

Research Science Institute (RSI)  
Mathematics Section  
Abstracts of Final Research Papers  
M.I.T.  
Summer 2014

Kavish Gandhi

Minimal Saturated Subgraphs of the Hypercube

under the direction of Mr. Chiheon Kim

**Abstract**

Within the hypercube  $Q_n$ , we investigate bounds on the *saturation number* of a forbidden graph  $G$ , defined as the minimum number of edges in a subgraph  $H$  of  $Q_n$  that is both  $G$ -free and has the property that the addition of any  $e \in E(Q_n) \setminus E(H)$  creates  $G$ . For all graphs  $G$ , we find a lower bound based on the minimum degree of non-leaves. For upper bounds, we first examine general graphs and derive conditions that, if satisfied, allow us to bound the saturation number. We also study specific cases, finding improved bounds for paths, stars, and most caterpillars. In all of these cases, we find bounds that are  $O(2^n)$ , an interesting fact that we conjecture to hold for all trees  $T$ .

Petar Gaydarov

Vector Parking Functions and Tree Inversions

under the direction of Mr. Samuel Hopkins, Dr. Pavel Etingof

**Abstract**

We find a depth-first search version of Dhar's burning algorithm that gives a bijection between the parking functions of a multigraph and its spanning trees. Thus we extend a result by Perkinson, Yang and Yu in response to a problem posed by Stanley. We also find another variant of this algorithm which gives a bijection between vector parking functions and labeled spanning trees closely related to the rooted planar trees. Both bijections have the goal of establishing a relation between the degree of a parking function, the  $\kappa$ -statistic for inversions, and the edge labelling of a tree. In addition, we find intriguing formulas for the number of vector parking functions in two special cases of particular interest.

Noah Golowich

Acyclic Colorings and Subgraphs of Directed Graphs

under the direction of David Rolnick, Professor Jacob Fox,  
Professor Pavel Etingof

**Abstract**

The *acyclic chromatic number* of a directed graph  $D$ , denoted  $\chi_A(D)$ , is the minimum positive integer  $k$  such that there exists a decomposition of the vertices of  $D$  into  $k$  disjoint sets, each of which induces an acyclic subgraph. We show that for all digraphs  $D$  without directed 2-cycles, we have  $\chi_A(D) \leq \frac{4}{5} \cdot \bar{\Delta}(D) + o(\bar{\Delta}(D))$ , where  $\bar{\Delta}(D)$  denotes the maximum arithmetic mean of the out-degree and the in-degree of a vertex in  $D$ . This result significantly improves a bound of Mohar and Harutyunyan. A related question to finding  $\chi_A(D)$  is finding the maximum size of an acyclic induced subgraph of  $D$ . We partially resolve a conjecture of Harutyunyan that all planar digraphs on  $n$  vertices have an acyclic induced subgraph of size  $3n/5$ . We also improve several existing lower bounds on the size of an acyclic induced subgraph for general digraphs.

# Lev Kendrick

On the quotient of the second and third terms of the lower central series for certain quotients of a free algebra under the direction of Mr. Augustus Lonergan and Prof. Pavel Etingof

## Abstract

We take  $r$ ,  $s$ , and  $n$  in  $\mathbb{Z}^+$  such that  $(r, s) = 1$ . Let  $L_k(R)$  be the lower central series of a ring  $R$ , and let  $B_k = L_k/L_{k+1}$ . We prove that the homomorphism developed by Feigin and Shoikhet in 2007 induces an isomorphism from  $B_2(\mathbb{C}\langle x, y \rangle / \langle P \rangle)$  to  $\Omega_{(\mathbb{C}[x, y] / \langle P \rangle) / \mathbb{C}}^2$ , for quasihomogeneous polynomials  $P$  with abelianizations either of the form  $\alpha x^{nr} + \beta y^{ns}$ , the form  $x(\alpha x^{nr} + \beta y^{ns})y$ , or the form  $x(\alpha x^{nr} + \beta y^{ns})$ , where  $\deg x = s$ ,  $\deg y = r$  and  $\alpha, \beta \in \mathbb{C}^\times$ . We find bases for and the Hilbert-Poincaré series of these  $B_2$ .

Shashwat Kishore

Hermitian forms on quantum  $\mathfrak{sl}_2$  modules and  
applications to critical point theory

under the direction of Mr. Gus Lonergan and Prof. Pavel  
Etingof

**Abstract**

The representation theory of the Lie algebra  $\mathfrak{sl}_2$  often appears in particle physics. Of particular importance is the Clebsch-Gordan decomposition for the tensor product of two finite dimensional simple  $\mathfrak{sl}_2$  representations. This decomposition allows us to express the combination of the quantum states of two particles. In this paper we compute a  $q$ -analogue of the form-preserving Clebsch-Gordan decomposition for  $q$ -deformed  $\mathfrak{sl}_2$  representations. We then analyze a representation theoretic bound on the number of special critical points of the two-variable  $\mathfrak{sl}_2$  master function. We show that the representation theoretic bound is not tight as long as there are more than three parameters, and we prove a new bound using geometric methods. We extend our results to modified master functions and classify certain types of unitary representations of a tensor power of  $\mathfrak{sl}_2$ . Our work has applications to the representation theory of quantum groups, the Gaudin model in particle physics, and the critical point theory of Lie algebra master functions.

Yelena Mandelshtam

Arrangements and Amounts of Equal Minors in Totally  
Positive Matrices

under the direction of Miriam Farber, Dr. Tanya Khovanova,  
Dr. Pavel Etingof, and Dr. David Jerison

**Abstract**

Understanding the positive Grassmannian has been a highly significant question in mathematics for several decades. We discuss arrangements of equal minors in the totally positive Grassmannian. It was previously shown that arrangements of equal minors of largest value correspond to the simplices in Sturmfels triangulation. Here we discuss arrangements of equal minors of second largest value, and show that they correspond to the facets of Sturmfels triangulation. We then define the notion of cubical distance and obtain a general conjecture that provides a correspondence between minors of  $m$ -largest value and maximal simplices of certain cubical distance in Sturmfels triangulation. We prove this conjecture for the case  $m = 3$ , and also for certain cases in  $Gr^+(2, n)$ .

David Stoner

On the Minimal Reset Words of Synchronizing  
Automata

under the direction of Mr. Chiheon Kim

**Abstract**

Cerny's conjecture is a 50 year old conjecture which concerns the combinatorial field of synchronizing automata. In particular, it postulates that the maximal length of the minimal reset word among all  $n$ -state automata is  $(n - 1)^2$ . We present a proof for Pin's Theorem, which applies Cerny's conjecture to  $p$ -state automata consisting of a cycle and a non-permutation, where  $p \geq 3$  is an odd prime. We also introduce families of the form  $F(p, k)$  of automata which consist of a cycle and a group of  $k$  simple merging arcs, and we define  $C(p, k)$  to be the maximal length of minimal reset words within these families. We provide a lower bound of  $C(p, k)$  for general  $k$ , and we find with proof the exact value of  $C(p, 2)$ .

Peter Tian

Pattern avoidance in  $d$ -dimensional 0-1 matrices and  
ordered hypergraphs

under the direction of Mr. Jesse Geneson

**Abstract**

Pattern avoidance is a central topic in combinatorics and graph theory. A  $d$ -dimensional zero-one matrix  $A$  avoids another  $d$ -dimensional zero-one matrix  $P$  if no submatrix of  $A$  can be transformed into  $P$  by changing some ones to zeros. The fundamental problem is to study the maximum number, denoted by  $f(n, P, d)$ , of ones in a  $d$ -dimensional  $n \times \cdots \times n$  matrix that avoids  $P$ . A related problem in graph theory is to find the maximum possible value of the sum of the sizes of all edges, denoted by  $ex_i(n, G)$ , in a hypergraph on  $n$  vertices that avoids another hypergraph  $G$ .

We improve the upper bound on  $\frac{f(n, P, d)}{n^{d-1}}$  from  $2^{O(k \log k)}$  to  $2^{O(k)}$  for all  $d$ -dimensional permutation matrices  $P$  of size  $k \times \cdots \times k$ . We show that this new upper bound is also true for all  $j$ -tuple permutation matrices of size  $jk \times k \times \cdots \times k$ . We also show that  $ex_i(n, G) = \Theta(n)$  for all tuple permutation matchings  $G$ . This tight bound is the lowest possible order for  $ex_i(n, G)$ . Finally we also show that there are infinitely many ordered hypergraphs  $G$  such that  $ex_i(n, G)$  is minimally nonlinear.

Kelvin Wang

Homomesy in Minuscule Posets

under the direction of David B Rush

**Abstract**

An action, known as rowmotion, defined on the order ideals of posets created by Duchet is analyzed in posets arising from the minuscule representations of complex simple Lie algebras. We consider the homomesies on this action in all minuscule posets from a combinatorial perspective through the use of a slope-based representation. We examine the cardinality statistic defined on the size of the order ideals in each rowmotion orbit and prove that the cardinality statistic is homomesic in all minuscule posets. A uniform characterization of rowmotion is given by investigation of a bijection between the weights of the minuscule poset and its corresponding sign word, in all classical minuscule posets. The cardinality homomesy is also shown through the uniform characterization.

Ingrid Zhang

Homomesy of Alignments in Perfect Matchings  
under the direction of Sam Hopkins

**Abstract**

We investigate the existence of a group action  $\tau$  that is homomesic with respect to alignments, a type of statistic in perfect matchings. Homomesy is defined as the consistency of an average, and perfect matchings are defined as the set of all partitions of  $1$  to  $2n$  into pairs. We take advantage of the bijection between labeled Dyck paths and perfect matchings to investigate the possibility of defining  $\tau$  inductively. We also relate perfect matchings to the oscillating tableaux and discover relationships that support the possibility of the existence of  $\tau$  but do not prove it. We find some surprisingly clean formulae for statistical averages of the analogue to the number of alignments for oscillating tableaux of arbitrary shape, which suggests that  $\tau$  may exist for more general contexts than just perfect matchings.

Jesse Zhang

Laplacians Associated with Regular Bipartite Graphs  
and Application to Finite Projective Planes

under the direction of David B Rush

**Abstract**

Internet search technology is a pervasively used utility that relies on techniques from the field of spectral graph theory. We present a novel spectral approach to investigate an existing problem: the critical group of the line graph has been characterized for regular nonbipartite graphs, but the general regular bipartite case remains open. Our approach aims to obtain the relationship between the spectra of the Laplacians of the graph  $G$  and its line graph  $\hat{G}$ . We obtain a theorem for the spectra of all regular bipartite graphs and demonstrate its effectiveness by completely characterizing the previously unknown critical group for a particular class of regular bipartite graphs, the incidence graphs of finite projective planes with square order. This critical group is found to be  $\mathbb{Z}_2 \oplus (\mathbb{Z}_{q+1})^{q^3-1} \oplus (\mathbb{Z}_{q^2+q+1})^{q^2+q-1}$ , where  $q$  is the order of the finite projective plane.