Homomesy in Minuscule Posets

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Abstract

An action, known as rowmotion, defined on the order ideals of posets created by Duchet is analyzed in posets arising from the minuscule representations of complex simple Lie algebras. We consider the homomesies on this action in all minuscule posets from a combinatorial perspective through the use of a slope-based representation. We examine the cardinality statistic defined on the size of the order ideals in each rowmotion orbit and prove that the cardinality statistic is homomesic in all minuscule posets. A uniform characterization of rowmotion is given by investigation of a bijection between the weights of the minuscule poset and its corresponding sign word, in all classical minuscule posets. The cardinality homomesy is also shown through the uniform characterization.

Summary

We look at the behaviors of certain types of grids through specific rearrangements. These types of grids model real world quantum systems. We examine properties that stay constant regardless of what types grids these rearrangements are performed on. Examining the properties of these rearrangements on these grids can aid in the creation of a topological quantum computer, which are based on these grids. We also provide a uniform classification that explains why the certain properties are constant by analyzing the origins of these grids.

1 Introduction

The study of rowmotion on the order ideals of a poset has flourished ever since its original introduction by Duchet in 1974 [1].This topic has been studied by authors including Fon-der-Flaass [2], Cameron [2], Brouwer [3], Schrijver [3], Striker [4], and Williams [4]. In particular, rowmotion on minuscule posets, which originate from the representations of Lie algebras, have been of particular interest. A useful application of this work is in the creation of topological quantum computers.

A quantum computer is one that uses quantum-mechanical effects to perform computations, usually much faster than current supercomputers. A topological quantum computer is built using anyons, quasi-particles, as threads and relying on braid theory to form stable logic gates. These anyons have topological bases that are directly connected to Lie algebras.

Minuscule posets have been seen to exhibit a variety of constant properties under the action of rowmotion. In particular, a property that has been extensively examined is the cardinality statistic which is equivalent to the average size of an order ideal as it goes through a rowmotion orbit. This cardinality statistic has been shown to be constant over all orbits of rowmotions in the minuscule poset of types A_n by Jim Propp and Tom Roby [5].

We extend the work of [5] and prove that this cardinality statistic stays constant under classical rowmotion in all other types of minuscule posets: B_n, C_n, D_n, E_6, and E_7 [6].

In Section 2 we present a more rigorous formulation of constant properties as homomesies. In addition, we introduce basic relations and definitions used in our results. In Section 3 an alternative definition of classical rowmotion is given in terms of the toggle group.

In Section 4, we prove the homomesy property in all types of minuscule posets. In Section 5 we outline two characterizations of rowmotion in type A_n and show that there exists a bijection between the two. Section 6 extends one of these characterizations of rowmotion across the classical minuscule posets: A_n, B_n, C_n, and D_n based on their weights in their corresponding weight poset. We illustrate that the weights also have a bijection to the sign-word. Section 7 demonstrates that the cardinality statistic is constant as a result of this characterization.

2 Background

2.1 Homomesy and Posets

The notion of a constant function can be made rigorous with a property called homomesy.

Definition 1. *Given a set $\mathcal{S}$, an invertible map τ from $\mathcal{S}$ to itself such that each τ-orbit is finite, and a statistic $f : \mathcal{S} \to k$ taking values in some field k of characteristic zero, we say the triple $(\mathcal{S},\tau,f)$ exhibits* ***homomesy*** *if there exists a constant $c \in k$ such that for every τ-orbit $\mathcal{P} \subset \mathcal{S}$*

$$\frac{1}{|\mathcal{P}|}\sum_{x\in\mathcal{P}} f(x) = c.$$

We call the function $f : \mathcal{S} \to K$ ***homomesic*** *under the action of τ on $\mathcal{S}$, or more specifically c-**mesic**.*

Before moving on, we must first define what a poset is.

Definition 2. *A set P is called a* ***poset*** *if there exists a binary relation $\leq$ over the set P which satisfies the following criteria:*

1. *$x \leq x$ (reflexivity)*
2. *If $x \leq y$ and $y \leq x$ then $x = y$ (antisymmetry)*
3. *If $x \leq y$ and $y \leq z$ then $x \leq z$ (transitivity)*

for all x, y, $z \in P$.

If $x, y \in P$, then we say that x **covers** y, denoted by $x \gtrdot y$, if $x > y$ and there exists no element $k \in P$ that satisfies $x > k > y$. Define an **order ideal** of a poset P to be a subset $I \subseteq P$ such that if we have an element $k \in I$ and there exists an element $m \leq k$, then $m \in I$. The set $J(P)$ is the poset whose elements are all the order ideals of the poset P, ordered by inclusion. So, an order ideal I is $\geq$ an order ideal I_1 if $I_1 \subseteq I$. A **maximal** element of an order ideal I is an element $m \in I$ such that if $m \leq i$, then $m = i$. The **minimal** element is defined using the opposite binary relation.

Now we define the rowmotion on the order ideals of a poset P:

Definition 3. *Let M be the set of minimal elements of $P \backslash I$. Given a poset P and its corresponding set of order ideals $J(P)$, the* ***rowmotion*** *Φ is a map on $J(P)$, where the minimal elements of $P \backslash I$ are now the maximal elements of $\Phi(I)$.*

Additionally, define Φ_J to be an orbit of rowmotions Φ on an order ideal in $J(P)$. Likewise, let $|\Phi_J|$ to be the number of distinct order ideals within a rowmotion orbit on J(P).

Figure 1 illustrates rowmotion Φ on a minuscule poset, where the large dots indicate the new maximal elements that have been added through Φ, and the smaller dots indicate elements of the order ideal that are preserved through each successive application of Φ. The

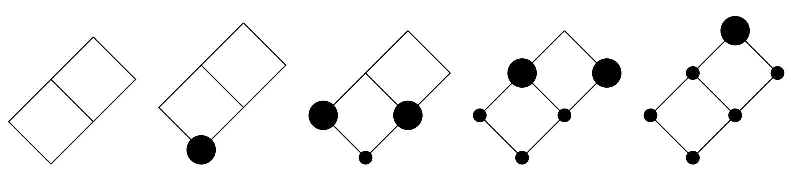


Figure 1: Rowmotion on a Type A Minuscule Poset

minuscule posets are types of posets that originate from the representation theory of Lie algebras. We elaborate on some background to help give a better understanding of where these minuscule posets come from.

2.2 Lie algebras and Minuscule Representations

A Lie algebra is defined to be a vector space $\mathfrak{g}$ over some field F together with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket, which satisfies the following axioms:

1. $[x, x] = 0$
2. $[ax + by, z] = a[x, z] + b[y, z]$ and $[z, ax + by] = a[z, x] + b[z, y]$
3. $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$

for all x, y, z in $\mathfrak{g}$. We define a subspace $h \subseteq \mathfrak{g}$ to be an **ideal** if $x \in \mathfrak{g}$, $y \in h$ implies $[x, y] \in h$. Additionally, we define a subspace $h \subseteq \mathfrak{g}$ to be a **subalgebra** iff $x \in h$, $y \in h$ implies that $[x, y] \in h$. A Lie algebra $\mathfrak{g}$ is considered to be **simple** iff the only ideals that $\mathfrak{g}$ has are trivial and that $\mathfrak{g}$ is non-abelian. Define the **dual space**, k^* of a space k defined over some field F to be the set of linear functionals $\phi : k \to F$. For the rest of this paper, let $\mathfrak{g}$ denote a complex simple Lie algebra. A **representation** of a Lie algebra $\mathfrak{g}$ is a function $\rho : \mathfrak{g} \to \mathrm{End}(V)$, where $\mathrm{End}(V)$ is the set of endormorphisms of a vector space V, such that $\rho_{[x,y]} = [\rho_x, \rho_y] = \rho_x\rho_y - \rho_y\rho_x$. Additionally, we define an **adjoint representation** of a Lie Algebra is to be a representation whose function is the adjoint action, defined below.

Definition 4. *Given an element x of a Lie algebra $\mathfrak{g}$, the **adjoint action** of x on $\mathfrak{g}$ is the map $ad_x : \mathfrak{g} \to \mathfrak{g}$ with $ad_x(y) = [x, y]$ for all $y \in \mathfrak{g}$.*

Given a Lie Algebra $\mathfrak{g}$, and corresponding vector space and representation $V : \mathfrak{g} \to \mathrm{End}(V)$, we define the **weight space** of a linear functional $\lambda \in \mathfrak{g}^*$ to be the space $V_\lambda = \{v \in V : \forall \xi \in \mathfrak{g}, \xi \cdot v = \lambda(\xi)v\}$. The linear functional is considered to be a **weight** iff the space V_λ is non-empty.

We define $\mathfrak{h}$ to be the **Cartan subalgebra** if it is the subalgebra of $\mathfrak{g}$ that is maximal and abelian. The adjoint representation is preserved to the Cartan subalgebra $\mathfrak{h}$. The weights of the adjoint representation restricted to $\mathfrak{h}$ are called **roots**, and the set of roots is called the **root system**. From this we write $\mathfrak{g}$ in a form called the Cartan Decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha.$$

Each root in a root system Φ can be characterized as either simple, positive, or negative.

Definition 5. *The set of* ***positive roots****, Φ^+, is a subset of the root system Φ such that*

- *For each root $\alpha \in \Phi$, exactly one of the roots α, -α is contained in Φ^+.*
- *For any two distinct roots α, $\beta \in \Phi^+$ such that $\alpha + \beta$ is a root, $\alpha + \beta \in \Phi^+$.*

Naturally the set of **negative roots** is the set -Φ^+. An element of Φ^+ is called a **simple root** if it cannot be written as a sum of two elements of Φ^+.

We now give the definition of a simple reflection:

Definition 6. *A* ***simple reflection*** *of a weight λ is a mapping that takes λ to $\lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$, where (x, y) is the inner product of the vector space and α is a simple root.*

The Weyl Group W is defined as the group generated by the simple reflections of all the simple roots of the corresponding Cartan Subalgebra $\mathfrak{h}$. Note that the simple reflections must satisfy the Coxeter relations and thus are also considered a Coxeter Group. From this, we define the **root order** to be a partial order on the roots, where $\lambda \gtrdot \mu$ if $\lambda - \mu$ is a simple root. We can now define a minuscule representation.

Definition 7. *A representation is considered* ***minuscule*** *if every weight is of the form $w\lambda$, where $w \in W$, the corresponding Weyl Group of the representation, and λ is the maximal weight with respect to the root order.*

This type of representation is equivalent to saying that the Weyl group acts *transitively* on the weights.

Definition 8. *An **irreducible minuscule lattice** or the **weight poset** is the set of weights of some minuscule representation of a simple Lie algebra that are ordered based on the root order.*

Let an element l of a lattice L be called **join-irreducible** if there does not exist s and t such that $l \gtrdot s$ and $l \gtrdot t$.

Definition 9. *The **minuscule poset** P is the poset of the join-irreducibles of the irreducible minuscule lattice ordered by inclusion.*

Note that in a minuscule poset P, there is an isomorphism of posets between the weight poset and $J(P)$.

3 The Toggle Group

It is well known that rowmotion can be characterized in terms of the **toggle group** of a finite poset. We give some definitions and highlight the toggle definition of rowmotion.

Definition 10. *Given an element p in a poset P, the toggle operation $\sigma_p : J(P) \to J(P)$ is defined on an order ideal as*

$$\sigma_p(I) = \begin{cases} I \diamondsuit \{p\} & \text{if } I \diamondsuit \{p\} \in J(P); \\ I & \text{otherwise,} \end{cases}$$

where $X \diamondsuit Y$ denotes the symmetric difference $X \backslash Y \cup Y \backslash X$.

Proposition 1. *([2]) Let P be a poset.*

(a) For every $p \in P$, σ_p is an involution, i.e., $\sigma_p^2 = 1$, the identity.

(b) For every $a, b \in P$ where neither a covers b nor b covers a, the toggles commute, i.e., $\sigma_a \sigma_b = \sigma_b \sigma_a$.

We call listing of elements of a poset P a **linear extension** on P if it is order-preserving. Note that there exists multiple linear extensions within all minuscule posets.

Proposition 2. *([2]) Let $a_1, a_2, \ldots, a_k$ be a linear extension of a poset P. Then the composite map $\sigma_{a_1}\sigma_{a_2}\cdots\sigma_{a_k}$ is equivalent to the rowmotion operator Φ.*

The linear extension that is often used when describing the toggle definition of rowmotion is toggling by rows from top to bottom.

4 Homomesy of the Cardinality Statistic in all Cartan Types

In this section, we show that the cardinality statistic is homomesic on all types of minuscule posets. For the rest of the paper we use the notation S_n to refer to the symmetric group of order n.

4.1 Type A_n: $[a] \times [b]$

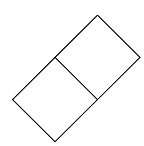


Figure 2: Structure for A_4

Figure 2 gives an example of the structures A_n, which are rectangles rotated 45 degrees counterclockwise. Jim Propp and Tom Roby [5] proved that the cardinality statistic is homomesic on the minuscule posets A_n for all positive integers n.

4.2 Type B_n: $([n] \times [n])/S_2$

Figure 3 gives an example of the structures B_n, which are the halves of an $n \times n$ diamond. Note that we can assign coordinates to the vertices of the squares. We define the bottom vertex of the minuscule poset to be $(0, 0)$, and every half-diagonal of a square is considered to be unit length on the coordinate plane. To prove the B_n case, we must first provide some definitions.

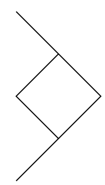


Figure 3: Structure for Type B_3

Definition 11. *For each order ideal $I \in ([n] \times [n])/S_2$, we associate a* ***lattice path*** *of length n, as a line that joins a distinct point that is part of the symmetric line of B_n, the line $x = 0$, and the right most vertex (n, n), such that the line traces over the top border of the order ideal.*

As an example, the lattice path of the empty order ideal and the order ideal of cardinality 1 for B_3 is shown in Figure 4 where each element is now represented by a square. This is different than the representation of B_3 in Figure 3.

Definition 12. *The lattice path that takes $(n, n) \to (0, k)$ for some even integer $k \in [0, 2n]$ is equivalent to the graph of the (real) piecewise-linear function h_I which outputs the y-coordinate of each point on the lattice path. We call h_I the* ***height function*** *representation of the order ideal I.*

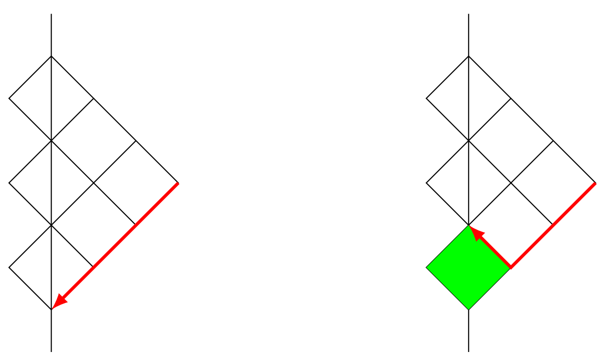
For a particular height function, we associate a word, a sequence of $+1's$ and $-1's$, whose ith term for $1 \le i \le n$ is $h_I(i) - h_I(i-1) = \pm 1$; we call this the **sign-word** associated with

Figure 4: Examples of Lattice Path

Type B_3: Empty Order Ideal Type B_3: Order Ideal of Cardinality 1

the order ideal I. Note that this sign-word simply describes the slopes of a lattice path as it goes from (n, n) to the symmetric line, and that this sign-word uniquely determines order ideal associated with it. Figure 5 illustrates the sign words associated with the lattice paths in Figure 4.

Figure 5: Corresponding Sign Word of Lattice Paths



Sign Word: $\{-1, -1, -1\}$ Sign Word: $\{-1, -1, +1\}$

Proposition 3. *Let $I \in ([n] \times [n])/S_2$ be characterized by the height function $h_I : [n, n] \to \mathbb{R}$. Then*

$$\sum_{k=n}^{0} h_I(k) = \frac{n(n+1)}{2} + 2|I|. \tag{1}$$

The proof is essentially only a matter of examining an invariant. As such, it is deferred to Appendix A.

Because there exists a bijection between the size of the order ideal and the height function sum, to prove that the cardinality statistic of the order ideal I is homomesic it suffices to show that the sum $h_I(k) + h_I(k-1) + \ldots + h_I(1) + h_I(0)$ is homomesic.

There is a very nice representation of the rowmotion orbit Φ_J in terms of the sign words. We show in Section 6 that this representation is also bijection.

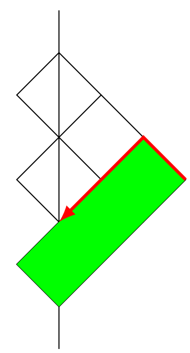


Figure 6: Order Ideal of Cardinality 3 on B_3

Given an order ideal and its corresponding sign-word, complete the square by reflecting the order ideal over the symmetric line, and extend the corresponding sign-word. The operation just described is shown through the order ideal in Figure 6. The resulting figure is shown in Figure 7.

The sign-word of Figure 6, $\{+1, -1, -1\}$, becomes $\{+1, -1, -1, +1, +1, -1\}$, the sign word of Figure 7, which we denote as the **extended sign word**.

Lemma 1. *The number of individual $+$'s and $-$'s is invariant in the extended sign word.*

The proof can be found in Appendix B.1. We define a **block** to be of the form $\{-1, +1\}$, and a **gap** to be every sequence of numbers between these blocks. We give a definition of rowmotion in terms of blocks and gaps.

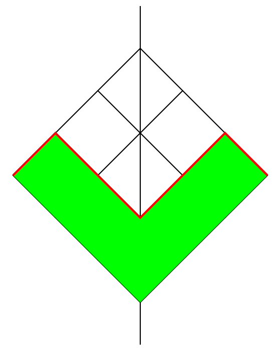


Figure 7: Reflected Order Ideal of Cardinality 3 on B_3

Lemma 2. *Rowmotion acts by the following rules:*

1. *Every block* $\{-1, +1\}$ *becomes* $\{+1, -1\}$.

2. *Every gap sequence is completely reversed.*

The proof can be found in Appendix B.2.

Following these rules, the sign word of Figure 7, $\{+1, -1, -1, +1, +1, -1\}$, becomes $\{-1, +1, +1, -1, -1, +1\}$. We give a lemma to describe the conjugate sign word across all rowmotion orbits.

Lemma 3. *The second half of the extended sign-word is always the conjugate to the first half, and this is conserved through the action of* Φ_J.

The proof is provided in Appendix B.3.

Note that the rules illustrate that Φ_J is equivalent to applying a 180° rotation on each lattice-path segment that corresponds to a block or a gap in the sign-word. Truncating the resulting extended sign word at the symmetric line gives us $\{-1, +1, +1\}$, which is our desired sign word that describes the next order ideal. However, note that we only need to truncate the extended sign-word if we want to see the order ideal. We now provide the proof of the cardinality homomesy in the case of B_n.

Theorem 1. *The cardinality statistic is homomesic under the action of rowmotion Φ_J on B_n, with all averages equal to $\frac{n(n+1)}{4}$.*

Proof. To show that $\#I$ is homomesic under the rowmotion, it is sufficient to prove that all increments $h_I(i) - h_I(i-1)$ are homomesic. Theorem 1.5 from Rush and Shi [7] demonstrates that the $|\Phi_J|$, the number of distinct order ideals in Φ_J, divides the Coxeter number of the corresponding Coxeter group. Note that the irreducible minuscule lattice of these minuscule posets are formed from the generators of its corresponding Weyl group, which is considered a Coxeter group. Therefore, because the Coxeter number of B_n is $2n$, it follows that $|\Phi_J|$ divides $2n$. The proof of Theorem 1 is now equivalent to showing that any single element of the extended sign-word is independent of the order ideal I under the action of Φ_J.

Create a rectangular array with $2n$ rows and $2n$ columns ($2n$ for the length of the sign-word and its corresponding conjugate). The rows contain the sign-words of I and its conjugate and each successive row is the image of the entire extended sign-word under the action of Φ. Now consider any two consecutive columns of the array, and the width-2 sub-array they form. There are only four possible combinations of values in each row of the sub-array: $(-1, -1)$, $(-1, +1)$, $(+1, -1)$, $(+1, +1)$. By Lemma 2 any sequence $\{-1, +1\}$ gets sent to $\{+1, -1\}$. The reverse occurs as a result of the toggle definition of rowmotion and the shape of the extended B_n structure. As a result, we have that the number of $\{+1, -1\}$ is the same as the number of $\{-1, +1\}$ between any two consecutive columns. Thus, the sum of any two consecutive columns is identical because the other two possible values increments the column sums by the same value, hence conserving the both column sums. Thus, all the columns have the same column sum. However, Lemma 3 implies that the sum of every row is 0. Thus, the total sum of the $2n \times 2n$ array is $0 \cdot 2n = 0$. As a result, the sum of each column is also 0. Since this is independent of which rowmotion orbit we are in, we have proved homomesy for elements of the sign-word of I as I varies over $([n] \times [n])/S_2$, and this finishes the homomesic part of Theorem 1. The computation of the average cardinality can be found in Appendix

D. ∎

4.3 Type C_n: $[2n-1]$

Figure 8: Structure for Type C_3

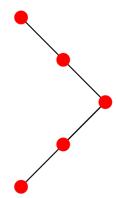


Figure 8 gives an example of the structures C_n, which are right triangles with sides of n vertices without their hypotenuses.

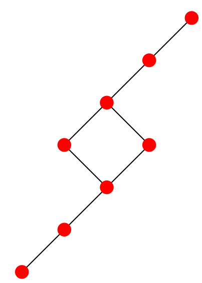
Theorem 2. *The cardinality statistic is homomesic under the action of rowmotion Φ_J on C_n, with all averages equal to $\frac{2n-1}{2}$.*

Proof. To go through all the order ideals of C_n, we only need one rowmotion orbit that starts with the empty order ideal. The cardinality of the order ideal at each successive step increments by one, so the general formula for the sum of the cardinalities of each order ideal on this orbit is

$$\sum_{i=0}^{2n-1} i = 2n^2 - n.$$

The cardinality statistic is the average of the cardinalities of the order ideals over an orbit. Thus, the average is $\frac{2n^2-n}{2n} = \frac{2n-1}{2}$. Because there is only one orbit, the cardinality statistic for C_n is homomesic. ∎

Figure 9: Structure for Type D_5

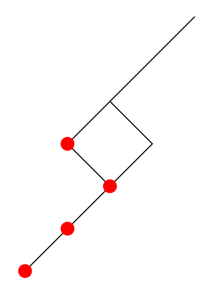


4.4 Type D_n: $([n-1] \times [n-1])/S_2$

Figure 9 gives an example of the structures D_n, which are 1×1 diamonds with two arms of length $n-3$ off the top and bottom vertices of the diamond.

Theorem 3. *The cardinality statistic is homomesic under the action of rowmotion Φ_J on D_n, with all averages equal to $n-1$.*

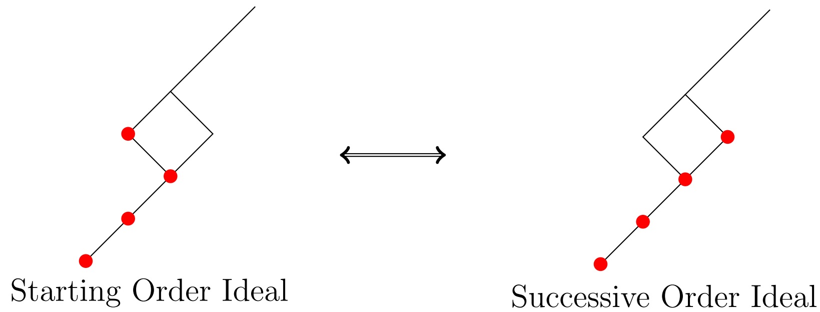
Figure 10: Starting Order Ideal for Second Orbit for D_5



Proof. There are two possible orbits for each structure D_n for some n to cycle through all possible order ideals of the structure D_n. One orbit starts with the empty order ideal which we will refer to as *orbit 1*, while the other orbit, which we will refer to as *orbit 2*, starts with the order ideal in Figure 10.

We start by examining *orbit 2*. The rowmotion orbit Φ_J corresponding to *orbit 2* is illustrated in Figure 11.

Figure 11: Rowmotion on Orbit 2



For D_5 the average cardinality of the order ideal through *orbit 2* is simply $\frac{4+4}{2} = 4$. Note that for all n, there exists an orbit similar to *orbit 2*, that contains two order ideals in D_n. This orbit's path is identical to the orbit path illustrated in Figure 11. The average cardinality of the order ideal in this type of orbit is simply $\frac{2(n-1)}{2} = n - 1$, since the size of each order ideal is $n - 1$.

We now examine *orbit 1*. The cardinality of the order ideal as it goes through the first orbit increments by one at each step until we reach the bottom vertex of the diamond in the structure D_n. At this step, we increment the cardinality of the order ideal by two, and at each successive step we increment by the cardinality of the order ideal similar to the way in which the cardinality was incremented in the first part of the orbit before the bottom vertex of the diamond was reached. So, for a general D_n, the total sum of the cardinalities of the order ideals over the first orbit is simply:

$$\left(\sum_{i=0}^{2n-2} i\right) - (n-1) = (n-1)(2n-2).$$

Note that $|\Phi_1|$ in D_n is simply $2n-2$ so, our average is $\frac{(n-1)(2n-2)}{2n-2} = n - 1$, which is identical to the average obtained in the second orbit. Thus for any structure in D_n, we have that the cardinality statistic is homomesic. $\square$

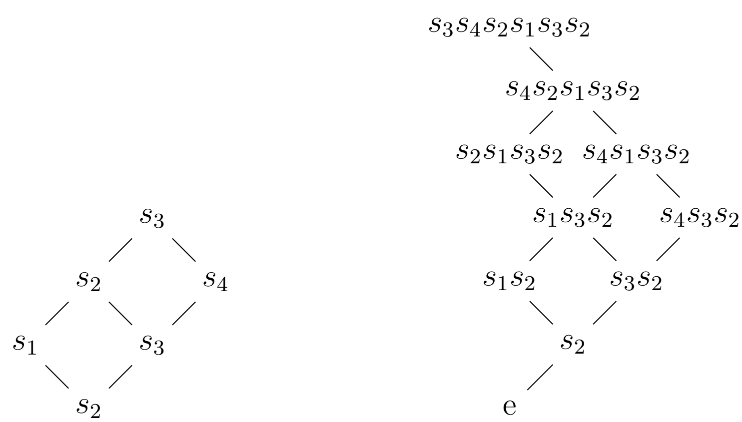
4.5 Types E_6: $J(([4] \times [4])/S_2)$, E_7: $J^2(([4] \times [4])/S_2)$

Theorem 4. *The cardinality statistic is homomesic under the action of rowmotion Φ_J on E_6, with all averages equal to* 8.

Theorem 5. *The cardinality statistic is homomesic under the action of rowmotion Φ_J on E_7, with all averages equal to* 13.5.

The proofs of Theorems 4 and 5 can be found in Appendix C.

5 Novel Characterization of Rowmotion on the Weights of A_n



Minuscule Poset A_4 Weight Poset A_4 with Generators

Figure 12: Structures for A_4

The minuscule poset A_n that was illustrated in Figure 2 has a corresponding weight poset based on the order ideals of the minuscule poset A_n. The elements of A_n are the join-irreducibles of its weight poset, but the minuscule poset is usually drawn with the generators of its corresponding Weyl Group, S_n, as its elements. The minuscule poset A_n with its elements and its corresponding weight poset is given in Figure 12.

$$
\begin{array}{ccccccc}
\varepsilon_4+\varepsilon_5 & & & & & & \\
 & \diagdown & & & & & \\
 & & \varepsilon_3+\varepsilon_5 & & & & \\
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 & \diagup & & & & & \\
\varepsilon_1+\varepsilon_2 & & & & & &
\end{array}
$$

Figure 13: The Weight Poset A_4

Each sequence of generators in the weight poset of A_n is a sequence of simple reflections under the corresponding simple root. The simple roots, α_i, of A_n are of the form $\alpha_i = \varepsilon_{i+1} - \varepsilon_i$ $(1 \leq i < n)$, and a generator s_i corresponds to the simple reflection under the simple root α_i. However, the simple reflection can be simplified.

Lemma 4. *In any minuscule poset P, concatenating a generator s_i with an weight λ, which corresponds to taking the weight λ to the weight $\lambda - \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$, is equivalent to adding the simple root α_i, where α_i is the corresponding simple root for generator s_i.*

The proof is given in Appendix E.1.

Combining the fact that the maximal weight in the weight poset of A_4 is $\varepsilon_4 + \varepsilon_5$ [6] with Lemma 4, we recreate the weight poset in terms of the positive roots. The resultant weight poset is shown in Figure 13. Using the roots of the weight poset of any minuscule poset of type A_n, we characterize rowmotion by a set of simple rules on the weights.

Theorem 6. *Given a weight $\varepsilon_{\alpha_1} + \varepsilon_{\alpha_2} + \cdots + \varepsilon_{\alpha_k}$, in A_n where the sequence $\alpha_1, \alpha_2, \ldots, \alpha_k$ is strictly increasing, the action of rowmotion Φ on it can be described as follows.*

Start with the leftmost term in the weight and apply the following steps.

1. *Given a term ε_{α_i}, increment α_i by 1.*

2. *For $i < k$, if $\alpha_i + 1 = \alpha_{i+1}$, then α_i becomes 1, otherwise $\alpha_i \to \alpha_i + 1$. If $i = k$ and $\alpha_k + 1 > n$, then α_k gets sent to 1.*

3. *For $2 \leq i < k$, if the new α_i is not greater than α_{i-1}, then keep incrementing α_i by 1 until the sequence $\alpha_1, \alpha_2, \ldots, \alpha_k$ is strictly increasing again. If $i = 1$ skip this step.*

4. *Move on to the next term in the weight.*

The proof is essentially applying the toggle definition of rowmotion. As such, it is deferred to Appendix E.2. Applying Theorem 6 to Figure 13, we get the following two orbits:

$$\varepsilon_1 + \varepsilon_2 \Rightarrow \varepsilon_1 + \varepsilon_3 \Rightarrow \varepsilon_2 + \varepsilon_4 \Rightarrow \varepsilon_3 + \varepsilon_5 \Rightarrow \varepsilon_4 + \varepsilon_5 \tag{2}$$

$$\varepsilon_2 + \varepsilon_3 \Rightarrow \varepsilon_1 + \varepsilon_4 \Rightarrow \varepsilon_2 + \varepsilon_5 \Rightarrow \varepsilon_3 + \varepsilon_4 \Rightarrow \varepsilon_1 + \varepsilon_5 \tag{3}$$

which is what we expect from the application of rowmotion on the minuscule poset A_4.

We now give the other characterization of rowmotion. For each weight of the weight poset in Figure 13, add the sum $\frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5)$. The weight poset in Figure 13 becomes the resulting weight poset in Figure 14. Notice that if one takes the signs of each ε_i from right to left, we obtain a sign word on the minuscule poset A_4 where each element represents a square. If we take the starting point of the sign word to be the rightmost vertex, we see that there exists a bijection between the transformed weights and the sign word on the minuscule poset A_4. In general, adding $\frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \cdots - \varepsilon_n)$ to each weight in A_n gives us this bijection as well. Figure 15 shows the bijection using the empty order ideal.

We now extend this bijection between the sign word and the weights to all classical types of minuscule posets.

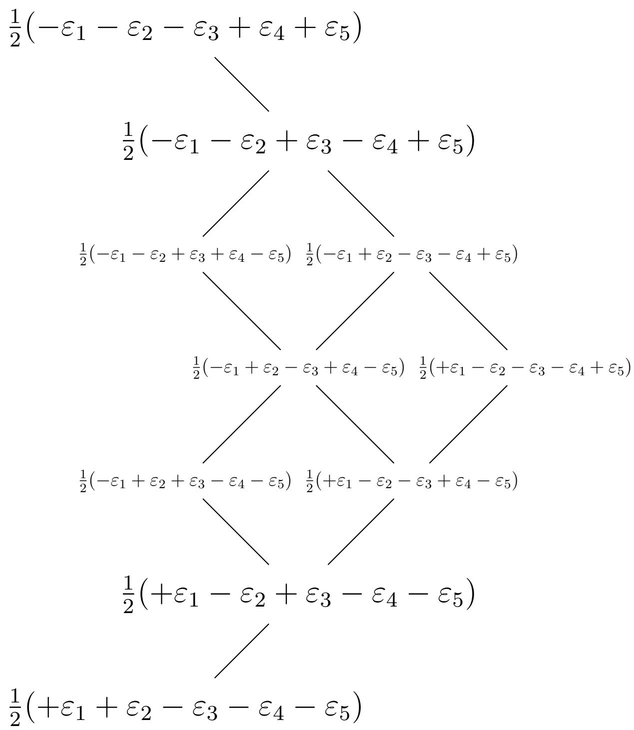

Figure 14: Transformed Weight Poset A_4

$$\varepsilon_1 + \varepsilon_2 \Rightarrow \frac{1}{2}(+\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5) \Longleftrightarrow$$

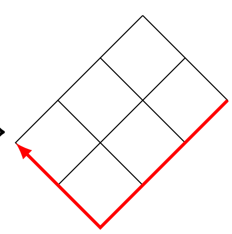

Figure 15: The positive root $\varepsilon_1 + \varepsilon_2$ gets sent to $\frac{1}{2}(+\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5)$ through the addition of $\frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5)$. When the signs of the resulting expression are taken from right to left, the obtained sign word is $\{-, -, -, +, +\}$ which is the sign word shown in the minuscule poset.

6 Extending the Bijection

Theorem 7. *There exists a bijection between the sign word and the weights in all classical types of minuscule posets.*

Proof. The bijection has already been shown to exist in type A_n. We now show the bijection in B_n.

The simple roots, α_i, of B_n are of the form $\alpha_1 = \varepsilon_1$, $\alpha_i = \varepsilon_i - \varepsilon_{i-1}$ $(2 \le i \le n)$ [6]. The maximal weight in the weight poset of B_n is given to be $\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)$. If one takes the signs of ε_i as one goes across this maximal weight, one gets the sign word that corresponds to the full order ideal based on the sign word associated with the lattice path defined in Definition 11. Note that the weight poset can also be written in terms of the generators of the corresponding Weyl Group, and that subtracting a generator from a generator sequence is the same as toggling out at the corresponding generator in the minuscule poset. Additionally, toggling out at each generator is equivalent to the rowmotion rule that sends every $\{-, +\}$ sub-word to the sub-word $\{+, -\}$. Toggling out at s_1 sends the $\{-, +\}$ sub-word that joins the conjugate sign word and the original sign word to $\{+, -\}$, which is equivalent to changing the sign of ε_1. Likewise, toggling out at s_i sends the $\{-, +\}$ sub-word that corresponds to the signs of ε_{i-1} and ε_i respectively to $\{+, -\}$. If we toggle by the reverse linear extension that is defined by going up the rows starting with the bottom vertex, by Proposition 2, its equivalent to a reverse rowmotion orbit. Since the action of toggling out by the reverse linear extension is equivalent to the reverse rowmotion rules on the weights and the reverse rowmotion rules on the sign word, there is a $1 - 1$ correspondence between the sign word and weights in B_n.

We now show the bijection in C_n. The simple roots, α_i, of C_n are of the form $\alpha_1 = 2\varepsilon_1$, $\alpha_i = \varepsilon_i - \varepsilon_{i-1}$ $(2 \le i \le n)$ [6]. The maximal weight in the weight poset of C_n is given by ε_n. The weight poset of C_n is one line that has the sequence $\{\varepsilon_n, \varepsilon_{n-1}, \ldots, \varepsilon_1, \varepsilon_{-1}, \ldots, \varepsilon_{-n}\}$

starting from the top vertex and going down by row. For the first half of the weight poset, $\{\varepsilon_n, \varepsilon_{n-1}, \dots, \varepsilon_1\}$, we add the sum $\frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \dots - \varepsilon_n)$, and for the second half of the weight poset $\{\varepsilon_{-1}, \dots, \varepsilon_{-n}\}$, we add the sum $\frac{1}{2}(+\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n)$. This creates two bijections between differently defined sign words on the minuscule poset C_n where each element is now a square. Examples of the bijections in C_5 are shown in Figure 16.

$$\varepsilon_3 \Rightarrow \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 - \varepsilon_5) \Longleftrightarrow$$

$$-\varepsilon_3 \Rightarrow \frac{1}{2}(+\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4 + \varepsilon_5) \Longleftrightarrow$$

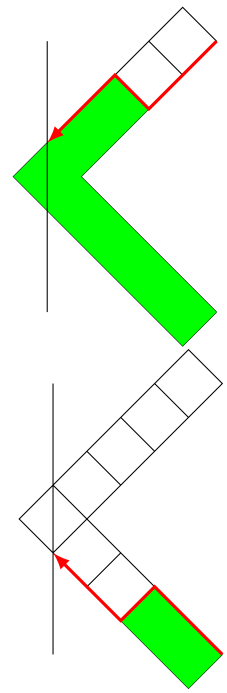

Figure 16: The Two Bijections in C_n

For each bijection, the corresponding sign word for each weight follow the rowmotion rules. Although there are two bijections, the weights still have a $1-1$ correspondence to sign words.

For D_n the two bijections are analogous to those in C_n. An example of the bijections is shown in Figure 19 in Appendix E.3. $\square$

7 Showing the Homomesy using the Uniform Bijection

For type A_n, there exists a bijection between the sign word, defined on the lattice path that starts from the rightmost vertex and travels to the leftmost vertex along the top border of an order ideal, and the roots in the weight poset of A_n. We apply the rowmotion rules on the signs of the weights and each resulting weight corresponds to the next order ideal after the rowmotion Φ has been applied. Using a technique similar to that used in the proof in Theorem 1, where we put the signs of each ε_i from right to left in a row, we show that each row sum is $b - a$. This makes the grand sum $(a + b)(b - a)$, and each column sum $b - a$ no matter which rowmotion orbit is taken. Therefore, the homomesy can be shown in type A_n.

Since there exists a bijection between the sign word and the weights in B_n, where the sign word is identical to taking the signs of the ε_i's from right to left in each weight, we can apply the same proof that was used in Theorem 1, where each row will now contain the signs of the ε_i's.

Since there exists a bijection between the first half of the weights and a sign word in C_n, we apply a reasoning similar to that used in the proof of Theorem 1 and show that the column sums are all the same in the array corresponding to first half of the orbit. Likewise, since there exists another bijection between the second half of the weights and a sign word in C_n, the column sums in the array corresponding to the second half of the orbit are the same. Therefore, if we combine the two arrays by placing the array of the first half of the orbit on top of the array of the second half of the orbit, we also have that each column sum is the same. By reasoning similar to that used in the proof of Theorem 1, the cardinality statistic in C_n is homomesic as well.

The proof of the homomesy in D_n is analogous to the proof of C_n because in both distinct orbits of D_n, we can split the array that describes the entire rowmotion orbit into two arrays; one array that describes rowmotion on the positive roots, and another that describes rowmotion on the negative roots. Likewise, we can show that the column sums in each array are the same and therefore, the columns sums in the original array are also all the same. Hence the homomesy of the cardinality statistic can be proven from the roots of D_n.

8 Concluding Remarks

We have shown that the cardinality statistic is homomesic in all existing minuscule posets, through a variety of methods. Using the sign word as a motivation, we characterized rowmotion uniformly across the classical minuscule posets, A_n, B_n, C_n, and D_n, in terms of the weights of each Cartan type. We plan to attempt to extend the characterization of rowmotion to the exceptional minuscule cases, E_6 and E_7, and once a totally uniform characterization is complete, we can investigate not only the cardinality homomesy but perhaps other types of homomesies that appear in rowmotion as well.

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A Proof of Height Function Formula

Proposition 3. *Let $I \in ([n]\times[n])/S_2$ be characterized by the height function $h_I : [n,n] \to \mathbb{R}$. Then*

$$\sum_{k=n}^{0} h_I(k) = \frac{n(n+1)}{2} + 2|I|.$$

Proof. When the order ideal has size 0, the height function is summing the y-coordinates of the lattice points on the lattice path that takes the point (n,n) to $(0,0)$ in a straight line. This sum is

$$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$$

Each time the size of an order ideal increases by 1, there exists either a 3-point sub-line in the lattice path, say $(i,j) \to (i-1,j-1) \to (i-2,j+1)$, or a 2-point sub-line, $(i,j) \to (i-1,j-1)$, that gets changed to $(i,j) \to (i-1,j+1) \to (i-2,j+1)$, or $(i,j) \to (i-1,j+2)$, respectively. In both cases, the overall sum of $h_I(k)$ is increased by 2. Thus, the total sum of the height function increments by 2 every time the cardinality of $|I|$ increases by 1 and equation 1 is obtained. $\square$

B Rowmotion through the Extended Sign Word

B.1 Proof of Lemma 1

Lemma 1. *The number of individual $+$'s and $-$'s is invariant in the extended sign word.*

Proof. The lattice path that the extended sign word describes is a path that travels from the leftmost vertex to the rightmost vertex of a $[n]\times[n]$ Type A_n poset. If we rotate the diamond $45°$ clockwise, we see that the lattice path is equivalent to getting to the bottom-right corner of a $n \times n$ square from the upper-left one, moving only down or right. In any such path, the number of down and right movements must always be equal to n. Hence, since the down and right moves are bijective to $-$ and $+$ moves, the proof is complete. □

B.2 Proof of Lemma 2

Lemma 2. *Rowmotion acts by the following rules:*

1. *Every block* $\{-1, +1\}$ *becomes* $\{+1, -1\}$.

2. *Every gap sequence is completely reversed.*

Proof. Note that every sub-word $\{-1, +1\}$ represents a minimal element, and every sub-word $\{+1, -1\}$ represents a maximal element, of the order ideal corresponding to total sign-word. Therefore, by Definition 3, every $\{-1, +1\}$ sub-word must become $\{+1, -1\}$ which proves *Rule 1*. Now we only have to consider the maximal elements in the original order ideal.

Note that every maximal element that exists in the original order ideal must exist within a gap sequence. Additionally, only one maximal element can be in a gap sequence or else there must exist a place within the gap sequence where the sub-word $\{-1, +1\}$ appears which contradicts the definition of a gap sequence. Therefore, a gap sequence must be a nonnegative number of $+$'s followed by a nonnegative number of $-$'s. If a gap sequence contains only either $+$'s or $-$'s, there exists no maximal element within this gap and therefore this type of gap sequence is preserved, which follows *Rule 2*. If there is a maximal element in the gap sequence, by Definition 3 every maximal element in the original order ideal cannot be maximal anymore, so the $\{+1, -1\}$ sub-word in the gap sequence must change. The three possible sub-words that it can become are $\{+1, +1\}$, $\{-1, -1\}$, or $\{-1, +1\}$. However, the first two contradicts Lemma 1. Hence it must become $\{-1, +1\}$. This is equivalent to toggling out the maximal element. However, this creates other $\{+1, -1\}$ sequences that also need to be changed. Therefore, the way to eliminate all maximal elements in a gap sequence is to successively change each $\{+1, -1\}$ to a $\{-1, +1\}$, until there exists no $\{+1, -1\}$ sub-words anymore. Note that this action will not terminate until following every $+$ is another $+$. The only possible sequence where this is possible is a sequence that has a nonnegative number of $-$'s followed by a nonnegative number of $+$'s. Note that this action has to be finite or it implies that there exists an infinite number of maximal elements that can be toggled out in a finite poset which is a contradiction. This transformation of the original gap to such a sequence is identical to *Rule 2* and the proof is complete. □

B.3 Proof of Lemma 3

Lemma 3. *The second half of the extended sign-word is always the conjugate to the first half, and this is conserved through the action of* Φ_J.

Proof. Note that the action of rowmotion sends symmetric order ideals to other symmetric order ideals. This can be seen if we divide the symmetric order ideal along its symmetric line. If a minimal or maximal element exists on one side of or on the symmetric line, then another minimal or maximal element also exists correspondingly on the other side of or on the symmetric line. Hence, rowmotion acts identically on both sides of the symmetric line and conserves the symmetry of the order ideal. Since the extended sign word is defined to be the reflection of an order ideal in B_n over the symmetric line of B_n, the order ideal formed is symmetric, and is always symmetric through any rowmotion orbit. Hence, since the second half's conjugation is representative of the symmetry, it is also conserved. □

C Appendix: Proof of E_6 and E_7

C.1 Proof of Type E_6

Theorem 4. *The cardinality statistic is homomesic under the action of rowmotion* Φ_J *on* E_6*, with all averages equal to* 8.

Proof. This section was solved by hand by drawing out all three possible orbits that successfully covered all 27 possible order ideals. The average cardinality is 8 and the size of the orbits are 12, 12, and 3 respectively. □

C.2 Proof of Type E_7

Theorem 5. *The cardinality statistic is homomesic under the action of rowmotion Φ_J on E_7, with all averages equal to* 13.5.

Proof. This section was solved by hand by drawing out all four possible orbits that successfully covered all 56 possible order ideals. The average cardinality is 13.5 and the size of the orbits are 18, 18, and 18, and 2 respectively. □

D Computation of Average Cardinality in B_n

$$
\begin{array}{l}
0 \\
0 + 1 \\
0 + 1 + 1 \\
0 + 1 + 1 + 2 \\
0 + 1 + 1 + 2 + 2 \\
0 + 1 + 1 + 2 + 2 + 3 \\
0 + 1 + 1 + 2 + 2 + 3 + 2 \\
0 + 1 + 1 + 2 + 2 + 3 + 2 + 2 \\
0 + 1 + 1 + 2 + 2 + 3 + 2 + 2 + 1 \\
0 + 1 + 1 + 2 + 2 + 3 + 2 + 2 + 1 + 1
\end{array}
$$

Figure 17: Sizes on Odd Orbit Defined by Empty Order Ideal in B_5

The average cardinality of $\frac{n(n+1)}{4}$ is derived from the orbit defined by the empty order ideal. We can prove this average cardinality by splitting B_n into cases with even or odd n. We can examine the summation in easier terms if we arrange the size of the order ideal in a bottom left triangle arrangement like in Figure 17 and Figure 18, where each successive row

$$
\begin{array}{l}
0 \\
0 + 1 \\
0 + 1 + 1 \\
0 + 1 + 1 + 2 \\
0 + 1 + 1 + 2 + 2 \\
0 + 1 + 1 + 2 + 2 + 2 \\
0 + 1 + 1 + 2 + 2 + 2 + 1 \\
0 + 1 + 1 + 2 + 2 + 2 + 1 + 1
\end{array}
$$

Figure 18: Sizes on Even Orbit Defined by Empty Order Ideal in B_4

represents the next order ideal in the rowmotion orbit starting with the empty order ideal. In Figure 17, each number from 1 to $\frac{n-1}{2}$ appears symmetrically twice by column on both sides of the column with entries $\frac{n+1}{2}$, where $n = 5$. If we look at the columns immediately before and after the the column with entries $\frac{n+1}{2}$ and count the number of entries we see that it is 10 or $2n$, where $n = 5$. In general for odd n, the number of entries in two columns that are symmetric across the column with entries $\frac{n+1}{2}$, always is $2n$. Additionally, the number of times the entry $\frac{n+1}{2}$ appears is n in the case of odd n. Therefore, for odd n the sum is $(4n)(1 + 2 + \ldots + \frac{n-1}{2}) + (n)(\frac{n+1}{2})$. The sum simplifies to $\frac{(n)^2(n+1)}{2}$. Since the total number of order ideals in the odd case is $2n$ the average cardinality is $\frac{(n)(n+1)}{4}$ as desired. We look at the even case through the bottom left triangle arrangement in Figure 18. In Figure 18 we see that a similar symmetric structure to that in Figure 17 appears. From this, we can deduce that for even n our sum is $(4n)(1 + 2 + \ldots + \frac{n-2}{2}) + (3n)(\frac{n}{2})$. The sum simplifies to $\frac{(n)^2(n+1)}{2}$, the same sum in the odd case. The total number of order ideals in the even case is also $2n$, and thus the average cardinality is $\frac{(n)(n+1)}{4}$ as desired, which proves the second part of Theorem 1.

E Showing the Representation Theory

E.1 Proof of Lemma 4

Before we begin the proof we must first give a crucial definition:

Definition 13. *A subgroup of W, say W_J, is a* ***maximal parabolic subgroup stabilizing*** *λ, where λ is the maximal weight when the weights of W are ordered by the root order, if $W_J \subset W$ and if for all elements $w_J \in W_J$, $w_J\lambda = \lambda$. The standard notation for a maximal parabolic subgroup stabilizing λ is W_J.*

We can now give the proof of the lemma.

Lemma 4. *Applying the generator s_i that corresponds to taking a weight λ to the weight $\lambda - \frac{2(\lambda,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$, is equivalent to adding the simple root α_i.*

Proof. Through Definition 7, Definition 13, and Proposition 4.1 from [8], we can give an alternate description of the weight poset in Definition 8.

Let W_J be the maximal parabolic subgroup of W stabilizing λ, and let W^J be the set of minimum-length coset representatives for the quotient W/W_J. Then there exists bijection such that

$$W^J \to W\lambda.$$

This bijection implies that there exists a map that takes $\omega \mapsto \omega_0\omega\lambda$, where ω is an element of W, and ω_0 is the longest element of W. This mapping is an isomorphism between the root order on $W\lambda$ and *strong Bruhat order* on W restricted to W^J. For more information on the *Bruhat order* the author refers the reader to [8].

Therefore, there exists an isomorphism between the weight poset in Figure 12 and the weight poset in Figure 13, and hence the covering relations between the posets are isomorphic. The covering relation in Figure 12 is defined on by the *strong Bruhat order* where $s_iw \gtrdot w$, only if both w and s_iw are reduced words. The covering relation in Figure 13 is defined on the root order, where $\lambda \gtrdot \mu$ if $\lambda - \mu$ is a simple root. Because of the isomorphism, the concatenation of a generator s_i to a reduced word in the Bruhat lattice, is equivalent to adding the corresponding simple root to the corresponding root in the minuscule lattice and the proof is complete. $\square$

E.2 Proof of Theorem 6

Theorem 6. *Given a weight $\varepsilon_{\alpha_1} + \varepsilon_{\alpha_2} + \cdots + \varepsilon_{\alpha_k}$, in A_n where the sequence $\alpha_1, \alpha_2, \ldots, \alpha_k$ is strictly increasing, the action of rowmotion Φ on it can be described as follows.*

Start with the leftmost term in the weight and apply the following steps.

1. *Given a term ε_{α_i}, increment α_i by 1.*

2. *For $i < k$, if $\alpha_i + 1 = \alpha_{i+1}$, then α_i becomes 1, otherwise $\alpha_i \to \alpha_i + 1$. If $i = k$ and $\alpha_k + 1 > n$, then α_k gets sent to 1.*

3. *For $2 \leq i < k$, if the new α_i is not greater than α_{i-1}, then keep incrementing α_i by 1 until the sequence $\alpha_1, \alpha_2, \ldots, \alpha_k$ is strictly increasing again. If $i = 1$ skip this step.*

4. *Move on to the next term in the weight.*

Proof. The set of rules described in Theorem 6 is equivalent to toggling across the diagonals of the minuscule poset A_n starting with the top vertex. Note that this path is a linear extension as it is order-preserving. By Proposition 2, this toggling is equivalent to an action of rowmotion Φ on an order ideal and the proof is complete. $\square$

E.3 Bijections in D_n

This section illustrates the bijections that occurs in D_n. The bijections are almost identical to C_n, except for the extra element that is included in the order ideals corresponding to the positive roots in D_n.

$$\varepsilon_3 \Rightarrow \tfrac{1}{2}(-\varepsilon_1 - \varepsilon_2 + \varepsilon_3 - \varepsilon_4 - \varepsilon_5) \Longleftrightarrow$$

$$-\varepsilon_3 \Rightarrow \tfrac{1}{2}(+\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4 + \varepsilon_5) \Longleftrightarrow$$

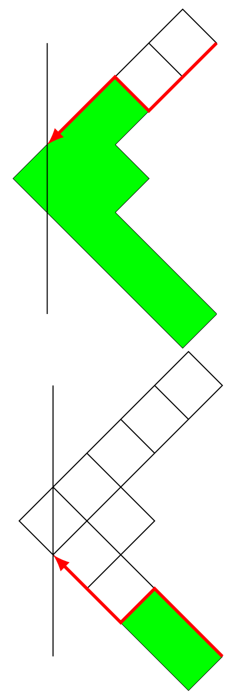

Figure 19: The Two Bijections in D_n