 \approx \frac{\log 1/3}{\log 7/9} \leq \alpha < \frac{\log 6}{\log 3/2} \approx 4.42$, the second interval is completely included in the first, so we have $\left[0, \frac{1}{9}\right]$.

If $4.42 \approx \frac{\log 6}{\log 3/2} \leq \alpha < \frac{\log 1/2}{\log 6/7} \approx 4.50$, we can combine the first two intervals into $\left[0, \left(\frac{3}{7}\right)\left(\frac{7}{9}\right)^{\alpha+1}\right]$.

If $4.50 \approx \frac{\log 1/2}{\log 6/7} \leq \alpha < \frac{\log 1/2}{\log 8/9} \approx 5.88$, we can combine the first three intervals into $\left[0, \left(\frac{9}{7}\right)\left(\frac{7}{9}\right)^{\alpha+1}\right]$.

For each of the four cases, we check that the upper bound of the combined interval is less than each of the lower bounds for the remaining intervals. Therefore, the union excludes some interval from $[0,1]$ for all α in the range $\frac{\log 3/2}{\log 8/7} < \alpha < \frac{\log 1/2}{\log 8/9}$. □

Since we have proven the sharp range of bad α that we conjectured from $f(C_2 \times C_2)$, we now compute $f(C_3 \times C_3)$ with MATLAB. The results, as seen in Figure 4, give the approximate intervals of bad α.

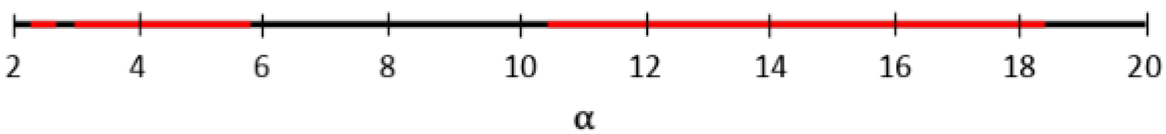

Figure 4: Bad α for $2 < \alpha < 20$ (in red)

From the MATLAB program, it appears that $f(C_n \times C_n)$ covers all of $[0,1]$ for all sufficiently large α. The intervals of bad α for $f(C_3 \times C_3)$ are distributed throughout $[2,20]$, but it is difficult to compute the bounds precisely.

8 Conclusion

In this paper, we considered two problems. The first examined $|S_N|$ for the three polynomial equations $c = a + b$, $c = a - b$, and $c = ab$ on the truncated middle-thirds Cantor set $\mathcal{C}(A)$ for $A = \{0,1\}$ and $A = \{1,2\}$. We found formulas for the first two polynomial equations and computed $|S_N|$ for $N = 1, 2, ..., 12$ for the last equation. These results are important because they provide information towards the Minkowski dimension of the solution set of the continuous version of this problem. In the second problem, we focused on a theorem proved in 2017 by Athreya, Reznick, and Tyson, which states $f(C \times C) = [0,1]$ when $f = x^2y$. Generalizing their proof to $x^\alpha y$, we found that this result holds for all α in the range $\frac{\log 2}{\log 3/2} \leq \alpha \leq 2$. We then directly computed $f(C_2 \times C_2)$ to find a range of bad α: $\frac{\log 3/2}{\log 8/7} < \alpha < \frac{\log 1/2}{\log 8/9}$.

Future directions for the first problem include exploring $|S_N|$ for $P(a, b) = ab$ when A is $\{0, 2\}$ and $\{1, 2\}$. For the second problem, we can use $f(C_n \times C_n)$ for $n > 2$ to find more necessary conditions for good $\alpha > 2$ and evaluate what happens for very large α.

9 Acknowledgments

I would first like to thank Yuqiu Fu, a graduate student in the MIT Math Department, for mentoring me and giving me valuable insight throughout the entire research process. I would like to thank Prof. Larry Guth of the MIT Math Department for suggesting the original problem. Further thanks go to our head mentor Dr. Tanya Khovanova for giving me potential directions of exploration and general guidance on my project and paper. Many thanks to Dr. John Rickert, my tutor, for his advice on my presentation and paper along the way. Miles Edwards and Sean Elliot also provided helpful feedback on my paper. I also want to acknowledge the Research Science Institute, the Center for Excellence in Education, and the Massachusetts Institute of Technology for giving me the opportunity to research this topic. I am very grateful to my sponsors, the Department of Defense, Ms. Catherine Y. Li, and Mr. Kalani Oshiro for their support that made my experience possible.

References

[1] K. J. Falconer. On the minkowski measurability of fractals. *Proceedings of the American Mathematical Society*, 123(4):1115–1124, 1995.

[2] J. S. Athreya, B. Reznick, and J. T. Tyson. Cantor set arithmetic. *The American Mathematical Monthly*, 126(1):4–17, 2019.