Research Science Institute (RSI)

Mathematics Section

Abstracts of Final Research Papers

M.I.T.

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Matthew Babbitt

On Visibility Graphs — Upper Bounds and Classification of Special Types

under the direction of Jesse Geneson

Abstract

We examine several types of visibility graphs: bar and semi bar k-visibility graphs, arc and circle k-visibility graphs, and compact visibility graphs. We improve the upper bound on the thickness of bar k-visibility graphs from $2k(9k-1)$ to $6k$, and prove that the upper bound must be at least $k+1$. We also show that the upper bound on the thickness of semi bar k-visibility graphs is between $\lceil \frac{2}{3}(k+1) \rceil$ and $2k$. We find bounds on the number of edges and the chromatic number of arc and circle k-visibility graphs. Finally, we relate two conjectures on compact visibility graphs, prove that every n-partite graph in the form $K_{1,a_1,a_2,\ldots,a_{n-1}}$ is a compact visibility graph, and classify all (but one) graphs with at most six vertices as compact visibility graphs.

Surya Bhupatiraju

On the Complexity of the Marginal Satisfiability Problem

under the direction of Alex Arkhipov

Abstract

The marginal satisfiability problem (MSP) asks the following question: Given desired marginal distributions D_S for every subset S of c variable indices from $\{1, \ldots, n\}$, does there exist a distribution D over n-tuples of values in $\{1, \ldots, m\}$ with those S-marginals D_S? We solve MSP for $n = 2$, and completely characterize the complexity of three closely related variants of MSP.

Joshua Brakensiek

Bounds on the Size of Sound Monotone Switching Networks Accepting Permutation Sets of Directed Trees

under the direction of Aaron Potechin

Abstract

Given a directed graph G as input with labeled nodes s and t, the ST-connectivity problem asks whether s and t are connected. The memory efficiency of an algorithm which solves this problem can be analyzed using sound monotone switching networks. This paper concerns bounds on the size of sound monotone switching networks which are restricted to the case that the input graph G is isomorphic to a given graph H. Previously, tight bounds had been found in the cases that H is a special kind of tree and that H is a collection of disjoint paths from s to t. This paper improves these results to find a nearly tight bound which applies to all directed trees. If we let n be the number of vertices in the graph, ℓ be the length of the path from s to t in the tree, and C_1 and C_2 be variables which depend on the distances of the vertices from s and t, then an upper bound on the size is on the order of $n^{\log\log\ell}C_1^{\log\ell}$ and the lower bound is on the order of $C_2^{\log\ell}$. These two bounds are within $\log\log\ell$ times a constant factor in the exponent.

Katherine Cordwell

Lower central series quotients of finitely generated algebras over the integers

under the direction of Teng Fei and Pavel Etingof

Abstract

We study the lower central series $L_1, \dots, L_k$ for unital associative algebras A over $\mathbb{Z}$. Specifically, we consider the quotients $B_k = L_k/L_{k+1}$, each of which is graded and can be written as the direct sum of graded components. Each of these components is a finitely generated abelian group and may be further decomposed into a free component F and a torsion component T. We determine the structures of F and T, thereby finding the exact decomposition of each graded component of B_i. The structure of B_i depends on the structure of the underlying algebra A; using *Magma*, we gather data and make conjectures for the B_i over various A. We mainly consider algebras $A \cong A_n(\mathbb{Z})/\langle f \rangle$ where f is a homogeneous relation and A_n is the free associative algebra on n generators. We completely describe B_i for free algebras modulo a relation of the form $f = xy - qyx$, $q \in \mathbb{Z}$.

Kevin Garbe

Patterns in the Coefficients of Powers of Polynomials Over a Finite Field

under the direction of Dorin Boger

Abstract

We examine the patterns, counts, and behavior of coefficients of powers of polynomials in a finite field. We use two methods to produce matrices that let us quickly calculate the total number of occurences of any coefficient, and examine some properties of the matrices. We form conjectures on the maximum eigenvalues of these matrices and methods for reducing their size. Furthermore, we examine the blocks of coefficients that appear in the powers of polynomials when reduced modulo some prime, and derive recursive formulae for the number of blocks of varying sizes can appear for certain polynomials and primes.

Simanta Gautam

A Novel Approach to the Spherical Codes Problem

under the direction of Dmitry Vaintrob

Abstract

The spherical codes problem asks for the maximum number of n-dimensional unit vectors such that the maximum pairwise angle between two vectors is at least α. We present a novel computational approach that focuses on symmetry groups acting on the configuration of vectors. Using this method, we were able to reproduce configurations for most known spherical codes. Furthermore, we provide a method to improve known lower bounds using our algorithm in dimension 16.

Gil Goldshlager

Characterizing Outerplanar and x-Monotone Thrackles

under the direction of Aaron Potechin

Abstract

A *thrackle* is defined as a graph drawing in the plane in which every edge intersects every other edge exactly once. Two special cases of thrackles are the focus of this paper: (i) *outerplanar thrackles*, in which the vertices lie on a circle and the edges are contained inside the circle, and (ii) *x-monotone thrackles*, in which the vertices may be anywhere, but each edge may intersect any vertical line in the plane at most once. For both of these cases, we determine the set of graphs that can be drawn in the specified manner. The results obtained here may be a step towards solving the general problem of characterizing all thrackles.

Jacob McNamara

Shortest Vectors in Principal Ideals of CM Number Fields

under the direction of Dmitry Vaintrob

Abstract

We define a family of norms on the rings of integers $\mathcal{O}_F$ of a CM number field F by weighting the k different ways of embedding F into the complex numbers with positive real numbers. These norms are related to an automorphic form that contains the theta functions of the ring of integers and all of its principal ideals and can be defined for any CM number field. By studying the shortest vectors for this family of norms, we can understand the behavior of this automorphic form at $i\infty$. Our main result is a bound, in terms of the norms to $\mathbb{Q}$ of the units in $\mathcal{O}_F$, on the vectors that can be shortest for some choice of weights. We also apply our result to the more concrete Craig's Difference Lattice Problem, and reduce the number of points which have to be checked to find the minimum length vectors to a finite number.

Charles Pasternak

Random Error Models in Quantum Error Correction

under the direction of Matthew Coudron

Abstract

We examine the performance of quantum error correcting codes subjected to random Haar distribution transformations of weight t. Rather than requiring correction of all errors, we require some high probability that an error is corrected. We find that, for any integer i and arbitrarily high probability $p < 1$, there are codes which perfectly correct errors up to weight t and can correct errors up to weight $t + i$ with probability at least p. We also find an analog to the quantum Hamming bound for the new error model.

Lilly Shen

Forbidden Patterns in 0-1 Matrices

under the direction of Jesse Geneson

Abstract

The extremal function $ex(n, P)$ counts the maximum number of 1's in an $n \times n$ matrix which avoids a pattern P. Cibulka and Kynčl defined a new function $ex_k(m, P)$ which counts the maximum number of columns that a matrix with m rows and k 1's per column can contain such that the matrix avoids P. We find bounds on $ex_k(m, P)$ for elementary operations on one or two patterns. We determine that L_1 is linear in the modified extremal function as well. Previously, bounds have been determined on the original extremal functions of rectangular patterns. We prove that for rectangular configurations with dimensions $r \times c$, $ex_k(m, P) = \theta(m^r)$. To improve and find new bounds on our original extremal function, we determine the relation $ex(m, n, P) \leq k(ex_k(m, P) + n - 1)$, which holds for range-overlapping patterns. Finally, we define a new pattern Q and bound $ex_4(m, Q)$ and $ex_5(m, Q)$.

Maurice Shih

The Hit Problem over the Steenrod Algebra

under the direction of John Ullman

Abstract

We study the problem of whether the cohomology of MO(2) splits as a module over the Steenrod algebra. To do this, we consider various finite-dimensional subalgebras $\mathcal{A}(n)$. The cohomology of MO(2) is similar to the symmetric algebra on two generators except that the classes are multiplied by ω, the Thom class of the universal two dimensional vector bundle. We prove that MO(2) splits over $\mathcal{A}(0)$ and $\mathcal{A}(1)$, and find a minimal generating set over $\mathcal{A}(2)$.

Kathleen Zhou

On Successive Quotients of Lower Central Series Ideals for Finitely Generated Algebras

under the direction of Teng Fei and Pavel Etingof

Abstract

This paper examines the behavior of successive quotients of lower central series ideals of finitely generated associative algebras over $\mathbb{Z}$. In particular, it studies patterns in the ranks and the torsion of these quotients for homogeneous relations, including $x^m + y^m$, q-polynomial relation $yx - qxy$, and x^2 in multiple variables. This paper includes a complete description of N_i for the q-polynomial algebra, $A\langle x, y\rangle / (yx - qxy)$. In addition, the appendix of this paper includes data tables for the free and torsion components for N_i with various relations, which have never been calculated before.