Generalization of Bridge Length to other Cartan-Killing Forms

Roshan Warman

Abstract

In 2006, Postnikov derived the combinatorics of the nonnegative grassmannian in his famously dubbed paper “Total Positivity.” In it, Postnikov was able to describe the bridge length (similar to the notion of length in a permutation) of A-type Weyl groups, which had the capacity to fully describe the dimensional theory of the nonnegative Grassmannian. However, these results have been limited to characterizing only elements of type A Weyl groups. We present similar characterizations for the elements of the remaining types of Weyl groups ($B_n, C_n, D_n, E_{6-8}, G_2, F_4$) and explicit bounds for their bridge lengths.

1 Introduction

Since Postnikov's seminal paper [1] detailing the total positivity of the Grassmannian, the field of algebraic combinatorics has seen a burgeoning interest in the combinatorial connections with inherently topological objects. Initially, Hermann Grassmann motivated the Grassmannian to efficiently describe continuity of a linear space by looking at it as a topology, and the advent of algebraic geometry led to the study of the Grassmannian in a lens unrelated to combinatorics, let alone permutations.

The Grassmannian is a way to look at a single fibre of a root subspace with continuity and homotopy. Specifically, it is a set of k dimensional subspaces of an n dimensional vector space, V. In an effort to efficiently parametrize the Grassmannian, one injects $G(k, n) \to \mathbb{P}\big(\bigwedge^k V\big)$, which is the projectivization of the exterior power of V, referred to as the Plücker Coordinates. In this case, the nonnegative Grassmannian is exactly the restriction to an image greater than of all k-multivectors with positive divisors.

In his proof, Postnikov [1] showed that these nonnegative Grassmannians can generate graphical networks. Briefly consider this set of networks generated by a Grassmannian with sources, S, and sinks T and order $|V|$. Call this set $\mathrm{Net}(k, n)$, where $|S| = k$ and $|V| = n$, and let the weight of an edge be $w(e)$. Now, consider all paths in this network, and define the boundary measurement, $M_{ij} = \sum_{p:S\to T} \prod_{e\in P} w(e)$, where $\{M_{ij}\}$ will represent a matrix.

The Lindström–Viennot lemma [2] was able to express these determinants as $\det(M) = \sum_{\text{disjoint } \mathcal{P}} \prod_{i=1}^{n} \prod_{e\in p_i} w(e)$, where the criteria for disjoint paths prevents considering two minors different if they are equivalent by action on $GL(n)$.

In combination, one can completely genearte the Grassmannian from a combinatorial point of view. The combinatorics were elucidated when Postnikov initiated the study of these networks embedded in a disk known as *plabic* graphs (Figure 1).

A Plabic graph is planar, and by convention, bi-colored, meaning its vertices are 2-colored

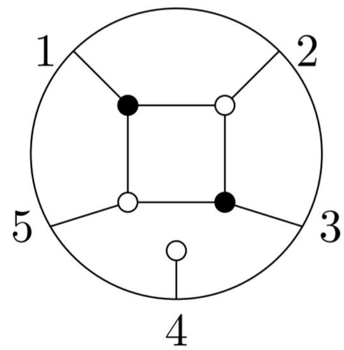

Figure 1: Plabic graph with the vertices colored to indicate right turns on black and left turns on white.

to indicate direction. A subclass of these plabic graphs are known as bridge graphs (Figure 2), which are in bijection with permutations such that each edge in the graph switches the two elements at the edge's endpoints much like a transposition would. These permutations describe type A Weyl group elements. In this paper, we expand these known graphical connections to the other classical types of Weyl groups and investigate a property of these Weyl groups that describe the dimension of the Grassmannian and many other quantities: the bridge length. Postnikov [1] was able to find an expression to find the bridge length of type A Weyl group elements and relate these lengths to the positive Grassmannian.

However, for all other Cartan-Killing types, the elements are more abstract than permutations, and as a result much less concretely grounded in the networks generated by the nonnegative Grassmannian. In this paper, we look at all other classical types of Weyl groups (B_n, C_n, D_n) as well as some of the exceptional cases to determine explicitly, their bridge lengths and connection with the positive Grassmannian.[1]

[1]Throughout this paper, we will mention the word "root." Though it may seem different, root and transposition convey the same idea, but in some cases, we defer to root as to clarify the idea that transpositions only apply to permutations (and not to the more abstract elements of Weyl groups for types other than A_n) is false.

2 Preliminaries

Weyl Groups

Fix a finite dimensional $\mathbb{Q}$-vector space E equipped with a symmetric bilinear form $(,)$. For a nonzero vector $v \in E$, let r_v denote the reflection about the hyperplane orthogonal to v. Given a root system R, its *Weyl group* is defined to be the subgroup of $\mathrm{GL}(E)$ generated by reflections $r_\alpha : E \to E$ with $\alpha \in R$. Fix a generic linear form $f : E \to \mathbb{Q}$. Then $\{\alpha \in R : f(\alpha) > 0\}$ is the set of positive roots, R^+, and $\Pi \subset R^+$ are the simple roots, such that all the positive roots can be written as a nonnegative linear combination of simple roots. The length, $\ell(w)$ of an element of a Weyl group is the minimum number of reflections of a root over a hyperplane required to create that element. Note that because the generators have finite orbits, the Weyl group is finite. The area between two roots of a root system is known as a Weyl chamber and for type A these Weyl chambers have $n!$ isometries which makes $W_a \cong S_n$.

Definition 2.1 (Root poset)**.** For $\alpha, \beta \in R^+$, define $\alpha \leq \beta$ if $\beta - \alpha$ can be written as a nonnegative linear combination of simple roots. This gives a partial order on all positive roots, which form the *root poset*.

We now introduce the most important definitions in this paper. Also, for an unspecified type of Weyl group, we shall write W_t.

Definition 2.2 (Bridge decomposition)**.** Let $w \in W_t$, be an element of a certain type Weyl group. Then $w = r_{s_1} r_{s_2} \dots r_{s_\ell}$ is called the bridge decomposition if in its root poset $s_1, \dots, s_\ell \in R^+$ and for $i < j$, either $s_i \leq s_j$ (non-decreasing), or $(s_i) \cap (s_j) = \emptyset$ (disjoint).

The non-decreasing and disjoint criteria will be referred to as the bridge conditions.

Definition 2.3 (Bridge length)**.** For $w \in W_t$, define its *bridge length* $\ell_t(w)$ to be the smallest ℓ such that w can be written as $r_{s_1} \cdots r_{s_\ell}$ and forms a valid bridge decomposition.

The author should refer to Appendix A for the Hasse diagrams of the root posets for all classical types of Weyl groups. An element's bridge decomposition is diagrammatically represented with a bridge diagram, such as Figure 2. Each root corresponds to a horizontal bridge (edge between white and black vertex) in the decomposition, which moves downward until there are no more roots that satisfy the non-decreasing or disjoint property. For example, the first bridge in Figure 2 switches 2 and 3, and corresponds to the root $e_2 - e_3$. Further, note that by the restriction placed on by the bridge conditions, we see that once a root is used, every root after that must have a disjoint order or larger placement in the poset.

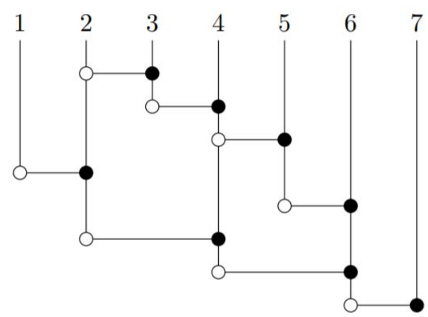

Figure 2: A bridge diagram (Bd_n) of a type A Weyl group element, specifically $1234567 \mapsto 3542671$. Counting the bridges, we see this element has bridge length 8.

Permutations and Circular Diagrams

Embedding the Bridge diagram of an element of W_a in D^2 creates a plabic graph [3]. The reader should realize that the planarity of the plabic graph is a result of the disjoint order property from the bridge condition, so no two bridges intersect anywhere except for endpoints. To help construct these bridge diagrams, we guide the construction with a *Circular Diagram.*

Given a $w \in W_a$ one can construct a circular diagram by drawing each of the n elements on the perimeter of a circle and drawing a directed chord from i to j if $w(i) = j$. For example

Figure 3 is the permutation 2413 where the direction of the arrow points $i \to w(i)$. In the case where $w(i) = i$, we refer it as a self-loop.

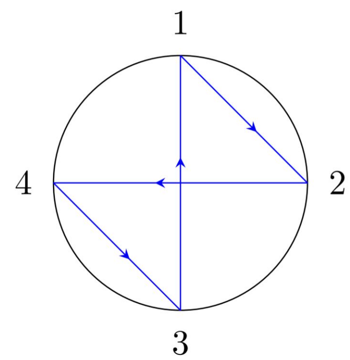

Figure 3: The permutation 2413 of CB_4

Postnikov described a procedure to systematically generate these bridge decompositions from the circular diagram [1]. It entails making crossings, which are chords that intersect, into alignments, which are chords that don't intersect and point in the same direction. Formally, the uncrossing is a root, $e_i - e_j$, that takes $i \mapsto w(j)$ and $j \mapsto w(i)$. From Figure 3, $\overline{12}$ and $\overline{43}$ make an alignment, $\overline{31}$ and $\overline{12}$ form a degenerate crossing and $\overline{24}$ and $\overline{31}$ form a crossing. This procedure of exchanging the elements so that a crossing becomes an alignment forms a bridge in the diagram corresponding to a root in the root poset for type A. The process is depicted in Figure 4.

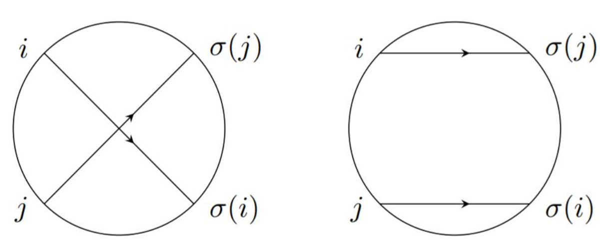

Figure 4: A crossing turned into an alignment through a transposition of i and j. This would represent a bridge in the bridge decomposition of the permutation.

Finding the bridge length with the circular diagram amounts to using appropriate roots in the root poset for type A satisfying the bridge conditions of w until the circular diagram has only self-loops ($w(i) = i$) referring back to itself (the identity permutation). These loops become fixed points and must have a specified direction, called decoration, as the direction of the loops can affect the number of alignments. One can alternatively look at a loop as a degenerate chord that begins and ends at the same point to look at alignments more easily. Additionally, note that a counterclockwise (ccw) and clockwise (cw) decoration makes an alignment.

Total Positivity of the Grassmannian

Let $Gr(k, n)$ be the grassmannian of k-dimensional subspaces of an n-dimensiona space. Its elements are rank k homomorphisms modulo action on $GL(k)$, which makes $Gr(k, n) = GL_k/\text{Mat}_{kn}^{\times}$, where $\text{Mat}_n^{\times}$ is the space of $k \times n$ matrices of rank k. This leaves the $\dim Gr(k, n) = k(n - k)$.

Combinatorially, the basis vectors form the columns of an element as a matrix, we define the matroid associated with the grassmannian, $\mathcal{M} \subseteq \binom{[n]}{k}$. For a given $\mathcal{I} \in \mathcal{M}$ and for $V \in Gr(k, n)$, A is the associated matrix with columns as the basis vectors of the matroid generated by $\mathcal{M}_V$ such that $\Delta_{\mathcal{I}}(A) \neq 0$, where $\Delta_{\mathcal{I}}(A)$ is the minor with column indices specified by the elements in $\mathcal{I}$. Formally, we can create a correspondence between spaces of the Grassmannian and partitions of the matroid, such that:

$$S_{\mathcal{M}} = \{V \in Gr(k, n) : \mathcal{M}_V = \mathcal{M}\}. \quad (1)$$

Flag Varieties of a Combinatorial Object

Let $Gr(1, \mathbb{R}^n)$ be the space of lines passing through the origin in n-dimensional Euclidean space. Suppose we have a set of flags $\mathcal{F}(\mathbb{R}^n)$, $0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{R}^n$. For a given V_p, we can look at the basis vectors of a subspace and the position of the pivots of the basis creates a unique permutation $w \in S_{\dim(V_p)} \cong W_a$. Each of these permutations forms a single cell structure, known as the Schubert cell, Ω_w. These Schubert cells are subcomplexes of the Grassmannian and are locally homeomorphic to $\mathbb{R}^{\ell_A(w)}$ (recall $\ell_a(w)$ is the bridge length of $w \in W_a \cong S_n$ [1]. This decomposition results in the following: $Gr(k,n) = \bigcup_{w \in S_{\dim(V_p)}} \Omega_w \cong \bigcup_{w \in S_{\dim(V_p)}} \mathbb{R}^{\ell_A(w)}$. The Grassmannian is also a variety structure since it is the intersection of multiple projective hypersurfaces (through the Plücker embedding). In fact, this is also apparent from the minors of an element.

Definition 2.4. (Positive Grassmannian) The Positive Grassmannian, $Gr_p(k,n)$, is the Grassmannian whose entire image under the Plücker embedding is greater than 0.

Currently, these positive Grassmannians are only known to be in equivalence with the bridge length of type A Weyl group elements, so an aim of this project is to find their relationships with other types of Weyl groups.

Previous Work

Most combinatorial work with the Grassmannian is due to Postnikov. We summarize the relevant results below. Note that $\text{codim} S^p_{\mathcal{M}} = \dim(Gr_p(k,n)) - \dim(S^p_{\mathcal{M}})$, and $S^p_{\mathcal{M}} = S_{\mathcal{M}} \cap Gr_p(k,n)$, so $\dim S^p_{\mathcal{M}} = \dim Gr_p(k,n) - \text{codim} S^p_{\mathcal{M}}$. The analog of these dimensions to the bridge length of permutations are represented by a flag within this stratification.

Theorem 2.5 (Postnikov)**.** *For a Weyl group element of type A,* $\dim S^p_{\mathcal{M}} = \ell_a(w) = k(n-k) - A(w)$*, where* $A(w)$ *is the number of alignments in the circular diagram of* w*, and* k *is* $|\{i \mid w^{-1}(i) < i\}|$.

Proof. Recall that the Grassmannian, $Gr(k,n)$ has dimension $k(n-k)$. Additionally, the Grassmannian is expressible as a decomposition of Schubert cells, which are in bijection with permutations. From this decomposition, we can generate an entire set of flags $\mathcal{F}(\mathbb{R}^n) = \bigsqcup_{w \in S_n} \Omega_w$. One can construct a bijection [1] between the permutations present in the symmetric group by constructing flags with bases given by the rows of the matrix equal to the

Schubert cell. By the decomposition presented in Section 1.3, we can reconstruct the Grassmannian, but this union still preserves the bijection between the cells and the stratifications so we are done. □

Example 2.6. In Figure 3, we see that $n = 4, k = 2$, and $A(w) = 1$. So through computation of Theorem 2.5, $\ell_A(2413) = 3$.

3 Results for types B and C

Weyl groups of type B and C

W_b is the group of all permutations w of $\{\pm 1, \ldots, \pm n\}$ such that $w(i) = -w(-i)$ for $1 \leq i \leq n$. From the Dynkin diagrams, the only difference between type B_n and C_n are the relative lengths of the roots, and by linearty of the reflection maps, they form the same group (i.e., $W_b \cong W_c$.) although their root posets are different.

A Bridge diagram for Types B, C, and D

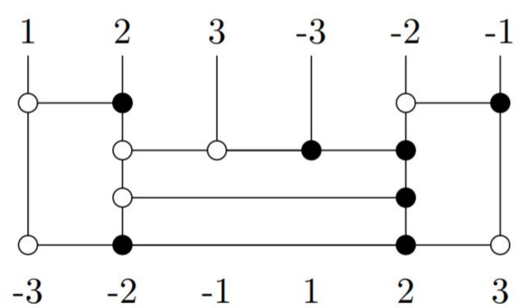

Figure 5: This is a permutation of the Weyl group type C on three elements. Note the similarities to that of W_a on $2n$ elements. From first to last, we see $e_1 - e_2$, $e_2 + e_3$, $2e_2$, and $e_1 + e_2$. We know this decomposition is of type C, since it is the only type to have a root that switches 2 and -2 ($2e_2$) that is larger than the rot that switches the 2 and -3.

We will use Figure 5 to describe how to construct such a bridge diagram. We introduce a new representation of a bridge decomposition that is analogous to that of type A. Order the elements from $1 \dots n$ and backward $-n \cdots -1$. A root of the form $e_i - e_j$ switches the i and j and $-i$ and $-j$. A root of the form $e_i + e_j$ switches i and $-j$ and $-i$ and j, and finally a root of the form $2e_i$ switches i and $-i$. Most importantly, however, is that this decomposition of roots follows the bridge decompositions. We shall prove this more rigorously next, though it is immediately clear if one changes $w \in W$ from the definition presented above (in the Preliminaries) to $w \in W \rtimes \mathbb{Z}_2$

Note that this decomposition follows the order ideal criteria by having no crossings from smaller bridges. The edges intersecting in the middle of 3 and -3 are demonstrative of the roots higher in the Hasse diagram for type A root poset. In the decomposition, that root is $e_2 + e_3$, but because of the $w(i) = -w(-i)$ condition, we construct this dual bridge. This idea of the dual bridge is what makes this diagram a valid representation.

Revisiting the Circular Diagrams

For other Weyl group types, there is a similar way to construct the bridge diagram from their circular diagrams. Laying $\{1, \dots, n\}$ on one half and with opposites directly across $\{-n, \dots, -1\}$, the alignments and crossings are then defined exactly as they are for type A. Therefore, the standard procedure Postnikov detailed to form the bridge diagrams applies exactly here with the additional roots presented by the higher structure of Weyl groups for type B and C.

Bounds of lengths

From our bridge decomposition construction, we can provide an exact bound that details the lengths.

Theorem 3.1. *Let $\ell_c'(w)$ be a length of a type C Weyl group element satisfying the bridge* *conditions but using only* $R^+/\{e_i + e_j\}$*. Then* $\ell'_c(w) = \frac{n^2 - A(w) + |\{i>0 \mid w(i)<0\}|}{2}$*, where* n *is the number of positive values in* w *and* $A(w)$ *is the number of alignments in the circular diagram of* w*.*

Proof. Consider an element of the Weyl group of type C. We will construct a bridge decomposition of this element that only uses root types $e_i - e_{i+1}$ and $2e_n$. Suppose we have a decomposition that is represented similarly as the bridge diagram of types B and C. Suppose that there were a root in this valid decomposition of the form $e_i - e_j$ that couldn't validly represent such a decomposition. Then there must have been a root above that is nondisjoint since the less than condition can't apply as the root appears after. This would imply that there was a fixed point which directly contradicts the existence of this invalid $e_i - e_j$ root, so these roots can validly form a decomposition. Similarly, for $2e_i$ roots, the elements must be nondisjoint, which once again contradicts the existence of these invalid $2e_i$ roots. Therefore, such a decomposition exists.

Recall that constructing the bridge decomposition from the circular diagram requires that in the end there are only self-loops that represent the identity permutation. This means for type C, all elements on the circular diagram are fixed. These fixed points must be decorated by specifying a direction, and because $w(i) = -w(-i)$, there must be an equal number of clockwise (cw) points as counterclockwise (ccw) points. Without loss of generality, suppose all positive n are cw and all negative n are ccw. This forms a total of n^2 alignments, which implies there must be exactly $n^2 - A(w)$ new alignments formed by the last transposition that makes id_n. Whenever we make a crossing an alignment through a root $e_i - e_j$, we must make sure that all elements in between are fixed points. Suppose not, then after the crossing is made into an alignment, for all $i \le k \le j$, there must eventually be a root that transposes k to a fixed point. But by our bridge conditions, this would mean However, this would mean there is a root, $e_k - e_q$ or $2e_k$ that are larger than $e_i - e_j$, or if the transposition is $(i, -i)$, $2e_i$, both of which contradict $i < k$, which implies that the roots are non-disjoint so all k

must be fixed.

Notice that this means for every simple uncrossing with $e_i - e_j$ the number of alignments increases by 2, one for (i, j) and one for $(-i, -j)$. Similarly, the number of alignments incrases by 1 using a $2e_i$ root. Call the set of all uncrossings by $e_i - e_j$, T and the set of all uncrossings by $2e_i$, $\tilde{T}$. In the reduced decomposition of $w = t_1 t_2 \ldots t_\ell$, there is no specified interval for T and $\tilde{T}$, so there is no way to partition the sizes easily. However, notice that a root of the form $(e_i - e_j)$ cannot move $i < 0 \mapsto i > 0$, so asymptotically, $\{i > 0 \mid w(i) < 0\} = o(|\tilde{T}|)$. Further, note that there can be a maximum increase in $A(w)$ by 1, from $2e_i$ and that $\ell = |T| + |\tilde{T}|$. Now, notice that $2|T| + |\tilde{T}| \geq n^2 - A(w)$ and $|\tilde{T}| = |\{i > 0 \mid w(i) < 0\}|$. Adding the two, we get $\ell_c(w) = \frac{n^2 - A(w) + |\{i>0 \mid w(i)<0\}|}{2}$.

However, since this was for a specific construction, we can't conclude that this length is definteively minimal. For that we need to prove a lower bound. We prove this by looking at a minimal and ideal bridge diagram representation. We can look at the number of alignments needed to be formed as once again $n^2 - A(w)$, but this time, we can't guarantee that the number of alignments increases by exactly 2 or 1 this time, since it could still be ideal to make an alignment a crossing to use a different root (effectively reducing the number of alignments). This implies that for our set T, the number of alignments increases by at most 2, and $\tilde{T}$, at most 1. This means that for a given $|T|$ times we apply a root of the form $e_i - e_j$, the alignments increase by at most $2|T|$ times and similarly $|\tilde{T}|$, exactly $|\tilde{T}|$ more alignments. Then we get $2|T| + |\tilde{T}| \geq n^2 - A(w)$. Now since we can't give a precise amount number of $|\tilde{T}|$, we must say that $|\tilde{T}| \geq |\{i > 0 \mid w(i) < 0\}|$. Adding the preceding inequalities gives our lower bound. □

One might wonder if because of the similar root posets of type B and C, there may be a relationship between their lengths, and that is precisely so:

Lemma 3.2. *Deleting any $r \in R^+$ maintains the hierarchy in a root system.*

Proof. Recall the generation of the Hasse Diagram from a root system. Since the first layers are all simple roots, their deletion must result in erasing everything above it, so the hierarchy is preserved.

Next, consider a positive root. Since this is a nonnegative linear combination of simple roots, look at an ordering of these elements, $a_1 \lessdot a_2 \lessdot a_3 \lessdot \cdots \lessdot a_n$, and consider a 3-progression, $a_1 \cdots \lessdot a_{m-1} \lessdot a_m \lessdot a_{m+1} \ldots a_n$ and delete a_m. The hierarchy doesn't break since as roots they must be expressible as *unique* combinations. Therefore, if $(a_{m+1}) \cap (a_m) = \emptyset$, then the a_{m+1} root can't be written as a unique combination without a_m. And if it is smaller in the poset, then if it wouldn't be possible to express a_{m+1} without appending a root of a_m, so a_m would have to exist. In the case we delete it, then a_m and $\{a_i | a_m \lessdot a_i\}$ must also be deleted as to not make a larger non disjoint exist in the root system, so the hierarchy is preserved. □

Corollary 3.3. *For $w \in W_{a-c}$, $\ell_b(w) = \ell_c(w)$ for $\Pi = \{e_i - e_j, 2e_i\}$.*

Proof. Let H_b and H_c denote the hasse diagrams for types B and C, respectively. Further, for $r \in R^+$ let $\{r\}$ denote the set of all roots of that form. From Lemma 3.2, deletion will construct a valid root poset, then $\overline{\bigcap}_{r \in \{e_i + e_j\}} H_b \cap H_c \equiv H_a \cup \{2e_i\}$. Since the e_i and $2e_i$ roots are in the same hiearchy after the deletion of the $\{e_i + e_j\}$ roots, distinguishing between them is only geometrically different. So we can conclude that with the same $w \in W_a, W_b, and W_c$, $\ell_b(w) = \ell_c(w)$. □

4 Equalities among types A, B, and C

Recall that the root systems of B and C are larger than that of A, so we can form an inclusion $H_a \hookrightarrow H_b$, which allows us to see for $w \in W_a,\ \ell_a(w) \geq \ell_b(w) = \ell_c(w)$. Note that here, even though w is a type A Weyl group, one can look at it's bridge length using roots of a type B or C root system. This could make the bridge length smaller, and consequently, the process of simplifying the circular diagrams more 'efficient,' in the sense it will require less roots. Of more interest is when $w \in W_a$ has equal lengths for all bridge types (i.e., $\ell_a(w) = \ell_b(w) = \ell_c(w)$).

Theorem 4.1. *In the reduced decomposition of the minimum length $w \in W_a$, the first root in the decomposition cannot be a simple root.*

Proof. Suppose for some $w \in W_a$, $\ell_a(w) > \ell_c(w)$. Let S be the set of all w such that $\ell_a(w) > \ell_b(w)$. By the Well-Ordering Principle, S must have a least element. Let r be the smallest element in S, by measure of its length. Decomposing r, we get $r = t_1 t_2 \dots t_{\ell_c(w)}$. Then $t_1^{-1} r = t_2 t3 \dots t_l$ and $\ell_c(t_1^{-1} r) = l - 1$. Furthermore, note that since $t^{-1} w \in S_n$, and $\ell_c(t_1^{-1} w) < \ell_c(r)$, by the minimality of r, $\ell_a(t_1^{-1} r) = \ell_c(t_1^{-1} r) = l - 1$. From the beginning, $\ell_a(r) \geq \ell_c(r) + 1 = l + 1$, which further implies that $\ell_a(r) - \ell(t^{-1} r) \geq 2$.

Now consider when t_1 is a simple transposition $(i, i+1)$ from its CB_n, we know that it either forms a crossing from an alignment or an alignment from a crossing. In the former, the length increases by 1 and in the latter it decreases by 1. In either, case $\pm 1 < 2$, so t_1 cannot be a simple transposition. □

Lemma 4.2. *For all four classical root systems A, B, C, and D the maximum increase in alignments from an $e_i - e_j$ root is $4n - 6$.*

Proof. Consider a circular diagram on $2n$ vertices. Since for B_n—D_n we must have $w(i) = -w(-i)$, we look at the case presented in Figure 6. Note that for any transposition, we must have $1 \leq i < j \leq n$. Note that a single transposition, $e_i - e_j$ can only form more than one alignment if there are chords that begin $i < k < j$ and end $w(i) < w(k) < w(j)$. This means that in order to form the most alignments, one must have the most chords lying in between the values i and j. To optimize this, we allow for the largest difference between i and j that is possible so that more ks can fit in between. From above, we see that this bound is when $i = 1$ and $j = n$. This accounts for a difference of $j - i - 1 = n - 1 - 1 = n - 2$.

This means there can be a maximum of $n-2$ chords lying in between that can form alignments when we apply $e_i - e_j$. Suppose all these chords take $k \mapsto -k$. After we apply $e_i - e_j$, two more horizontal chords are formed that take $i \mapsto -i$ and $j \mapsto -j$. Call them I and J, respectively. To enumerate, we note that there are $n-2$ alignments formed with I and likewise with J. In addition, we must account for the alignment created between I and J and the additional alignments created by $w(i) = -w(-i)$, which doubles the total. The total is then $2(1 + 2(n-2)) = 4n - 6$.

□

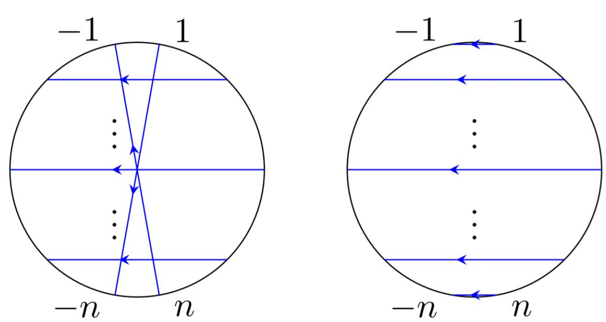

Figure 6: Left: Before transposition. Right: After transposition.

Corollary 4.3. $\ell_D(w) \leq \frac{n^2 - A(w)}{4n-6}$

Proof. This follows from Lemma 4.2. □

5 Cyclic Shifts for Extending the Grassmannian

Grassmannian construction for B and C

From the tight connection with decorated permutations above, it is natural to ask whether there is a nontrivial connection for types outside of type A. We sketch out a constructive method to arrive at exact bounds: Observe that the Weyl groups for type B and C are specific sub-cases of type A, with the added constraint that for a $w \in W_{B,C}$, $w(i) = -w(-i)$. Therefore, for a stratification of the positive Grassmannian, we can extend the results from A to types B and C. Consider a function, f in with a zero locus defined to be precisely the length of a Weyl group element of types B, C, and D. Recall from type A that $\dim S^p_{\mathcal{M}} = k(n-k) - A(w) = \dim Gr_p(k,n) - \text{codim} S^p_{\mathcal{M}}$. Since $S^p_{\mathcal{M}}(v.l.) \subset S^p_{\mathcal{M}} \subset Gr_p(k,n) \hookrightarrow \mathbb{RP}^{\binom{n}{k-1}}$, we can construct similar bijections between the stratification of the Grassmannian under zero locus of the between the lengths of other types of Weyl groups and the stratificatios.

Cyclic Shifts are invariant for the Grassmannian

We conjecture a natural bijection between $\dim S^p_{\mathcal{M}}$ under the vanishing locus and the bridge lengths. Less formally, we will attempt to elucidate its existence by showing that there is a similar intersection as present in type A between the Schubert cells and the Grassmannian.

Definition 5.1 (Cyclic shift)**.** Let $\tilde{c} : Gr_p(k,n) \to Gr_p(k,n)$ be the map such that $[v_1, \ldots, v_n] \mapsto [v_2, v_3, \ldots, v_n, (-1)^{k-1} v_1]$.

In [1], Postnikov was able to show that such a cyclic shift leaves the coordinates of the Grassmannian invariant, but more interestingly, we note that

Lemma 5.2. *The cyclic shift induces a free action of $\mathbb{Z}/n\mathbb{Z}$ on $Gr_p(k,n)$).*

From [1], we know that the positivity of the Grassmannian is closed under action of cycling, and the action cycles through the Grassmannian $2n$ times before it stabilizes again. This action $2n$ is repeated, however, with the negative component of the last element after a cycle (i.e., $(-1)^{k-1}$), and the Grassmannian's Plücker relations are not invariant under sign changes, so the cyclic shift actually forms an action on $\mathbb{Z}/n\mathbb{Z}$. The action is free because the Grassmannian is only invariant if all the signs of its Plücker relations change, which is trivially an identity.

This implies the action is free on characteristic n or $2n$, so the cyclic action is free on higher exterior powers of the Plücker relations. Recall that B_n is the wreath product of P_2 and S_n $(\pm)$, so positive shifts correspond to positive shifts and negative shifts correspond to negative shifts. Then by free action a positive shift of a class must equal the inverse of the negative class, which is exactly the invariance present in type B_n and C_n. Systematically, looking at the cyclic shifts of the rows of each elements will rearrange the ordering of the pivot columns of the Schubert cell, we form the following proposition.

Proposition 5.3. *The Schubert decomposition is invariant under the action of a cyclic shift.*

The positive part of the Plücker coordinates in the matroid structure, referred to as its positroid has a nontrivial intersection of the Schubert cells (after the cyclic shift) and the positive Grassmannian, which is indicative of the $\dim S^p_{\mathcal{M}}(v.l.) = \ell_b(w)$.

More formally, [1] and [4] described this intersection once again in terms of a (positroid) cell complex, $\Pi_{\mathcal{M}}$:

$$\Pi_{\mathcal{M}} = \bigcap_{i=0}^{n-1} \tilde{c}^i(\Omega_{I_i} \cap Gr^{\geq 0}(k,n)),$$

where $\tilde{c}^i$ is the cyclic shift applied i times. And because of this nontrivial intersection, we know that a function exists.

6 Discussion and Future Work

In addition, we pose the two following open conjectures based on the work presented in this paper.

Conjecture 6.1. There is a bijection $g : \Omega_\lambda \to \ell(W)$ between the dimension of the Schubert cells that decompose the totally positive Grassmannian and the bridge lengths of other types.

Conjecture 6.2. Using an $e_i + e_j$ root in a type B or C element cannot reduce the bridge length from what was presented in Theorem 3.1. In other words, Theorem 3.1 is the bridge length for type B and C Weyl groups.

In the future, we will work on a dictionary of lengths which entails finding the number of elements with bridge length x where x is any of the possible lengths. Additionally, there are applications in the Quantum Field Theory of scattering amplitudes of particle wave lengths and the Grassmannian that can be explained by the bridge length, which are left out because of the page limit, though [4] and [5] are deferred to the reader for clear applications.

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References

[1] A. Postnikov. Total positivity, grassmannians, and networks. *arXiv preprint math/0609764*, 2006.

[2] S. Carrozza, T. Krajewski, and A. Tanasa. Using grassmann calculus in combinatorics: Lindstr\" om-gessel-viennot lemma and schur functions. *arXiv preprint arXiv:1604.06276*, 2016.

[3] R. Karpman. Bridge graphs and deodhar parametrizations for positroid varieties. *Journal of Combinatorial Theory, Series A*, 142:113–146, 2016.

[4] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. B. Goncharov, A. Postnikov, and J. Trnka. Scattering amplitudes and the positive grassmannian. *arXiv preprint arXiv:1212.5605*, 2012.

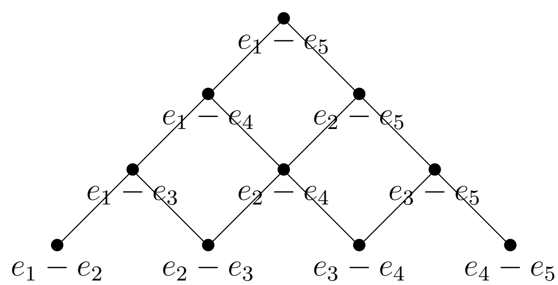
[5] L. Williams. A positive grassmannian analogue of the permutohedron. *Proceedings of the American Mathematical Society*, 144(6):2419–2436, 2016.

A Root Systems for $(A_n — D_n)$

Type A_{n-1}

- $R = \{e_i - e_j : i \neq j\}$,
- $R^+ = \{e_i - e_j : i < j\}$,
- $\Pi = \{e_i - e_{i+1}\}$,
- $W = \mathfrak{S}_n$,
- $r_{e_i - e_j} = (i, j)$.

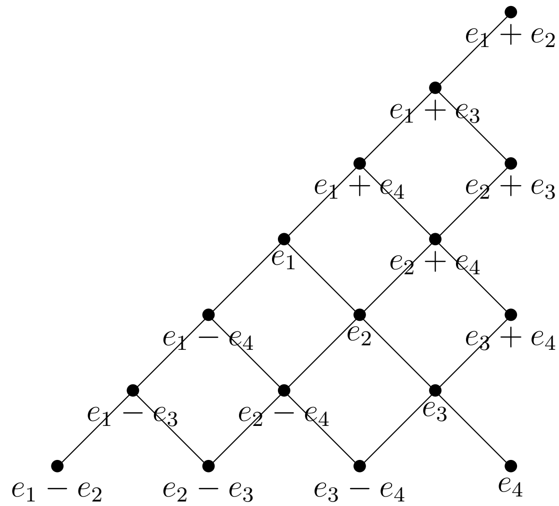
The root poset looks like the following ($n = 5$):



Type B_n

- $R = \{e_i - e_j : i \neq j\} \cup \{\pm(e_i + e_j) : i \neq j\} \cup \{\pm e_i\}$,
- $R^+ = \{e_i - e_j : i < j\} \cup \{e_i + e_j : i \neq j\} \cup \{e_i\}$,
- $\Pi = \{e_i - e_{i+1}\} \cup \{e_n\}$,
- $W = \{w \in \mathfrak{S}_{1,\ldots,n,-n,\ldots,-1} : w(i) = -w(-i)\}$,
- $r_{e_i - e_j} = (i, j)(-i, -j)$, $r_{e_i + e_j} = (i, -j)(j, -i)$, $r_{e_i} = (i, -i)$.

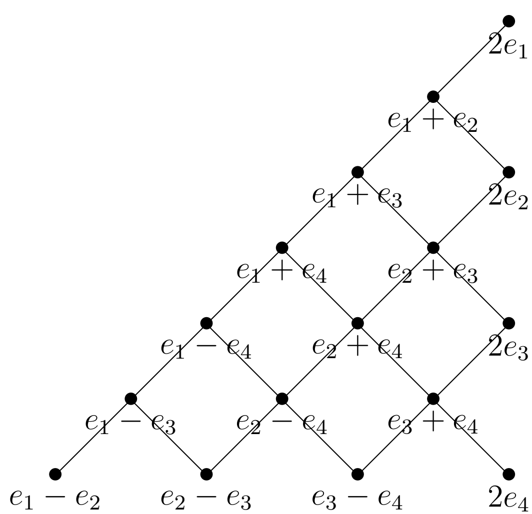
The root poset looks like the following ($n = 4$):



Type C_n

- $R = \{e_i - e_j : i \neq j\} \cup \{\pm(e_i + e_j) : i \neq j\} \cup \{\pm 2e_i\}$,
- $R^+ = \{e_i - e_j : i < j\} \cup \{e_i + e_j : i \neq j\} \cup \{2e_i\}$,
- $\Pi = \{e_i - e_{i+1}\} \cup \{2e_n\}$,
- $W = \{w \in \mathfrak{S}_{1,\dots,n,-n,\dots,-1} : w(i) = -w(-i)\}$,
- $r_{e_i - e_j} = (i, j)(-i, -j)$, $r_{e_i + e_j} = (i, -j)(j, -i)$, $r_{2e_i} = (i, -i)$.

The root poset looks like the following ($n = 4$):



Type D_n

- $R = \{e_i - e_j : i \neq j\} \cup \{\pm(e_i + e_j) : i \neq j\}$,
- $R^+ = \{e_i - e_j : i < j\} \cup \{e_i + e_j : i \neq j\}$,
- $\Pi = \{e_i - e_{i+1}\} \cup \{e_{n-1} + e_n\}$,
- $W = \{w \in \mathfrak{S}_{1,\ldots,n,-n,\ldots,-1} : w(i) = -w(-i)$ and $w(j) < 0$ for an even number of $j > 0\}$,
- $r_{e_i - e_j} = (i,j)(-i,-j)$, $r_{e_i+e_j} = (i,-j)(j,-i)$.

The root poset looks like the following ($n = 4$):