

On the Spectral Invariance of Ellipses in Convex, Planar Domains

Caleb He

under the direction of
Ethan Yale Jaffe
Department of Mathematics
Massachusetts Institute of Technology

Research Science Institute
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Abstract

Consider Ω , an open subset of \mathbb{R}^2 with a convex and smooth boundary. The set of eigenvalues λ for the Dirichlet problem on Ω with clamped boundary is known as the *spectrum* of Ω . We investigate the question posed by M. Kac, which asks whether the shape of a drum Ω is *spectrally invariant*, or determined by the spectrum. Specifically, we ask the question when Ω is an ellipse. Marvizi and Melrose found a family of spectral invariants I_k that were able to prove the spectral invariance of the disk and are believed to determine the spectral invariance of ellipses. They also showed that these invariants can be calculated from the coefficients in the asymptotic expansion of $\lim_{n \rightarrow \infty} \mathcal{L}_n$, where \mathcal{L}_n is the length of a *closed geodesic* of length n , or a path governed by the billiard ball map β . We compute β explicitly on the ellipse and provide a precise description of calculating the asymptotic expansion of \mathcal{L}_n .

Summary

In 1966, Mark Kac popularized the question, “Can One Hear the Shape of a Drum?” The question asks whether one can determine the shape of a drum if given sound that it produces. The answer to Kac’s question, in general, is no; however, for drums that are planar and convex, the answer is still unknown. We investigate Kac’s question for planar and convex drums, specifically looking to determine if the shape of the ellipse can be determined by the sound it produces. We do so by studying the length of billiard ball paths that traverse the boundary of our ellipse. Our main result is a precise method to calculate such lengths, bringing us a step closer to determining whether one can hear the shape of an ellipse.

1 Introduction

Consider a resonating drum. It is known that the drum will vibrate only with a discrete set of frequencies, known as the *spectrum* of the drum. The specific values of these frequencies are determined by the drum's shape, material, and tension.

A mathematical analogue exists for the drum scenario. Consider a drum Ω . There exists constants λ such that one can find a wave function that is a solution to the wave equation (1) that corresponds to Ω . An illustration of a wave function is given in Figure 1. The set of all such λ , known as *eigenvalues* of the wave equation, are analogous to the spectrum of frequencies at which the physical drum will vibrate. It is known that the set of eigenvalues of the wave equation is uniquely determined by the shape of Ω .

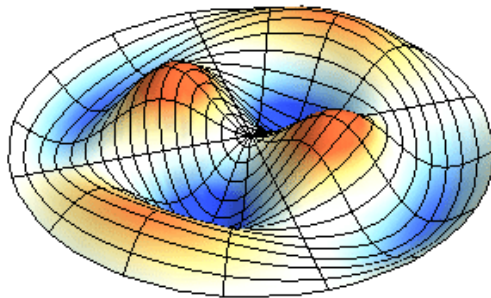


Figure 1: A solution to the wave equation on a planar drum [9].

A natural question to ask is the reverse: is the shape of Ω uniquely determined by its eigenvalues? Or equivalently, as coined by Mark Kac [1] in 1966, can one hear the shape of a drum? The study of this question has long intrigued mathematicians and helped further the field of *spectral theory*, which has played a key role in describing quantum mechanics and formulating Hilbert spaces.

The answer to Kac's question turns out to be negative for general domains of Ω as well as some restricted domains (such as drums with planar dimension). Explicit examples have been

found of nonisometric yet isospectral drums. For example, after Kac's paper was published, John Milnor [4] found an explicit example of two 16-dimensional tori whose geometries were different but whose spectra were identical. Despite this counterexample, some geometric data about Ω can be recovered from the spectrum. For example, Weyl's law [3] gives the volume and area of Ω in terms of the limit distribution of its eigenvalues. Such geometric data that can be determined by the spectrum are known as *spectral invariants*. Kac's question asks if the shape of a drum is a spectral invariant.

We study Kac's question when Ω is restricted to the convex, planar domain with smooth boundary, in which the answer is still unknown. What is known is that the spectrum of Ω can be described using the asymptotics of the lengths of *close geodesics*, which are the paths of billiard balls as they bounce off the boundary of Ω and eventually return to their starting position. An illustration of a closed geodesic is given in Figure 2. The lengths of geodesics serve as a more tractable description of the spectrum, and thus serves as our focus of study.

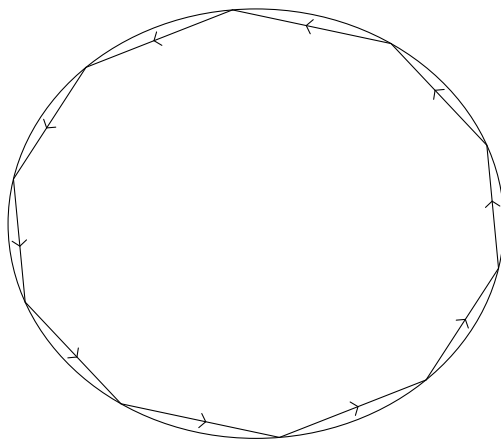


Figure 2: A closed geodesic on an ellipse.

Marvizi and Melrose [2] computed a family of integral invariants I_k that were used to prove the spectral invariance of the disk in the convex, planar domain. These integral invariants were proved to be not only spectrally invariant but calculatable from the coefficients in the asymptotic expansion of the lengths of closed geodesics. In particular, they found that

maximizers of I_1 were disks, thus implying the disk's spectral invariance.

Our overall goal is to determine the spectral invariance of the ellipse, which is a natural generalization of the disk, by calculating the integral invariants derived by Marvizi and Melrose. It is not yet known whether the ellipse is spectrally invariant, or whether it can be classified by the integral invariants in the same way as the disk. Our main result is the explicit calculation of the asymptotic expansion of the lengths of closed geodesics on the ellipse, which has yet to be done and serves as the first step in calculating the integral invariants for the ellipse. We state our result in the following theorem:

Theorem 1. *Suppose Γ is an ellipse with unit length semimajor axis and eccentricity e . Then the asymptotic expansion of the length \mathcal{L}_n of a closed geodesic on Γ with n reflection points can be written as $L(\Gamma) + \sum_{i=1}^{\infty} c_k n^{-2k}$ as $n \rightarrow \infty$, where $L(\Gamma)$ denotes the perimeter of Γ and the coefficients c_k are derived from the power series expansions of elliptic functions.*

In Section 2, we provide mathematical background to the problem. In Section 3, we compute the billiard ball map through a coordinate change. In Section 4, we use the coordinate change from Section 3 to prove our main theorem. In Section 5, we discuss future research.

2 Mathematical Background

Consider Ω , an open subset of \mathbb{R}^2 with a convex and smooth boundary $\partial\Omega$. A constant λ is an *eigenvalue* of the Dirichlet problem if there exists $u \in C^\infty(\Omega)$ such that

$$\begin{aligned} \Delta u &= \lambda^2 u \text{ in } \Omega, \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{1}$$

where Δ is the Laplace operator given by $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$. Here, u is the vertical displacement of the wave at a specific position on Ω .

Pieces of information that are determined by the eigenvalues of the Dirichlet problem are known as *spectral invariants*. Marvizi and Melrose [2] computed a family of spectral invariants I_k in terms of the integral of functions of the curvature of $\partial\Omega$. These integral invariants are useful for determining the spectral invariance of drums; for example, maximizers of I_1 were found to be disks, proving the disk's spectral invariance. Our goal is to calculate the integral invariants of ellipses to determine their spectral invariance.

The direct computation of the integral invariants, however, proves to be difficult. Yet, Marvizi and Melrose found an indirect method to compute the invariants by studying the lengths of closed geodesics, which we discuss now.

Consider a billiard ball on Ω that is reflected off the boundary according to the Law of Reflection and is traveling in the counterclockwise direction. The path of the ball is a *geodesic*. If the ball travels m times around the boundary and returns to its starting position after n reflections, then the path it traversed is a *closed geodesic*. We denote the family of such closed geodesics $\Gamma(m, n)$, in which m denotes the *winding number* and n denotes the *reflection number*. A result due to Poincaré states that $\Gamma(m, n)$ is non-empty on any strictly convex compact C^∞ region [8]. We also define the *length* of a closed geodesic to be the total distance that the ball travels before returning to its original position.

Let $\{\mathbf{g}_{1,n}\}$ be any sequence of closed geodesics such that for each n , we have $\mathbf{g}_{1,n} \in \Gamma(1, n)$. Marvizi and Melrose found that the values of the integral invariants are algebraically related to the values c_k , which are coefficients in the asymptotic sum

$$L(\mathbf{g}_{1,n}) \sim L(\partial\Omega) + \sum_{k=1}^{\infty} c_k n^{-2k} \quad \text{as } n \rightarrow \infty,$$

where L denotes length. Thus, by computing $\lim_{n \rightarrow \infty} L(\mathbf{g}_{1,n})$ for the ellipse, we can then compute the integral invariants from the coefficients c_k .

We now define the billiard ball map β that maps the position and velocity of the billiard

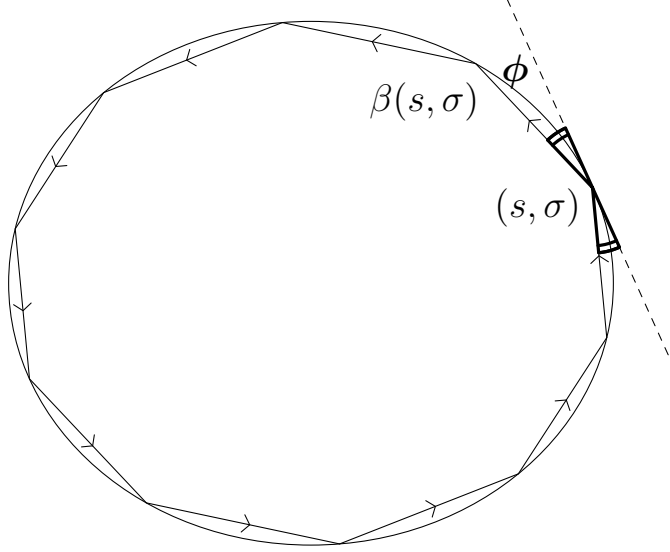


Figure 3: A closed geodesic on an ellipse described using the billiard ball map β .

ball from each point on $\partial\Omega$ to the next along the path of a closed geodesic. With β , we are able to describe the movement along closed geodesics symbolically. In the definition of β in [2], the coordinates of β are parametrized as (s, σ) , where $s \in [0, L]$ is the arc length parameter of $\partial\Omega$ and σ denotes the cosine of the angle ϕ measured inward from the forward tangent to the velocity vector. We define $\beta(s, \sigma)$ to be the next position and velocity pair in our closed geodesic. An illustration of a closed geodesic described by β is given in Figure 3.

It is known that β preserves the area 2-form $d\sigma \wedge ds$, a proof of which is given by Tabachnikov [7]. Using this fact, Marvizi and Melrose proved the existence of a function ζ that can be used to locally describe the vector flow of β in the following theorem.

Theorem 2 (Marvizi and Melrose). *If β is the billiard ball map of a strictly convex C^∞ planar domain, there exists a function $\zeta \in C^\infty(T^*(\partial\Omega))$ defined on the positive half of the cosphere bundle $B^*(\partial\Omega)$ such that*

$$\beta \circ \exp\left(+\zeta^{\frac{1}{2}}H_\zeta\right) = Id$$

locally, where H_ζ denotes the Hamiltonian vector field of ζ on $T^*(\partial\Omega)$.

Using the power series of ζ , Marvizi and Melrose determined the algebraic relationship between the coefficients c_k in the asymptotic expansion of the length of closed geodesics and the integral invariants I_k . However, ζ is difficult to compute; as a result, we directly calculate β in this paper instead.

In section 3, we compute β on the ellipse; however, unlike [2], we use the radial coordinate θ instead of the arc length coordinate s to determine position on the ellipse. More precisely, we compute $\beta : (\theta, \sigma) \mapsto (\theta', \sigma')$, where $\theta \in [0, 2\pi]$ parametrizes the points on the ellipse with semimajor axis length a and semiminor axis length b as $(a \cos \theta, b \sin \theta)$.

3 Change of Coordinates

Consider the special case when β acts on the disk parametrized by $(r \cos \theta, r \sin \theta)$. It is evident that β translates the radial coordinate θ by a constant under each iteration due to the symmetrical geometry of the disk. However, the same parallel translation in the radial coordinate θ does not hold true for β when it acts on the ellipse.

Thus, to compute β for the ellipse, we introduce a new coordinate $u(\theta)$ such that $\beta : (u, \sigma) \mapsto (u + c, \sigma')$ translates u by a constant c , similar to behavior of β on the radial coordinate of the disk. To begin, we consider the geometry of closed geodesics on the ellipse. The following theorem, a proof of which is given in Tabachnikov [7], allows us to characterize our closed geodesics in terms of confocal *caustics*, or curves to which the billiard ball forever remains tangent.

Theorem 3. *A geodesic inside an ellipse forever remains tangent to a fixed confocal conic. If a segment of the geodesic intersects the segment determined by the two foci of the ellipse, then the ball's path will forever remain tangent to a confocal hyperbola. Otherwise, the ball's path will remain tangent to a confocal ellipse.*

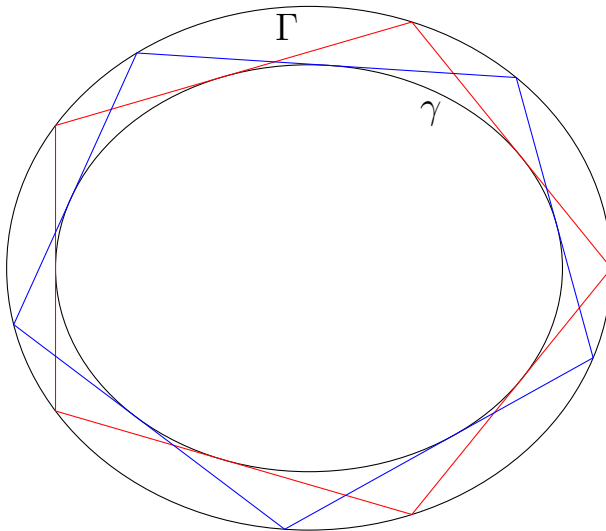


Figure 4: Two closed geodesics on Γ (blue and red) with confocal caustic γ .

It is also known that the reflection number of a closed geodesic that is tangent to a fixed confocal caustic remains invariant under a change of the starting position; a proof is given in [7]. As a result, we can parametrize the closed geodesics in $\Gamma(1, n)$ by the eccentricity of the confocal caustic and assume an arbitrary starting position for our closed geodesic. We also note that the confocal caustic must be an ellipse to have a closed geodesic with winding number 1.

An equivalent statement of Theorem 3 is that the billiard ball map on the ellipse is *integrable*, or that there exists a function $F(\theta, \sigma)$ that remains invariant under the billiard ball map. This fact is further expounded upon in [7]. We have that the integrability of β implies the existence of our coordinate u that is parallelly translated by a constant under β .

We consider β acting on $\Omega = \Gamma$, where Γ is an ellipse with eccentricity e with semimajor axis of unit length and centered at the origin. Let γ be confocal to Γ with eccentricity $e < t < 1$ such that the billiard ball path is always tangent to γ . We parametrize points on Γ with θ as $(\cos \theta, \sqrt{1 - e^2} \sin \theta)$. Then we have the following proposition.

Proposition 1. *Suppose β acts on the ellipse Γ such that the billiard trajectory remains*

tangent to the ellipse γ that is confocal with Γ . The coordinate u such that β maps (u, σ) to $(u + c_t, \sigma')$ for a constant c_t dependent on t is given by

$$u(\theta, t) = \frac{F\left(\theta, \frac{ti}{\sqrt{1-t^2}}\right)}{F\left(2\pi, \frac{ti}{\sqrt{1-t^2}}\right)}, \quad (2)$$

where $F(\phi, k) = \int_0^\phi \frac{d\tau}{\sqrt{1 - k^2 \sin^2 \tau}}$.

Proof. We use methods similar to those in [7]. We seek a coordinate u such that du remains invariant under β . To do so, we find a function f such that $f(\theta)d\theta$ remains invariant under β , and then integrate to recover u .

Let $A_1 : (x, y) \mapsto (x, \frac{y}{\sqrt{1-e^2}})$ be the affine transformation that takes Γ to the unit disk, and choose a point θ on the disk. Tangency is preserved under affine transformations, so we denote $\alpha = A_1\beta(\theta)$ to be the second intersection of the tangent from θ to $A_1\gamma$ with the disk. Denote by $R_{A_1\gamma}(\theta)$ and $L_{A_1\gamma}(\theta)$ the lengths of the positive (right) and negative (left) tangent segments from θ to $A_1\gamma$. Consider θ' infinitesimally close to θ . Let ω be the angle determined by $\overline{\theta\alpha}$ and $\overline{\theta'\alpha'}$, as illustrated in Figure 5. By the Law of Sines, we have

$$\frac{|\theta\theta'|}{R_{A_1\gamma}(\theta)} = \frac{\sin \omega}{\sin \angle \alpha \theta \theta'} = \frac{\sin \omega}{\sin \angle \theta' \alpha' \alpha} = \frac{|\alpha\alpha'|}{L_{A_1\gamma}(\alpha)},$$

which implies that

$$\frac{d\theta}{R_{A_1\gamma}(\theta)} = \frac{d\alpha}{L_{A_1\gamma}(\alpha)}.$$

Now let $A_2 : (x, y) \mapsto (x, \frac{\sqrt{1-e^2}}{\sqrt{1-t^2}}y)$ be the affine transformation that takes $A_1\gamma$ to a circle. Then we have that the right and left tangents from a point p to $A_2A_1\gamma$ are of equal length. We denote this common length as $D_{A_2A_1\gamma}(p)$. Because the ratio of parallel lengths are preserved

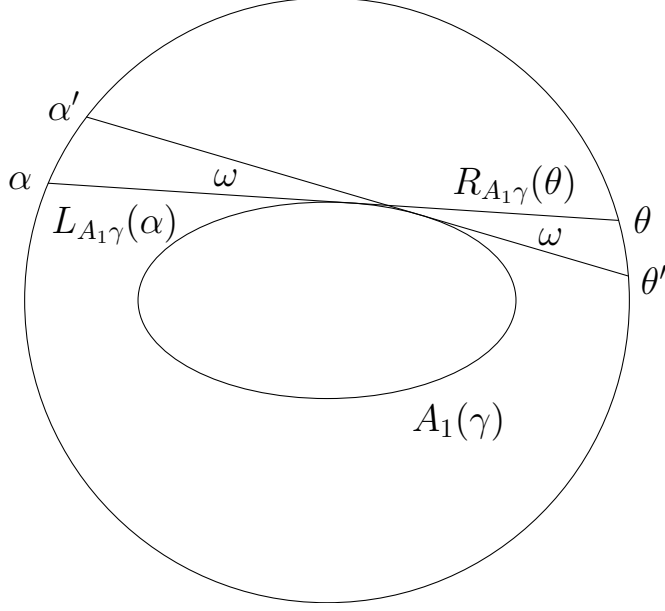


Figure 5: Construction of the function f which is invariant under β .

under affine transformations, we have that

$$\frac{R_{A_1\gamma}(\theta)}{L_{A_1\gamma}(\alpha)} = \frac{D_{A_2A_1\gamma}(A_2\theta)}{D_{A_2A_1\gamma}(A_2\alpha)}.$$

Thus, if one sets $f(\theta) = \frac{1}{D_{A_2A_1\gamma}(A_2\theta)}$, then the desired invariance is achieved. We now compute $D_{A_2A_1\gamma}(A_2\theta)$ explicitly.

The coordinates of $A_2\theta$ are given by $(\cos \theta, \frac{\sqrt{1-e^2}}{\sqrt{1-t^2}} \sin \theta)$, and $A_2A_1\gamma$ is a disk centered at the origin with radius $\frac{t}{e}$. Using the Power of a Point Theorem, we have that

$$D_{A_2A_1\gamma}(A_2\theta) = \sqrt{\left(1 - \frac{e^2}{t^2}\right) \left(1 + \frac{t^2}{1-t^2} \sin^2 \theta\right)}$$

after simplification. We now solve for u such that $du = f(\theta)d\theta$ and with u normalized onto $[0, 1]$ for θ on $[0, 2\pi]$, which yields

$$u(\theta, t) = \frac{F\left(\theta, \frac{ti}{\sqrt{1-t^2}}\right)}{F\left(2\pi, \frac{ti}{\sqrt{1-t^2}}\right)},$$

where F is known as the incomplete elliptic integral of the first kind. □

Having found u , we now solve for c_t .

Proposition 2. *Given ellipse γ with eccentricity t confocal to ellipse Γ with eccentricity e , the constant c_t for which $\beta : u \mapsto u + c_t$ maps u parallelly along the billiard ball path tangent to γ is given by $c_t = u(\theta_t, t)$, where*

$$\theta_t = \cos^{-1} \left(\frac{-t^2 + 2e^2 - e^4}{t^2 - 2e^2t^2 + e^4} \right).$$

Proof. We compute θ_t , which is the first image of $\theta = 0$ under the billiard ball trajectory with caustic γ . It follows that c_t is the image of θ_t under the coordinate change u . An illustration

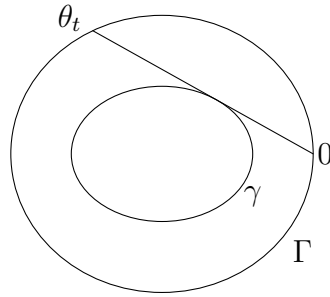


Figure 6: Solving for $c_t = u(\theta_t)$ utilizing one iteration of β .

of this proof is given in Figure 6. Let ϕ be such that $(\frac{e}{t} \cos \phi, \frac{e\sqrt{1-t^2}}{t} \sin \phi)$ is the point of tangency on γ from $(1, 0)$. Differentiation gives us the slope of the tangent as

$$-\sqrt{1-t^2} \cot \phi, \tag{3}$$

while the slope formula gives us the slope as

$$\frac{e\sqrt{1-t^2} \sin \phi}{e \cos \phi - t}. \tag{4}$$

Equating (3) and (4), we find that $\phi = \cos^{-1}(\frac{e}{t})$ and the slope of the tangent is

$$\frac{-e\sqrt{1-t^2}}{\sqrt{t^2-e^2}}.$$

Thus, the equation of the tangent to γ can be written as

$$y = \frac{-e\sqrt{1-t^2}}{\sqrt{t^2-e^2}}(x-1). \quad (5)$$

The equation of Γ is

$$x^2 + \frac{y^2}{1-e^2} = 1. \quad (6)$$

Substituting Equation 5 into the Equation 6 yields an x -coordinate of

$$\frac{-t^2 + 2e^2 - e^4}{t^2 - 2e^2t^2 + e^4}, \quad (7)$$

which is equivalent to $\cos\theta_t$. Thus, taking \cos^{-1} of (7) yields θ_t . □

Remark 1. *The eccentricity t_n of the confocal caustic that corresponds to the family of geodesics $\Gamma(1, n)$ can be found by solving $c_t = \frac{1}{n}$ for t using Proposition 2.*

Remark 2. *If $\phi(x, t)$ is such that $\phi(u(\theta, t), t) = \theta$ (i.e. the inverse of u), then the radial coordinate for the k^{th} point (starting with 0 and going counterclockwise) of a closed geodesic in $\Gamma(1, n)$ is precisely $\phi(\frac{k}{n}, t_n)$. An illustration is given in Figure 7. Because u is one-to-one, ϕ is well-defined. Moreover, from the definition of u , we have that ϕ is infinitely differentiable with no singularities and satisfies the property $\phi(x+1, t) = \phi(x, t) + 2\pi$.*

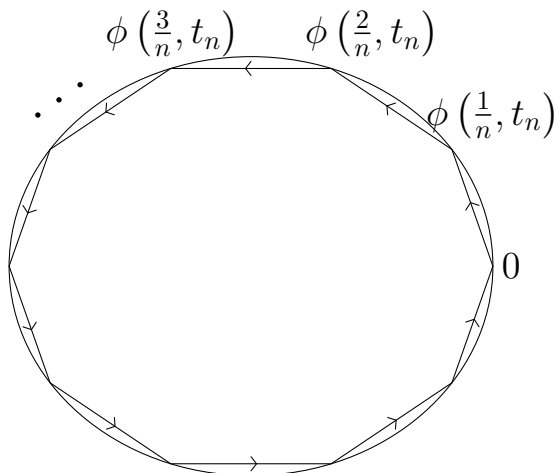


Figure 7: The angular coordinates for points on a closed geodesic of length n .

4 Length of Closed Geodesic

We now determine the length of our closed geodesic using the results from Section 3. We define $l(\alpha, \beta)$ to be the Euclidean distance between the points on Γ determined by radial coordinates α and β . We define the function $g_n(x) := l(\phi(\frac{x}{n}, t_n), \phi(\frac{x-1}{n}, t_n))$ to be the length of the segment between the two points determined by u coordinates $\frac{x}{n}$ and $\frac{x-1}{n}$. Then the length of our closed geodesic, \mathcal{L}_n , is given as

$$\mathcal{L}_n = \sum_{i=1}^n g_n(i).$$

We seek the asymptotic expansion of \mathcal{L}_n in terms of n . We begin with the following lemma.

Lemma 1. *The function g_n is n -periodic.*

Proof. We have that $\phi(x+1, t) = \phi(x, t) + 2\pi$ from Remark 2. Using this fact, we have

$$\begin{aligned} g_n(x+n) &= l\left(\phi\left(\frac{x}{n} + 1, t_n\right), \phi\left(\frac{x-1}{n} + 1, t_n\right)\right) \\ &= l\left(\phi\left(\frac{x}{n}, t_n\right) + 2\pi, \phi\left(\frac{x-1}{n}, t_n\right) + 2\pi\right) \end{aligned}$$

$$\begin{aligned}
&= l\left(\phi\left(\frac{x}{n}, t_n\right), \phi\left(\frac{x-1}{n}, t_n\right)\right) \\
&= g_n(x).
\end{aligned}$$

The second-to-last equality holds true since l is a function of the sine and cosine of its two arguments and is thus 2π -periodic. \square

With Lemma 1, we now prove the following proposition.

Proposition 3. *The length of our closed geodesic \mathcal{L}_n on ellipse Γ is asymptotically given as*

$$\mathcal{L}_n - \int_0^n g_n(x) dx = O\left(\frac{1}{n^p}\right) \forall p \in \mathbb{Z}^+.$$

Proof. The Euler-Maclaurin formula for \mathcal{L}_n yields the following for all positive integers p ,

$$\sum_{i=1}^n g_n(i) = \int_0^n g_n(x) dx + \frac{g_n(n) - g_n(0)}{2} + \sum_{k=1}^p \frac{B_{2k}}{(2k)!} (g_n^{(2k-1)}(n) - g_n^{(2k-1)}(0)) + R, \quad (8)$$

where B_i denotes the i^{th} Bernoulli number and the remainder term is bounded as

$$|R| \leq \frac{2\zeta(p)}{(2\pi)^p} \int_0^n |g_n^{(p)}(x)| dx,$$

where ζ denotes the Riemann zeta function [10]. However, $g_n(x)$ and all of its derivatives are periodic with period n from Lemma 1. Thus, all terms on the right hand side of (8) vanish except for the integral and the remainder term.

It now suffices to prove that $|R| \leq \frac{c_p}{n^p}$ for a constant c_p independent of n . Let $g_1(x, t_n) = \cos(\phi(x, t_n))$ and $g_2(x, t_n) = \sin(\phi(x, t_n))$. We prove that the integral in the bound for the

remainder term is $O(\frac{1}{n^p})$. Using Taylor's theorem, we have

$$\begin{aligned}
g_n(x) &= \sqrt{\left(g_1\left(\frac{x}{n}, t_n\right) - g_1\left(\frac{x-1}{n}, t_n\right)\right)^2 + (1-e^2)\left(g_1\left(\frac{x}{n}, t_n\right) - g_1\left(\frac{x-1}{n}, t_n\right)\right)^2} \\
&= \sqrt{\left(-\frac{1}{n}g_1'\left(\frac{x}{n}, t_n\right) + O\left(\frac{1}{n^2}\right)\right)^2 + (1-e^2)\left(-\frac{1}{n}g_1'\left(\frac{x}{n}, t_n\right) + O\left(\frac{1}{n^2}\right)\right)^2} \\
&= \frac{1}{n}\sqrt{g_1'\left(\frac{x}{n}, t_n\right)^2 + (1-e^2)g_2'\left(\frac{x}{n}, t_n\right)^2 + O\left(\frac{1}{n}\right)}.
\end{aligned}$$

The last equality is true since both g_1' and g_2' are finitely bounded for $x \in [0, 1]$, allowing for the $O(\frac{1}{n})$ term to appear. Then, we now have

$$\begin{aligned}
g_n(x) &= \frac{1}{n}\sqrt{g_1'\left(\frac{x}{n}, t_n\right)^2 + (1-e^2)g_2'\left(\frac{x}{n}, t_n\right)^2 + O\left(\frac{1}{n}\right)} \\
&= \frac{1}{n}\sqrt{\phi'\left(\frac{x}{n}, t_n\right)^2\left(\sin^2\left(\phi\left(\frac{x}{n}, t_n\right)\right) + (1-e^2)\cos^2\left(\phi\left(\frac{x}{n}, t_n\right)\right)\right) + O\left(\frac{1}{n}\right)}. \quad (9)
\end{aligned}$$

We now prove that g_n has the form $\frac{1}{n}\sqrt{C + O(\frac{1}{n})}$ for a positive constant C that is finitely bounded. Such a fact would imply the smoothness of g as well as a uniform upper bound on all derivatives of g in x and n . Given that the sine and cosine of the same argument cannot both be zero, it suffices to show that ϕ' is nonzero and bounded above uniformly in x and n . This is true since ϕ is infinitely differentiable with no singularities, as stated in Remark 2. Now, from (9), we see that the chain rule gives us the p^{th} derivative of g_n in the form $\frac{1}{n^{p+1}}h(x, n)$, where h is smooth and bounded above uniformly in x and n . Thus, we have

$$R \leq \frac{2\zeta(p)}{(2\pi)^p} \int_0^n |g_n^{(p)}(x)| dx = \frac{2\zeta(p)}{(2\pi)^p} \int_0^n \frac{1}{n^{p+1}} h(x, n) dx = \frac{2\zeta(p)}{(2\pi)^p} \int_0^1 \frac{1}{n^p} h(nx, n) dx < \frac{c_p}{n^p},$$

completing the proof. □

From here, the asymptotic expansion of \mathcal{L}_n can be written using the power series expansion

sion of g_n . We outline the means by which one can calculate the asymptotics for \mathcal{L}_n in the following theorem.

Theorem 1. *Suppose Γ is an ellipse with unit length semimajor axis and eccentricity e . Then the asymptotic expansion of the length \mathcal{L}_n of a closed geodesic on Γ with n reflection points can be written as $L(\Gamma) + \sum_{i=1}^{\infty} c_i n^{-2i}$ as $n \rightarrow \infty$, where $L(\Gamma)$ denotes the perimeter of Γ and the coefficients c_i are derived from the power series expansions of elliptic functions.*

Proof. The asymptotic expansion of \mathcal{L}_n can be derived from the asymptotic expansion of g_n through term-by-term integration. To recall, we have our expression for $g_n(x)$ as

$$\sqrt{\left(\cos \phi\left(\frac{x}{n}, t_n\right) - \cos \phi\left(\frac{x-1}{n}, t_n\right)\right)^2 + (1-e^2)\left(\sin \phi\left(\frac{x}{n}, t_n\right) - \sin \phi\left(\frac{x-1}{n}, t_n\right)\right)^2}.$$

The power series expansions for cosine and sine are well known, as is the means to take the square root of a power series. Thus, it suffices to derive the power series expansion for $\phi\left(\frac{x}{n}, t_n\right)$. Written explicitly, we have

$$\phi\left(\frac{x}{n}, t_n\right) = F^{-1}\left(\frac{x}{n} F\left(2\pi, \frac{t_n i}{\sqrt{1-t_n^2}}\right), \frac{t_n i}{\sqrt{1-t_n^2}}\right).$$

The power series for F , the incomplete elliptic integral of the first kind, is given as

$$F(z, m) = z + \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \sum_{j=0}^{k-1} \frac{\left(\frac{1}{2}\right)_k \binom{2k}{j} (-1)^{i-j+k} 2^{2i-2k+1} (j-k)^{2i} m^{2k}}{k!(2i+1)!} z^{2i+1},$$

where $(\)_k$ denotes the decreasing factorial of length k . The power series for F^{-1} can be determined using the Lagrange inversion theorem, which is stated in Appendix A. Thus, it remains to find the power series of t_n .

Recall t_n is the solution t to $c_t = \frac{1}{n}$ in Remark 1. Using the expression for c_t in Proposition

2, we write the series expansion in t for

$$c_t = \frac{F\left(\theta_t, \frac{ti}{\sqrt{1-t^2}}\right)}{F\left(2\pi, \frac{ti}{\sqrt{1-t^2}}\right)} = \frac{F\left(\cos^{-1} \frac{-t^2+2e^2-e^4}{t^2-2e^2t^2+e^4}, \frac{ti}{\sqrt{1-t^2}}\right)}{F\left(2\pi, \frac{ti}{\sqrt{1-t^2}}\right)}$$

using the standard operations on series. We then apply the Lagrange inversion theorem to the series for c_t and evaluate it at $\frac{1}{n}$, which produces the series expansion for t_n .

We note that the series expansion for θ_t produces even powers of t , implying even powers in the expansion of c_t as well. Furthermore, the Lagrangian inversion preserves the even powers of n for t_n . Moreover, $g_n(x)$ is implicitly a function of t_n , implying that only even powers of n appear in our asymptotic expansion of \mathcal{L}_n . \square

Remark 3. *Theorem 5.15 in [2] states that the asymptotic expansion of the length of geodesics on a general convex planar drum has only even powers of n . The proof of Theorem 1 serves as an alternate proof of this fact in the specific case of the ellipse.*

5 Conclusion

We studied the length of closed geodesics on the ellipse with the goal of determining whether ellipses are spectrally invariant in the convex, planar domain with smooth boundary. We computed a coordinate change that was parallelly translated under the billiard ball map that helped us find the precise calculation of the asymptotics of the lengths of closed geodesics on the ellipse.

The next step is to calculate the function ζ that describes the vector flow of the billiard map in Theorem 2. Calculating ζ would require an alternative coordinate to the velocity coordinate σ that can be written explicitly in terms of u and the eccentricity t of the confocal caustic. Using the power series of ζ , we could then determine the integral invariants from the coefficients of the asymptotic expansion of the length of our closed geodesic and thus have

the capability to analyze the spectral invariance of the ellipse.

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Appendix A Lagrange Inversion Theorem

Suppose $f(w) = z$, where f is analytic at a point a . The series for the inverse function g , such that $g(z) = w$, is given in the following theorem.

Theorem (Lagrange Inversion). *The series expansion of g is given by*

$$g(z) = a + \sum_{n=1}^{\infty} \left(\lim_{w \rightarrow a} \left(\frac{(z - f(a))^n}{n!} \frac{d^{n-1}}{dw^{n-1}} \left(\frac{w - a}{f(w) - f(a)} \right)^n \right) \right).$$

Alternatively, according to Morse and Feshbach [11], if the series expansion for f is given by

$$f(x) = \sum_{i=1}^n a_i x^i, \text{ then the series expansion for } g \text{ is given by } g(x) = \sum_{i=1}^n A_i x^i, \text{ where}$$

$$A_i = \frac{1}{i a_1^i} \sum_{s,t,u,\dots} \frac{n^{(s+t+u+\dots)}}{s!t!u!\dots} \left(\frac{a_2}{a_1} \right)^s \left(\frac{a_3}{a_1} \right)^t \left(\frac{a_4}{a_1} \right)^u \dots,$$

for $s + 2t + 3u + \dots = n - 1$. Here, $n^{(k)}$ denotes the rising factorial of length k .