Patterns in the Coefficients of Powers of Polynomials Over a Finite Field

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Abstract

We examine the behavior of the coefficients of powers of polynomials over a finite field of prime order. Extending the work of Allouche-Berthe, 1997, we study $a(n)$, the number of occurring strings of length n among coefficients of any power of a polynomial f reduced modulo a prime p. The sequence of line complexity $a(n)$ is p-regular in the sense of Allouche-Shalit. For $f = 1+x$ and general p, we derive a recursion relation for $a(n)$ then find a new formula for the generating function for $a(n)$. We use the generating function to compute the asymptotics of $a(n)/n^2$ as $n \to \infty$, which is an explicitly computable piecewise quadratic in x with $n = \lfloor p^m/x \rfloor$ and x is a real number between $1/p$ and 1. Analyzing other cases, we form a conjecture about the generating function for general $a(n)$. We examine the matrix B associated with f and p used to compute the count of a coefficient, which applies to the theory of linear cellular automata and fractals. For $p = 2$ and polynomials of small degree we compute the largest positive eigenvalue, λ, of B, related to the fractal dimension d of the corresponding fractal by $d = \log_2(\lambda)$. We find proofs and make a number of conjectures for some bounds on λ and upper bounds on its degree.

1 Introduction

It was shown by S. Wolfram and others in 1980s that 1-dimensional linear cellular automata lead at large scale to interesting examples of fractals. A basic example is the automaton associated to a polynomial f over $\mathbb{Z}/p$, whose transition matrix T_f is the matrix of multiplication by $f(x)$ on the space of Laurent polynomials in x. If $f = 1 + x$, then starting with the initial state $g_0(x) = 1$, one recovers Pascal's triangle mod p. For $p = 2$, at large scale, it produces the Sierpinski triangle shown in Figure 1. Similarly, the case of $f = 1 + x + x^2$, $p = 2$, and initial state $g_0(x) = 1$ produces the fractal shown in Figure 2.

The double sequences produced by such automata, i.e., the sequences encoding the coefficients of the powers of f, have a very interesting structure. Namely, if p is a prime, they are p-automatic sequences in the sense of [3]. In the case $f = 1+x$, this follows from Lucas' theorem that $\binom{n}{k} = \prod\limits_i \binom{n_i}{k_i}$ mod p, where n_i, k_i are the p-ary digits of n, k.

In [6, 7], S. Wilson studied this example in the case where f is any polynomial, and computed the fractal dimension of the corresponding fractal. The answer is $\beta = \log_p(\lambda)$, where $p \leq \lambda \leq p^2$ is the largest (Perron-Frobenius) eigenvalue of a certain integer matrix B associated to f (in particular, an algebraic integer). In terms of coefficients of powers of f, this number characterizes the rate of growth of the total number of nonzero coefficients in f^i for $0 \leq i < p^n$: this number behaves like n^β. The number of nonzero coefficients of each kind can actually be computed exactly at every step of the recursion, by using a matrix method similar to Wilson's; this is explained in the paper [3].

In this paper, we compute the eigenvalues λ and their degrees for $p = 2$ for Laurent polynomials f of small degrees, observe some patterns, and make a number of conjectures (in particular, that λ can be arbitrarily close to 4) in Section 3.3. We also prove an upper bound for λ depending on the degree of f.

The size of the matrix B (which is an upper bound for the degree of λ) is the number of accessible blocks (i.e., strings that occur in the sequence of coefficients of f^i for some i) of length $deg(f)$ (for $p = 2$). This raises the question of finding the number $a(n)$ of accessible blocks of any length n. The number $a(n)$ characterizes the so-called line complexity of the corresponding linear automaton, and is studied in the paper [1]. It is shown in [1],[5], and references therein that $C_1 n^2 \leq a(n) \leq C_2 n^2$, and that for $p = 2$ and $f = 1 + x$, one has $a(n) = n^2 - n + 2$. More generally, however, the sequence $a(n)$ does not have such a simple form, even for $f = 1 + x$ and $p > 2$. The paper [1] derives a recursion for this sequence, and we derive another one in Section 2.2.1, which is equivalent. These recursions show that the sequence $a(n)$ is p-regular in the sense of [2] (the notion of p-regularity is a generalization of the notion of p-automaticity, to the case of integer, rather than mod p, values). We then proceed to find a new formula for the generating function for $a(n)$ in Section 2.3, and use it to compute the asymptotics of $a(n)/n^2$ as $n \to \infty$ in Section 2.4. It turns out that if $n = \lfloor p^m/x \rfloor$, where x is a real number between $1/p$ and 1, then $f(n)/n^2$ tends to an explicit function of x, which is piecewise quadratic (a gluing together of 3 quadratic functions, which we explicitly compute). In Section 2.4 we also compute the maximum and minimum value of this function, which gives the best asymptotic values for C_1 and C_2. This gives us new precise results about the complexity of the Pascal triangle mod p. We also perform a similar analysis for $f = 1 + x + x^2$ and $p = 2$, and make a conjecture about the general case.

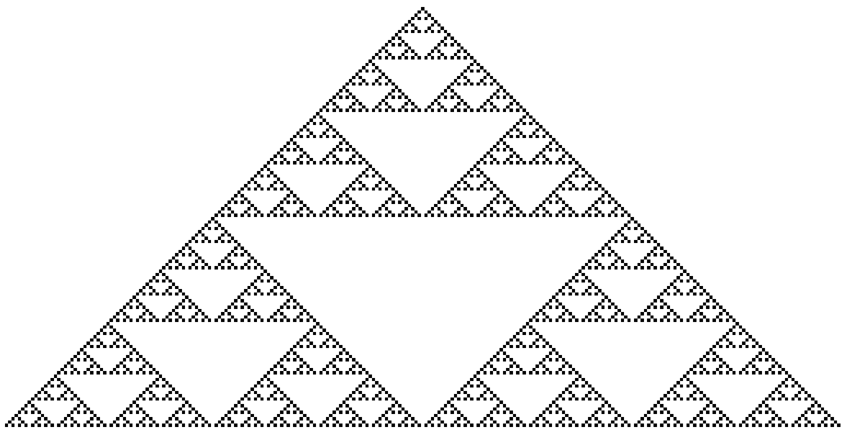


Figure 1: Fractal corresponding to $1 + x$ modulo 2 (Sierpinski's Triangle)

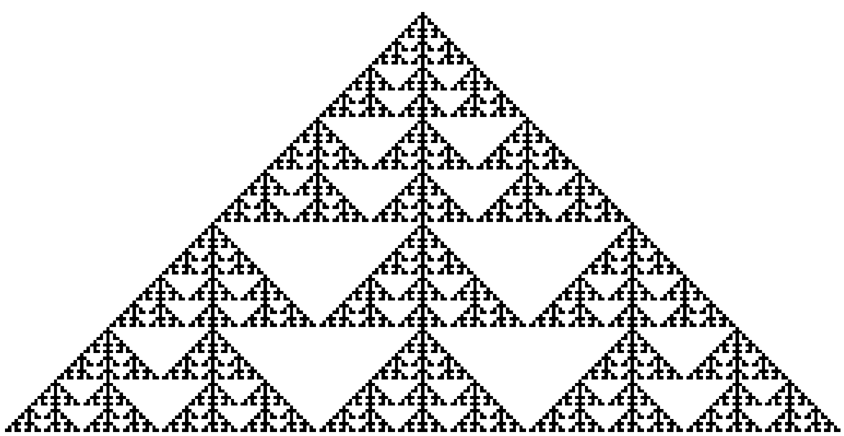


Figure 2: Fractal corresponding to $1 + x + x^2$ modulo 2

2 Accessible Blocks

2.1 Definitions

A block is a string of mod p digits. An m-block is a block with m digits. For example, the four 2-blocks modulo 2 are $00, 01, 11$, and 11.

For a polynomial $f(x)$ with integer coefficients reduced modulo p, an accessible m-block is an m-block that appears anywhere among the coefficients, ordered by powers of x, of powers of $f(x)$ modulo p. The number of accessible 0-blocks we define to be 1. Furthermore, we define row k for some $f(x)$ and p to be the coefficients of $f(x)^k$ reduced modulo p and define $a_{f(x),p}(m)$ to be the number of accessible m-blocks for the polynomial $f(x)$ and prime p.

Example 2.1. *For $f(x) = 1 + x$ and $p = 2$, the 4-blocks 1101 and 1011 are never a substring of any power of $1 + x$ reduced modulo 2. Every other 4-block appears in some power of $1 + x$ reduced modulo 2, so $a_{1+x,2}(4) = 14$.*

2.2 Recursion Relations for $a(n)$

We start with the well known fact in Lemma 2.2.

Lemma 2.2. *$f(x)^{k \cdot p} \equiv f(x^p)^k \pmod p$.*

Applying Lemma 2.2 to the accessible blocks, we have Corollary 2.3.

Corollary 2.3. *For any integer k, prime p, and polynomial $f(x)$, every row $k \cdot p$ for $f(x)$ mod p is of the form $b_1 0 \ldots 0 b_2 0 \ldots \ldots 0 b_{n-1} 0 \ldots 0 b_n$ where the entries b_i are the coefficients of $f(x)^k$, and where each string of zeros between two entries b_i and b_{i+1} is of length $p - 1$. Therefore, every accessible block from a row divisible by p is a subsection of $b_1 0 \ldots 0 b_2 0 \ldots \ldots 0 b_{n-1} 0 \ldots 0 b_n$.*

2.2.1 Accessible m-Blocks for $f(x) = 1 + x$ and General Prime p

The number of accessible m-blocks for $f(x) = 1 + x$ and any prime p, $a_{1+x,p}$, is defined by the recurrence relation in Theorem 2.4.

Theorem 2.4. *For $f(x) = 1 + x$ and any prime $p \geq 3$, for $0 \leq k \leq p-1$, the recursion relation with starting points $a_{1+x,p}(0) = 1, a_{1+x,p}(1) = p,$ and $a_{1+x,p}(2) = p^2$ is*

$$\begin{aligned} a_{1+x,p}(p \cdot n + k) =& \frac{(p-k)(p-k+1)}{2} \cdot a_{1+x,p}(n) + (kp + k - k^2 + \frac{p^2-p}{2}) \cdot a_{1+x,p}(n+1) \\ &+ \frac{k^2-k}{2} \cdot a_{1+x,p}(n+2) - (2p-1)(2p-2). \end{aligned}$$

Proof. From Corollary 2.3, every accessible block in a row r with $r \equiv 0 \pmod p$ is formed by adding $p-1$ zeros between every digit of an accessible block, then adding some number of zeros (possibly none) less than p to either side. Furthermore, because $f(x) = 1+x$, the coefficient of x^i in a row is the sum modulo p of the coefficients of x^i and x^{i-1} in the previous row. Because accessible blocks are subsections of a row, any accessible m-block comes from an accessible $(m+1)$-block. Table 1 provides the general forms of the $(p \cdot n + k)$-blocks for each row modulo p. To count the multiple additions of b in the forms, we define $g_i = \binom{p-1}{i}$.

The number of accessible blocks that lead into each form in Table 1 are the triangular numbers counting downwards for $a_{1+x,p}(n)$, the triangular numbers counting upward for $a_{1+x,2}(n+2)$, and because the total number of forms is p^2, we find $a_{1+x,p}(n+1)$ through subtraction. Namely, the factor of $a_{1+x,p}(n)$ starts at p for *row* congruent to 0 modulo p and k=0, and decreases as k and *row* increase, and the coefficient of $a_{1+x,p}(n+2)$ starts at 0 for *row* congruent to 0 and 1 modulo p and increases with k and *row*. An additional $(2p-1)(2p-2)$ must be subtracted to account for blocks that satisfy multiple forms. Therefore

$$\begin{aligned} a_{1+x,p}(p \cdot n + k) =& \frac{(p-k)(p-k+1)}{2} \cdot a_{1+x,p}(n) + (kp + k - k^2 + \frac{p^2-p}{2}) \cdot a_{1+x,p}(n+1) \\ &+ \frac{k^2-k}{2} \cdot a_{1+x,p}(n+2) - (2p-1)(2p-2). \end{aligned}$$

□

This is equivalent to Theorem 5.10 of Allouche-Berthe [1], reproduced below in Theorem 2.5.

	Blocks for $k =$				
Row mod p	0	1	2	$\cdots$	$p-1$
0	$b_1000\ldots00b_200\ldots\ldots\ 00b_n00\ \ldots000$	b_{n+1}	0	$\cdots$	0
	$0b_100\ldots000b_20\ldots\ldots\ 000b_n0\ \ldots000$	0	b_{n+1}	$\cdots$	0
	$00b_10\ldots0000b_2\ldots\ldots\ 0000b_n\ \ldots000$	0	0	$\cdots$	0
	$\vdots\ \vdots\ \vdots\ \ddots\ \vdots\ \vdots\ \vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	$0000\ldots b_10000\ldots\ldots b_{n-1}0000\ldots0b_n0$	0	0	$\cdots$	b_{n+1}
	$0000\ldots0b_1000\ldots\ldots0b_{n-1}000\ldots00b_n$	0	0	$\cdots$	0
1	$b_1000\ldots0b_2b_200\ldots\ldots\ 0b_nb_n00\ \ldots00b_{n+1}$	b_{n+1}	0	$\cdots$	0
	$b_1b_100\ldots00b_2b_20\ldots\ldots\ 00b_nb_n0\ \ldots\ 000$	b_{n+1}	b_{n+1}	$\cdots$	0
	$0b_1b_10\ldots000b_2b_2\ldots\ldots\ 000b_nb_n\ \ldots\ 000$	0	b_{n+1}	$\cdots$	0
	$\vdots\ \vdots\ \vdots\ \ddots\ \vdots\ \vdots\ \vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	$0000\ \ldots b_10000\ldots\ldots\ b_{n-1}0000\ \ldots\ b_n0$	0	0	$\cdots$	b_{n+1}
	$0000\ \ldots b_1b_1000\ldots\ldots b_{n-1}b_{n-1}000\ldots\ b_nb_n$	0	0	$\cdots$	b_{n+1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$p-1$	$b_1b_2(g_2b_2)\ \ldots\ldots(g_4b_{n+1})(g_3b_{n+1})(g_2b_{n+1})$	b_{n+1}	b_{n+2}	$\cdots$	(g_3b_{n+2})
	$(g_2b_1)b_1b_2\ \ldots\ldots(g_5b_{n+1})(g_4b_{n+1})(g_3b_{n+1})$	(g_2b_{n+1})	b_{n+1}	$\cdots$	(g_4b_{n+2})
	$(g_3b_1)(g_2b_1)b_1\ \ldots\ldots(g_6b_{n+1})(g_5b_{n+1})(g_4b_{n+1})$	(g_3b_{n+1})	(g_2b_{n+1})	$\cdots$	(g_5b_{n+2})
	$\vdots\ \ddots\ \vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	$(g_2b_1)(g_3b_1)(g_4b_1)\ldots\ldots\ (g_2b_n)b_nb_{n+1}$	(g_2b_{n+1})	(g_3b_{n+1})	$\cdots$	b_{n+1}
	$b_1(g_2b_1)(g_3b_1)\ \ldots\ldots\ (g_3b_n)(g_2b_n)b_n$	b_{n+1}	(g_2b_{n+1})	$\cdots$	(g_2b_{n+1})

Table 1: Forms of blocks for the general case $1+x$ with any prime p

Theorem 2.5. *For $0 \le k \le p-1$ and $n \ge 0$ such that $pn+k \ge 3$*

$$\begin{aligned} a_{1+x,2}(pn+k+1) - a_{1+x,2}(pn+k) =& (p-k)\Big(a_{1+x,2}(n+1) - a_{1+x,2}(n)\Big) \\ &+ k\Big(a_{1+x,2}(n+2) - a_{1+x,2}(n+1)\Big) \end{aligned}$$

with starting points $a_{1+x,2}(0) = 1$, $a_{1+x,2}(1) = p$, $a_{1+x,2}(2) = p^2$, and $a_{1+x,2}(3) = \frac{p^3+4p^2-5p+2}{2}$.

2.2.2 Accessible m-Blocks for $c + x + x^2$ and prime p

Table 2 provides $a_{c+x+x^2,p}(n)$ for small n and p.

Using a method similar to the one we used for Theorem 2.4, the recursion relations appear to be those shown in Table 3.

Prime	c	a(n)
2	1	2 4 8 4 25 36 53 70 92 114
3	1	3 9 25 43 71 109 157 207 259 313
3	2	3 9 25 61 105 165 233 321 417 533
5	1	5 25 121 393 673 929 1257 1761 2341 3097
5	2	5 25 125 393 689 953 1293 1801 2389 3145
5	3	5 25 117 385 657 905 1221 1713 2277 3017
5	4	5 25 101 169 253 353 509 721 989 1313
7	1	7 49 331 1285 2137 2881 3859

Table 2: $a(n)$ for $c + x + x^2$

p	c	Recursion	k	initial
2	1	2a(n)+2a(n+1) a(n)+2a(n+1)+a(n+2)	8	1,2,4,8,14,25
3	1	6a(n)+3a(n+1) 3a(n)+6a(n+1) a(n)+7a(n+1)+a(n+2)	20	1,3,9,25
3	2	4a(n)+4a(n+1)+a(n+2) 2a(n)+5a(n+1)+2a(n+2) a(n)+4a(n+1)+4a(n+2)	32	1,3,9,25,61,105
5	1	9a(n)+12a(n+1)+4a(n+2) 6a(n)+13a(n+1)+6a(n+2) 4a(n)+12a(n+1)+9a(n+2) 2a(n)+10a(n+1)+12a(n+2)+a(n+3) a(n)+12a(n+1)+10a(n+2)+2a(n+3)	152	1,5,25,121,393,673
5	2	9a(n)+12a(n+1)+4a(n+2) 6a(n)+13a(n+1)+6a(n+2) 4a(n)+12a(n+1)+9a(n+2) 2a(n)+10a(n+1)+12a(n+2)+a(n+3) a(n)+12a(n+1)+10a(n+2)+2a(n+3)	152	1,5,25,125,393,689
5	3	9a(n)+12a(n+1)+4a(n+2) 6a(n)+13a(n+1)+6a(n+2) 4a(n)+12a(n+1)+9a(n+2) 2a(n)+10a(n+1)+12a(n+2)+a(n+3) a(n)+12a(n+1)+10a(n+2)+2a(n+3)	152	1,5,25,117,385,657
5	4	15a(n)+10a(n+1) 10a(n)+15a(n+1) 6a(n)+18a(n+1)+a(n+2) 3a(n)+19a(n+1)+3a(n+2) a(n)+18a(n+1)+6a(n+2)	72	1,5,25,101,169

Table 3: Recursions for $c + x + x^2$

We see that for $p > 2$, $a_{c+x+x^2,p}(n) = a_{1+x,p}(n)$ if $c = \frac{1}{4} \pmod{p}$ because $c+x+x^2 = (1+x/2)^2$. Furthermore, we arrive at Conjecture 2.6.

Conjecture 2.6. *For $c \neq \frac{1}{4} \pmod{5}$, the recursion for $a_{1+x+x^2,p}(n)$ is independent of c. Only the initial terms of the recursion depend on c.*

2.3 Closed form for $a(n)$

Theorem 2.7. $a_{1+x,2}(m) = m^2 - m + 2$.

Proof. Theorem 2.4 provides the recursion relation of $a_{1+x,2}(2n) = 3a_{1+x,2}(n) + a_{1+x,2}(n+1) - 6$ and $a_{1+x,2}(n) = a_{1+x,2}(n) + 3a_{1+x,2}(n+1)$. We can find the starting points of $a_{1+x,2}(1) = 2$ and $a_{1+x,2}(2) = 4$ through inspection. This uniquely defines the sequence of accessible m-blocks. It is easy to show that the equation $a_{1+x}(m) = m^2 - m + 2$ satisfies both recursion relations through substitution, and also satisfies $a_{1+x,2}(1) = 2$ and $a_{1+x,2}(2) = 4$. □

This matches Remark 5.14 of [1].

2.3.1 Generating Functions for a(n)

Using recursion relations, we can find the generating functions $g_{f(x),p}$ for $p \geq 3$.

Theorem 2.8.

$$\begin{aligned} g_{1+x,p}(z) &= \sum_{n=0}^{\infty} a_{1+x,p}(n) z^n \\ &= \frac{1}{(1-z)^3} \Bigg(1 + (p-3)z + (p^2 - 3p + 3)z^2 \\ &\quad + z^2 \frac{(p-1)^2}{2} \sum_{i \geq 0} \left(pz^{p^i} - 2(p-1)z^{2p^i} + (p-2)z^{3p^i} \right) \Bigg). \end{aligned}$$

Proof. We have from Theorem 2.4 that for starting points $a(0) = 1$, $a(1) = p$, and $a(2) = p^2$ the recursion relation is defined for $pn+k>2$ as

$$\begin{aligned}a(pn+k) =&\frac{(p-k)(p-k+1)}{2}a(n)+(kp+k-k^2+\frac{p^2-p}{2})a(n+1)\\&+\frac{k^2-k}{2}a(n+2)-(2p-1)(2p-2).\end{aligned}$$

Adjusting for the $k=0,1$ cases by replacing k with $n+2$ gives

$$\begin{aligned}a(pn+k+2) =&\frac{(p-k-2)(p-k-1)}{2}a(n)+(kp-3k-k^2-2+\frac{p^2+3p}{2})a(n+1)\\&+\frac{(k+1)(k+2)}{2}a(n+2)-(2p-1)(2p-2).\end{aligned}$$

To adjust for the case when $p,k=0$, we define the recursion relation to have an additional term of $\frac{(p-2)(p-1)}{2}a(0)+\frac{(p-4)(p+1)}{2}a(1)-(2p-1)(2p-2)$ subtracted from the right hand side for only the case of $p,k=0$.

We multiply through by z^{pn+k}, then sum over $k=0$ to $p-1$, then $n=0$ to ∞. We also subtract from the right hand side of the sum the above mentioned additional term to account for the case of $p,k=0$. Defining $h(x)=\sum\limits_{n\geq 0}a(n+2)z^n$, we get

$$\begin{aligned}h(z) =&(1+z+z^2+\ldots+z^{p-1})^3h(z^p)+\frac{1}{2(1-z)^3}\Big(p^3z(1-z)^2+2p^2(1-z)(4-5z+2z^2)\\&+2(2-3z+3z^2-z^3-z^p)-p(12-19z+16z^2-5z^3-6z^p+2z^{2p})\Big)\\&-\frac{(2p-1)(2p-2)}{1-z}.\end{aligned}$$

Therefore $h(z)=\dfrac{(1-z^p)^3}{(1-z)^3}h(z^p)+Q(z)-(2p-1)(2p-2)\dfrac{1}{1-z}$ where

$$\begin{aligned}Q(z) =&\frac{1}{2(1-z)^3}\Big(p^3z(1-z)^2+2p^2(1-z)(4-5z+2z^2)+2(2-3z+3z^2-z^3-z^p)\\&-p(12-19z+16z^2-5z^3-6z^p+2z^{2p})\Big).\end{aligned}$$

We then define $u(z)=(1-z)^3h(z)$ and $R(z)=Q(z)(1-z)^3-(2p-1)(2p-2)(1-z)^2$. Iteratively substituting gives $u(z) = u(z^{p^\infty}) + \sum_{i\geq 0} R(z^{p^i}) = a(2) + \sum_{i\geq 0} R(z^{p^i})$, or $h(z) = \frac{1}{(1-z)^3}\Big(a(2) + \sum_{i\geq 0} R(z^{p^i})\Big)$. Note that

$$\begin{aligned}\sum_{i\geq 0} R(z^{p^i}) =& \sum_{i\geq 0} \frac{1}{2}\Big((p^3 - 2p^2 - 5p + 2)z - 2(p^3 - 3p^2 + 2p - 1)z^2 \\ &+ (p-2)(p-1)^2 z^3 + 2(3p-1)z^p - 2pz^{2p}\Big) \\ =& -\Big((3p-1)z - pz^2\Big) + \frac{(p-1)^2}{2}\sum_{i\geq 0}\Big(pz^{p^i} - 2(p-1)z^{2p^i} + (p-2)z^{3p^i}\Big).\end{aligned}$$

Therefore

$$\begin{aligned} g(z) =& a(0) + a(1)z + z^2 h(z) \\ =& 1 + pz + z^2 \frac{p^2 + \sum_{i\geq 0} R(z^{p^i})}{(1-z)^3} \\ =& \frac{1 + (p-3)z + (p^2 - 3p + 3)z^2 + z^2 \frac{(p-1)^2}{2} \sum_{i\geq 0}\Big(pz^{p^i} - 2(p-1)z^{2p^i} + (p-2)z^{3p^i}\Big)}{(1-z)^3}.\end{aligned}$$

□

Example 2.9. *Setting $p = 3$ in Theorem 2.8 and noting that the z^{3p^i} further reduces when $p = 3$ provides*

$$g_{1+x,3}(z) = \frac{1}{(1-z)^3}\Big(1 + 3z^2 - 2z^3 + 8z^2 \sum_{i=0}^{\infty}(z^{3^i} - z^{2\cdot 3^i})\Big).$$

Example 2.10. *Setting $p = 5$ in Theorem 2.8 provides*

$$g_{1+x,5}(z) = \frac{1}{(1-z)^3}\Big(1 + 2z + 13z^2 + 8z^2 \sum_{i=0}^{\infty}(5z^{5^i} - 8z^{2\cdot 5^i} + 3z^{3\cdot 5^i})\Big).$$

We can use a similar proof to find further generating functions $g_{x),p}(z)$ from the recursion relations for $a_{f(x),p}(n)$.

Theorem 2.11.

$$g_{1+x+x^2,2}(z) = \frac{1 + 2z^3 + 2z^5 - z^6 + \sum_{i=0}^{\infty}(z^{2^i} - z^{3\cdot 2^i})}{(1-z^2)(1-z)^2}.$$

Based on the recursions in Table 3 and the method provided in Theorem 2.8, we arrive at Conjecture 2.12, which is confirmed for $p = 3, 5$.

Conjecture 2.12. *For $c \neq \frac{1}{4} \pmod{p}$, the functional equation for the generating function $g_{c+x+x^2,p}(z)$ is*

$$g_{c+x+x^2,p}(z) = \frac{r(z^p)}{r(z)} g_{c+x+x^2,p}(z^p) - Q(z) - \frac{k}{1-z},$$

where $r(z) = (1-z^2)(1-z)^2$ and $Q(z)$ is some polynomial.

Conjecture 2.13. *For any $f(x)$ and p, the generating function $g_{f(x),p}(z)$ satisfies the equation $r(z)g_{f(x),p}(z) = r(z^p)g_{f(x),p}(z^p) + b(z)$ for some polynomials $r(z)$ and $b(z)$ depending on $f(x)$ and p.*

2.4 Limits of $\frac{a(n)}{n^2}$

Using the generating functions, we can find the asymptotic behavior of $a(n)$ as n goes to infinity. Inspired by the quadratic nature of Theorem 2.7, we examine the behavior of $\frac{a(n)}{n^2}$.

Theorem 2.14. *For $f(x) = 1 + x$ and any prime $p \geq 3$,*

$$\lim_{n\to\infty} \frac{a_{1+x,p}(n)}{n^2} = \begin{cases} \frac{p^2(p-5)(p-1)}{2(p+1)}\left(x + \frac{p+1}{p(p-5)}\right)^2 + \frac{(p-1)(p^2-7p+4)}{2(p-5)} & \frac{1}{p} \leq x \leq \frac{1}{3} \\ \frac{-(p-1)(7p^3-8p^2-9p+18)}{4(p+1)}\left(x - \frac{(p+1)(3p^2-7p+6)}{7p^3-8p^2-9p+18}\right)^2 \\ +\frac{(p-1)(p^5+5p^4-8p^3-15p^2+39p-18)}{2(7p^3-8p^2-9p+18)} & \frac{1}{3} \leq x \leq \frac{1}{2} \\ \frac{(p-2)(p-1)(p^2+2p+5)}{4(p+1)}\left(x - \frac{(p+1)^2}{p^2+2p+5}\right)^2 \\ +\frac{(p-1)(p^3+4p^2+3p-4)}{2(p^2+2p+5)} & \frac{1}{2} \leq x \leq 1 \end{cases}$$

where $n = \lfloor \frac{p^k}{x} \rfloor$ and the limit as $n \to \infty$ is with constant x and $k \to \infty$.

Remark 2.15. *The first polynomial from Theorem 2.14 corresponding to $\frac{1}{p} \leq x \leq \frac{1}{3}$ should be understood in the sense of the limit for $p = 5$ as we divide by $(p-5)$. In this case the polynomial is not quadratic but actually the linear polynomial $20x + 8$.*

Proof. Theorem 2.8 states that

$$\begin{aligned} g(z) &= \sum_{n\geq 0} a_{1+x,p}(n) z^n \\ &= \frac{1}{(1-z)^3}\Big(1 + (p-3)z + (p^2-3p+3)z^2 \\ &\quad + z^2 \frac{(p-1)^2}{2} \sum_{i\geq 0} \Big(p z^{p^i} - 2(p-1) z^{2p^i} + (p-2) z^{3p^i}\Big)\Big). \end{aligned}$$

Let $\sum\limits_{n\geq 0} b(n) z^n = \dfrac{z^2}{(1-z)^3} \sum\limits_{i\geq 0} \Big(p z^{p^i} - 2(p-1) z^{2p^i} + (p-2) z^{3p^i}\Big)$.

Therefore, with the limit of $n = \lfloor \frac{p^k}{x} \rfloor \to \infty$ taken with fixed x and $k \to \infty$, we have

$$\begin{aligned} \sum_{n\geq 0} a(n) z^n &= \frac{1 + (p-3)z + (p^2-3p+3)z^2}{(1-z)^3} + \sum_{n\geq 0} \frac{(p-1)^2}{2} b(n) z^n \\ \lim_{n\to\infty} a(n) &= \lim_{n\to\infty} \left(\frac{(p-1)^2 n^2}{2} + \frac{(p^2-1)n}{2} + p + \frac{(p-1)^2}{2} b(n) \right) \\ \lim_{n\to\infty} \frac{a(n)}{n^2} &= \frac{(p-1)^2}{2} + \frac{(p-1)^2}{2} \lim_{n\to\infty} \frac{b(n)}{n^2}. \end{aligned}$$

Therefore, because they act similarly, we can find the asymptotics of $\frac{a(n)}{n^2}$ by understanding the behavior $\frac{b(n)}{n^2}$. We can rewrite $\sum\limits_{n\geq 0} b(n) z^n$ as

$$\sum_{n=0}^{\infty} \Bigg(p \sum_{i=0}^{p^i \leq n} \frac{(n-p^i)(n-p^i-1)}{2} - 2(p-1) \sum_{i=0}^{2p^i \leq n} \frac{(n-2p^i)(n-2p^i-1)}{2} \\ + (p-2) \sum_{i=0}^{3p^i \leq n} \frac{(n-2p^i)(n-2p^i-1)}{2} z^n \Bigg).$$

From this we see that

$$\begin{aligned} b(n) =& p \sum_{i=0}^{p^i \leq n} \left(\frac{(n-p^i)(n-p^i-1)}{2} \right) - 2(p-1) \sum_{i=0}^{2p^i \leq n} \left(\frac{(n-2p^i)(n-2p^i-1)}{2} \right) \\ &+ (p-2) \sum_{i=0}^{3p^i \leq n} \left(\frac{(n-2p^i)(n-2p^i-1)}{2} \right). \end{aligned}$$

Therefore

$$\begin{aligned}\frac{b(n)}{n^2} =& \frac{p}{2}\sum_{i=0}^{p^i\le n}\left((1-\frac{p^i}{n})(1-\frac{p^i+1}{2})\right)-(p-1)\sum_{i=0}^{2p^i\le n}\left((1-\frac{2p^i}{n})(1-\frac{2p^i+1}{2})\right)\\ &+\frac{p-2}{2}\sum_{i=0}^{3p^i\le n}\left((1-\frac{3p^i}{n})(1-\frac{3p^i+1}{2})\right).\end{aligned}$$

Let $n=\lfloor\frac{p^k}{x}\rfloor$. We can neglect the 1 in the second factor (it creates a change that goes to zero as $k\to\infty$), so we get

$$\frac{b(n)}{n^2}=\frac{p}{2}\sum_{i=0}^{p^i\le n}(1-\frac{p^i}{n})^2-(p-1)\sum_{i=0}^{2p^i\le n}(1-\frac{2p^i}{n})^2+\frac{p-2}{2}\sum_{i=0}^{3p^i\le n}(1-\frac{3p^i}{n})^2.$$

Note that if $x\notin[\frac{1}{3},1]$ then there is $m\in\mathbb{Z}$ such that $p^m x\in[\frac{1}{p},1]$, so we can assume $\frac{1}{p}\le x\le 1$. Ignoring the floor for simplicity, we set $n=\frac{p^k}{x}$. Therefore we get

$$\frac{b(\frac{p^k}{x})}{(\frac{p^k}{x})^2}=\frac{p}{2}\sum_{i=0}^{p^i\le\frac{p^k}{x}}(1-p^{i-k}x)^2-(p-1)\sum_{i=0}^{p^i\le\frac{p^k}{2x}}(1-2p^{i-k}x)^2+\frac{p-2}{2}\sum_{i=0}^{p^i\le\frac{p^k}{3x}}(1-3p^{i-k}x)^2.$$

When examining the upper limits of the three sums, we find that we therefore have 3 cases: $\frac{1}{p}\le x\le\frac{1}{3},\frac{1}{3}\le x\le\frac{1}{2},\frac{1}{2}\le x\le 1$. For the first sum, $p^i\le\frac{p^k}{x}$ gives $i\le k+1$ for $x=\frac{1}{p}$, and $i\le k$ for $x=\frac{1}{3},\frac{1}{2},1$. For the second sum, $p^i\le\frac{p^k}{2x}$ gives $i\le k$ for $x=\frac{1}{p},\frac{1}{2},\frac{1}{3}$ and $i\le k-1$ for $x=1$. For the third sum, $p^i\le\frac{p^k}{3x}$ gives $i\le k$ for $x=\frac{1}{p},\frac{1}{3}$ and $i\le k-1$ for $x=\frac{1}{2},1$. Note that the limit is taken along the subsequences of the form $\lfloor\frac{p^k}{x}\rfloor$ with fixed x and $k\to\infty$. Also note that the limiting function does not change if x is replaced by $p\cdot x$.

For the first case of $\frac{1}{p} \leq x \leq \frac{1}{3}$, we find that

$$\begin{aligned}
\frac{b(\frac{p^k}{x})}{(\frac{p^k}{x})^2} =& \frac{p}{2}\sum_{i=0}^{k}(1-p^{i-k}x)^2 - (p-1)\sum_{i=0}^{k}(1-2p^{i-k}x)^2 + \frac{p-2}{2}\sum_{i=0}^{k}(1-3p^{i-k}x)^2 \\
=& (p-5)\frac{p^2-\frac{1}{p^{2k}}}{p^2-1}x^2 + 2\frac{p-\frac{1}{p^k}}{p-1}x \\
\lim_{n\to\infty}\frac{b(n)}{n^2} =& \lim_{k\to\infty}\left((p-5)\frac{p^2-\frac{1}{p^{2k}}}{p^2-1}x^2 + 2\frac{p-\frac{1}{p^k}}{p-1}x\right) \\
\lim_{n\to\infty}\frac{a(n)}{n^2} =& \frac{p^2(p-5)(p-1)}{2(p+1)}\Big(x+\frac{p+1}{p(p-5)}\Big)^2 + \frac{(p-1)(p^2-7p+4)}{2(p-5)}
\end{aligned}$$

For the case of $\frac{1}{3} \leq x \leq \frac{1}{2}$ we similarly find that because

$$\frac{b(\frac{p^k}{x})}{(\frac{p^k}{x})^2} = \frac{p}{2}\sum_{i=0}^{k}(1-p^{i-k}x)^2 - (p-1)\sum_{i=0}^{k}(1-2p^{i-k}x)^2 + \frac{p-2}{2}\sum_{i=0}^{k-1}(1-3p^{i-k}x)^2,$$

the limit of

$$\begin{aligned}
\lim_{n\to\infty}\frac{a(n)}{n^2} =& \frac{-(p-1)(7p^3-8p^2-9p+18)}{4(p+1)}\left(x-\frac{(p+1)(3p^2-7p+6)}{7p^3-8p^2-9p+18}\right)^2 - \frac{(p-4)(p-1)^2}{4} \\
&+ \frac{(p+1)(p-1)(3p^2-7p+6)^2}{4(7p^3-8p^2-9p+18)}.
\end{aligned}$$

Similarly for the case of $\frac{1}{2} \leq x \leq 1$ we find that because

$$\frac{b(\frac{p^k}{x})}{(\frac{p^k}{x})^2} = \frac{p}{2}\sum_{i=0}^{k}(1-p^{i-k}x)^2 - (p-1)\sum_{i=0}^{k-1}(1-2p^{i-k}x)^2 + \frac{p-2}{2}\sum_{i=0}^{k-1}(1-3p^{i-k}x)^2,$$

one has

$$\lim_{n\to\infty}\frac{a(n)}{n^2} = \frac{(p-2)(p-1)(p^2+2p+5)}{4(p+1)}\Big(x-\frac{(p+1)^2}{p^2+2p+5}\Big)^2 + \frac{(p-1)(p^3+4p^2+3p-4)}{2(p^2+2p+5)}.$$

$\square$

Corollary 2.16. *For the polynomial* $1+x$ *and* $p \geq 3$,

$$\liminf_{n\to\infty} \frac{a_{1+x,p}(n)}{n^2} = \frac{(p-1)(p^3+4p^2+3p-4)}{2(p^2+2p+5)}$$

$$\limsup_{n\to\infty} \frac{a_{1+x,p}(n)}{n^2} = \frac{(p-1)(p^5+5p^4-8p^3-15p^2+39p-18)}{2(7p^3-8p^2-9p+18)}$$

Proof. The maximum of Theorem 2.14 is when $x = \dfrac{3p^3-4p^2-p+6}{7p^3-8p^2-9p+18}$ and the minimum is when $x = \dfrac{p^2+2p+1}{p^2+2p+5}$. □

We can also apply this to other $a_{f(x),p}(n)$.

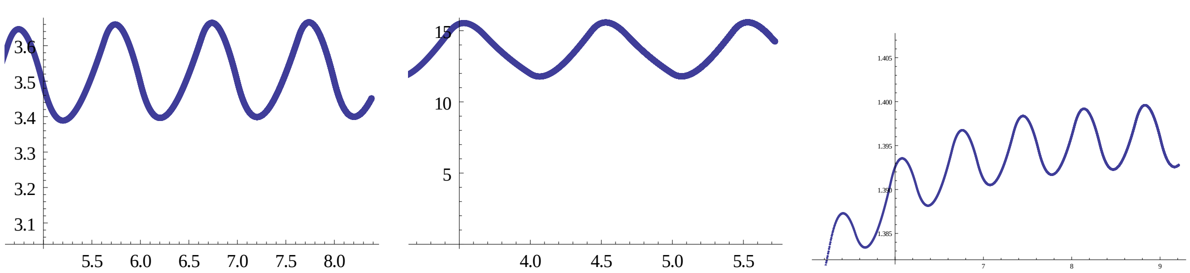
Theorem 2.17. *For polynomial* $1+x+x^2$ *and prime* 2,

$$\lim_{n\to\infty} \frac{a_{1+x+x^2,2}(n)}{n^2} = \begin{cases} \frac{5}{4}+\frac{1}{2}x-\frac{5}{12}x^2 & \frac{1}{2} \leq x \leq \frac{2}{3} \\ \frac{3}{2}-\frac{1}{4}x+\frac{7}{48}x^2 & \frac{2}{3} \leq x \leq 1 \end{cases}$$

Furthermore, the upper and lower limits of $\frac{a_{1+x+x^2}(n)}{n^2}$ *are* $\frac{7}{5}$ *and* $\frac{39}{28}$ *respectively.*

The proof of Theorem 2.17 is similar to the proof of Theorem 2.14.

Using the recursion relations, we computed the upper and lower limits of $\frac{a_{f(x),p}(n)}{n^2}$ for sufficiently large n for several $f(x)$ and p, The oscillatory nature of this sequence for large n stabilizing to a periodic function in $log(x)$ is illustrated by Figure 3.



(a) $1+x \pmod 3$ (b) $1+x \pmod 5$ (c) $1+x+x^2 \pmod 2$

Figure 3: $\dfrac{a_{f(x),p}(m)}{m^2}$ with the x axis showing $\log_p m$

This matches a prior result expressed in Lemma 5.15 by [1], which states that for large n, there exists constants c_1 and c_2 such that $c_1 n^2 \leq a(n) \leq c_2 n^2$. The limits given by Corollary 2.16 provide sharp values of c_1 and c_2. [1]

3 Counting Coefficients

3.1 Definitions

For a polynomial $f(x)$, prime p, and positive integer $\alpha \leq p-1$, we define $q_{f(x),p}(k,\alpha)$ to be the number of occurrences of α among the coefficients of $f(x)^k$ reduced modulo p. Similarly, we define $q_{f(x),p}(k)$ to be the total number of nonzero coefficients of $f(x)^k$. We then define $r_{f(x),p}(n,\alpha) = \sum_{i=0}^{n-1} q_{f(x),p}(i,\alpha)$ and $r_{f(x),p}(n) = \sum_{i=0}^{n-1} q_{f(x),p}(i)$. We search for a quick method for calculating both $q_{f(x),p}(k,\alpha)$ and the asymptotic behavior of $r_{f(x),p}(n,\alpha)$ for large n.

3.2 Willson Method

Willson [6] describes an algorithm for computing the value of $r_{f(x),2}(n)$, which is provided in Theorem 3.1.

Theorem 3.1 (Willson's Method)**.** *For some polynomial $f(x)$ with maximum degree d, there exists a matrix B, row vector u, and column vector v each of size $2^d - 1$ such that $u \cdot B^k \cdot v = r_{f(x),2}(2^k)$.*

Amdeberhan-Stanley [4] describes a similar and related algorithm for calculating the number of each coefficient α for any power k for general $f(x)$ and p, namely $q_{f(x),p}(k,\alpha)$. Willson also analyzed the case of $p > 2$ in [7].

Example 3.2. *For $1+x+x^2$ mod 2, $B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$. Note that the largest eigenvalue of this matrix is $1+\sqrt{5}$.*

Theorem 3.3. *The matrix B is the sum of four matrices, each of which corresponds to a self-mapping of the set $X = F_2[x]/x^d \setminus 0$.*

[1]Strictly speaking, fore these sharp values, we may not have $c_1 n^2 \leq a(n) \leq c_2 n^2$, but for any $\delta > 0$ we have $(c_1 - \delta)n^2 \leq a(n) \leq (c_2 + \delta)n^2$ for large enough n.

Theorem 3.3 follows easily from Willson [6].

Remark 3.4. *The size of the matrix B can be made smaller only by using accessible blocks, as explained in Wilson [6].*

3.3 Eigenvalue Analysis

The matrix B has nonnegative entries and is irreducible. Following Willson [6], define λ to be the Perron-Frobenius eigenvalue of B, i.e., the largest positive eigenvalue of B (it exists by the Perron-Frobenius theorem). We define $\lambda(f)$ to be the value of λ for the polynomial $f(x)$. We can approximate the value of $r_{f(x),p}(p^k, \alpha)$ with λ^k because the entries of B^k grow as a constant times λ^k.

Example 3.5. *For $f(x) = 1 + x$ and $p = 2$, $\lambda = 3$ because $B = [3]$. In this case λ corresponds exactly to the scaling of the number of nonzero coefficients when doubling the number of rows, namely $r_{1+x,2}(2k) = 3 \cdot r_{1+x,2}(k)$.*

When examining the eigenvalues, we note that there are multiple transformations of a polynomial that does not change λ.

Theorem 3.6. *We define the polynomials $f(x)$ and $g(x)$ to be similar if we can transform $f(x)$ into $g(x)$ through a combination of the transformations $f(cx)$ and $cf(x)$ with integer $1 < c < p$, $x^c f(x)$ with integer $c > 0$, $f(x^c)$ with integer $c > 1$, $x^{deg(f)} f(x^{-1})$, and $f(x)^c$ with integer $c > 1$. Any two similar polynomials have the same λ.*

Proof. Because the transformations $f(c \cdot x), f(x^c), x^c \cdot f(x), c \cdot f(x)$, and flipping a polynomial do not change the number of nonzero coefficients of a polynomial, λ do not change. Furthermore, because $f(x)^c$ is every c^{th} row, the ratios over the long term of the sums of total number of nonzero coefficients does not change, so λ is the same. Namely, let $q_{f(x)}(n)$ be the number of nonzero coefficients of $f(x)^n$. Therefore $q_{f(x)}(n+1) \leq C \cdot q_{f(x)}(n)$, where C is the number of nonzero coefficients of $f(x)$. This means that

$$r_{f(x)}(k \cdot n) = \sum_{j=0}^{k \cdot n-1} q_{f(x)}(j) \leq \sum_{j=0}^{n-1} (1 + C + \ldots + C^{k-1}) q_{f(x)}(j \cdot k) \leq (1 + C + \ldots + C^{k-1}) r_{f(x)^k}(n).$$

Polynomial	λ	d	Polynomial	λ	d
$1+x$	3	1	$1+x+x^6$	3.45686	20
$1+x+x^2$	3.23607	2	$1+x+x^2+x^6$	3.49009	20
$1+x+x^3$	3.31142	4	$1+x+x^3+x^6$	3.50478	10
$1+x+x^4$	3.33159	5	$1+x^2+x^3+x^6$	3.53521	20
$1+x+x^2+x^4$	3.3788	7	$1+x+x^2+x^3+x^6$	3.53141	19
$1+x+x^3+x^4$	3.47662	4	$1+x+x^4+x^6$	3.50468	17
$1+x+x^2+x^3+x^4$	3.45729	4	$1+x+x^2+x^4+x^6$	3.55002	19
$1+x+x^5$	3.35174	10	$1+x+x^3+x^4+x^6$	3.59415	16
$1+x^2+x^5$	3.46127	12	$1+x^2+x^3+x^4+x^6$	3.53665	15
$1+x+x^2+x^5$	3.49563	7	$1+x+x^2+x^3+x^4+x^6$	3.59043	11
$1+x+x^3+x^5$	3.45469	12	$1+x+x^5+x^6$	3.54536	14
$1+x^2+x^3+x^5$	3.46639	5	$1+x+x^2+x^5+x^6$	3.50809	18
$1+x+x^2+x^3+x^5$	3.5229	14	$1+x+x^2+x^3+x^5+x^6$	3.57066	17
$1+x+x^2+x^4+x^5$	3.47168	11	$1+x+x^2+x^4+x^5+x^6$	3.49995	6
$1+x+x^2+x^3+x^4+x^5$	3.52951	6	$1+x+x^2+x^3+x^4+x^5+x^6$	3.5598	6

Table 4: λ and the degree of its minimal polynomial for $p=2$ and $\deg(f(x)) \leq 6$

This implies that $\lambda(f) \leq \lambda(f^k)$. Similarly since $q_{f(x)}(j \cdot k - i) \geq C^{-i} q_{f(x)}(j \cdot k)$, we can show that $\lambda(f^k) \leq \lambda(f)$. Therefore $\lambda(f) = \lambda(f^k)$. □

3.3.1 Values of λ where $p = 2$

We calculate λ for polynomials with $p = 2$. We also find the minimal polynomial of λ. Provided are λ and the degree d of its minimal polynomial for non-similar polynomials with degree of up to 6, although we had calculated for $deg(f) \leq 9$.

We see that λ is between 3 and 4. We form several conjectures on the bounds of λ.

Conjecture 3.7. *When $p = 2$, $\lambda \geq 3$. Furthermore, $\lambda = 3$ only for polynomials similar to $1+x$. If $p = 2$ and $\lambda > 3$, then $\lambda \geq 1 + \sqrt{5}$. Furthermore, $\lambda = 1 + \sqrt{5}$ only if $f(x)$ is similar to $1 + x + x^2$.*

Question 3.8. *Is it true that $\lambda(f) = \lambda(g)$ if and only if $f(x)$ and $g(x)$ are similar in terms of the transformations described in Theorem 3.6?*

Theorem 3.9. *For some polynomial $f(x)$ with degree at most 2^k and $p = 2$,*
$\lambda(f) \leq 4(1 - \frac{1}{2^{k+2}})^{\frac{1}{k+1}}$.

Proof. Define k such that the degree of $f(x)$ is at most 2^k, with $p = 2$. From Theorem 3.3, we can draw an oriented graph whose vertices are elements of X and whose edges correspond to the four maps. Therefore there are exactly four edges coming out of each vertex. Therefore if $Q(n)$ is the number of paths in the graph of length n, we have $\log\lambda = \limsup_{n\to\infty} \dfrac{\log Q(n)}{n}$. From the definition of Willson's method, Theorem 3.1, two of the four mappings correspond to $g(x) \to g(x^2)$ and $g(x) \to x \cdot g(x^2)$. Assume $\deg(f(x)) = 2^k$. Then a path starting from any $g(x)$ and moving first to $x \cdot g(x^2)$ then alternating in any way between the two mappings leads to 0 after $k+1$ steps. So the number of such paths of length $k+1$ is 2^k. So the number of paths of length $k+1$ from any point that avoids 0 is at most $4^{k+1} - 2^k$. Thus the number of such paths of length $n \cdot (k+1)$ is at most $(4^{k+1} - 2^k)^n$. This gives us the bound of $\lambda \leq 4(1 - \frac{1}{2^{k+2}})^{\frac{1}{k+1}}$. □

For $k = 0$, the only polynomial is $1 + x$, so the bound $\lambda \leq 4(1 - \frac{1}{4})^1 = 3$ is sharp. However, for $k = 1$ the bound tells us that $\lambda \leq \sqrt{14}$ which is not sharp. Furthermore, this bound approaches 4 as k approaches ∞.

Conjecture 3.10. *Let Λ_k be the maximal $\lambda(f)$ for deg $f \leq k$. Then $lim_{k\to\infty}\Lambda_k = 4$.*

Remark 3.11. *Similarly for $p > 2$, one may conjecture that $lim_{k\to\infty}\Lambda_k = p^2$.*

Through computer analysis of λ for $p = 2$ and $deg\Big(f(x)\Big) \leq 9$, Conjecture 3.12 arises.

Conjecture 3.12. *The degree of the minimal polynomial of λ is less than or equal to $2^{deg(f)-1}$ for $p = 2$.*

4 Conclusion and Directions of Future Research

Natural goals for further study of the phenomena examined in this paper include the following:

- Obtain recursion relations, generating functions, and limiting functions as in Section 2 for $a_{f(x),p}(n)$ in the case $deg\Big(f(x)\Big) > 1$;
- Prove Conjecture 2.13 on the functional equation for the generating function for $a_{f(x),p}(n)$;

- Prove the conjectures in section 3 on the behavior of the eigenvalues λ and obtain better upper bounds;
- Find, tighten, and explore the upper bound mentioned in Conjecture 3.12;
- Study the algebras generated by the four transformations composing the Willson matrices and find analogs for larger p.

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