

On Visibility Graphs — Upper Bounds and Classification of Special Types

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Abstract

We examine several types of visibility graphs: bar and semi-bar k -visibility graphs, rectangle k -visibility graphs, arc and circle k -visibility graphs, and compact visibility graphs. We improve the upper bound on the thickness of bar k -visibility graphs from $2k(9k - 1)$ to $6k$, and prove that the upper bound must be at least $k + 1$. We also show that the upper bound on the thickness of semi-bar k -visibility graphs is between $\lceil \frac{2}{3}(k + 1) \rceil$ and $2k$. In addition, we give a method of finding the number of edges in a semi-bar k -visibility graph based on skyscraper puzzles. Analogous to bar and semi-bar k -visibility graphs, we establish bounds on the number of edges, chromatic number, and thickness of rectangle k -visibility graphs; as well as bounds on the number of edges and the chromatic number of arc and circle k -visibility graphs. Finally, we relate two conjectures on compact visibility graphs, prove that every n -partite graph in the form $K_{1,a_1,a_2,\dots,a_{n-1}}$ is a compact visibility graph, and classify all (but one) graphs with at most six vertices as compact visibility graphs.

1 Introduction

Many problems, such as wiring thousands of transistors together on the same chip, called Very Large Scale Integration (VLSI), navigating a robot through a set of obstacles, or establishing a temporary military base out of sight of an enemy base, stem from the basic problem of when some objects are visible to other objects. We can use the concept of a graph to mathematically model these scenarios in the following way: each vertex, visualized by a point, in the graph corresponds to an object in the system, and two vertices are connected by an edge, visualized by a line, whenever their corresponding objects are visible with respect to each other. These graphs are called *visibility graphs*. The above scenarios have different restrictions on the shape of the objects, and when two objects are determined to be visible. This gives rise to several different types of visibility graphs. In this paper, we study five types of visibility graphs, relating to bar visibility, semi-bar visibility, rectangle visibility, arc/circle visibility, and compact visibility. We also study the related problem when objects are able to see through exactly k other objects for some positive integer k . These visibility graphs are known as bar, semi-bar, rectangle, and arc/circle k -visibility graphs.

To discuss this topic while maintaining mathematical rigor, we must give a formal definition of visibility:

Definition 1.1. Given a collection of n pre-determined sets of points $\{S_1, S_2, \dots, S_n\}$, there exists a *sightline* ℓ between two sets S_j and S_k if the endpoints of ℓ are in S_j and S_k and $\ell \cap \cup_{i=1}^n S_i$ only has two points in it. Two regions are then said to be *visible* with respect to each other if there exists a sightline between them.

Two concepts that figure heavily in our discussion are the notions of the thickness and the chromatic number of graphs.

Definition 1.2. The *thickness* $\Theta(G)$ of a graph G is the minimum number of planar subgraphs whose union is G . That is, $\Theta(G)$ is the minimum number of colors needed to color

the edges of G such that no two edges with the same color intersect.

The thickness of a bar, semi-bar, or rectangle visibility graph is intimately related to VLSI. VLSI circuits are built in layers to avoid wire crossings which disrupt signals [8]. For each graph G with vertices corresponding to circuit gates and edges corresponding to wires, the thickness of G gives an upper bound on how many layers are needed to build the VLSI circuit.

Definition 1.3. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors needed to label the vertices of G such that any two adjacent vertices receive different colors.

The chromatic number is related to the problem of scheduling, where there are several jobs corresponding to vertices and pairs of jobs which cannot be completed at the same time corresponding to edges. By letting each color represent a distinct valid time slot, any coloring of the graph provides a valid schedule for the jobs and the chromatic number provides the minimum number of necessary time slots.

Dean *et al.* [2] previously placed upper bounds on the number of edges, the chromatic number, and the thickness of bar k -visibility graphs with n vertices in terms of k and n . Felsner and Massow [4] tightened the upper bound on the number of edges on bar k -visibility graphs and placed bounds on the number of edges, the thickness, and the chromatic number of semi-bar k -visibility graphs. Hartke *et al.* [7] found sharp upper bounds on the maximum number of edges in bar k -visibility graphs.

Lin, Lu, and Sun [9] determined an algorithm for plane triangular graphs G with n vertices which outputs a bar visibility representation of G no wider than $\lfloor \frac{22n-42}{15} \rfloor$ in time $O(n)$. Bar ends were on grid points in their model. Bose *et al.* [10] defined rectangle visibility graphs with horizontal and vertical visibility lines and proved several classification results about them. Hutchinson [5] defined and proved results on the structure of arc and circle visibility graphs, in which visibility lines are subsegments of diameters. Develin *et al.* [6]

proved several statements on the structure of compact visibility graphs and formulated a conjecture about their general structure.

In Section 3, we improve the upper bound on the thickness of bar k -visibility graphs from the old bound proven by Dean *et al.* [2], and show a lower bound on the maximal thickness. Furthermore we improve the bounds on the thickness of semi-bar k -visibility graphs and rectangle k -visibility graphs. We also show a method to count the number of edges of any semi-bar k -visibility graphs. Finally, we use the previous results to bound the thickness of rectangle k -visibility graphs. In Section 4, we place upper bounds on the number of edges and the chromatic number of arc and circle k -visibility graphs. In Section 5, we show that two open problems on the structure of compact visibility graphs conflict, and we show that a specific infinite class of graphs are all compact visibility graphs. Finally, we classify (nearly) every compact visibility graph with six or fewer vertices, and a comprehensive list of graphs is in Appendix A. Section 6 poses several open questions based on work in this paper.

2 Preliminaries

In this section, we define the various types of visibility graphs, and cover conditions we assume throughout the paper.

2.1 Bar visibility graphs

Bar visibility graphs are graphs that have the property that the vertices of the graph correspond to the elements of a given set of horizontal bars in such a way that two vertices of the graph are adjacent whenever there exists a vertical sightline between their corresponding line segments. The set of bars is said to be the *bar visibility representation* of the graph. This type of visibility graph was introduced in the 1980s by Duchet *et al.* [1] and Schlag *et al.* [3] mainly for its applications in the development of VLSI. Figure 1 gives an example of a set

of horizontal line segments and its corresponding bar visibility graph. Sightlines are drawn as dashed lines.

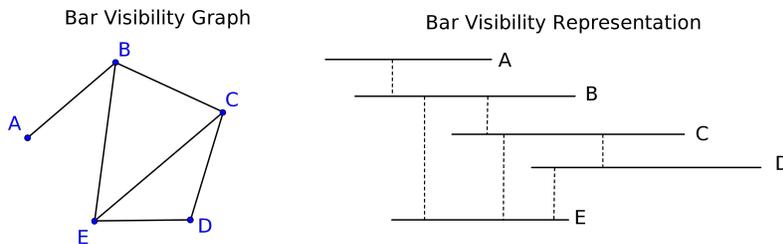


Figure 1: A bar visibility graph and its bar visibility representation.

Sometimes the requirement that the sightline must be a non-degenerate rectangle is added. This definition produces *bar ϵ -visibility graphs*. Though we only consider bar visibility graphs with sightlines only as line segments, usually called *strong bar visibility graphs*, we make the following assumption while bounding the thickness of bar k -visibility graphs:

Assumption 2.1. *If two endpoints of the bars have the same x -coordinate, we can elongate one of the bars by some real number ϵ such that no edges are deleted, and hence the thickness does not decrease, so we assume that the endpoints of the bars have different x -coordinates. This assumption ensures that the bar visibility graphs that we consider are all bar ϵ -visibility graphs. This assumption can also be extended to semi-bar k -visibility graphs.*

Dean *et al.* [2] recently extended the definition of bar visibility graphs, by adding the condition that sightlines are allowed to pass through at most k additional bars. These graphs are called *bar k -visibility graphs*. Figure 2 shows an example of a bar 1-visibility graph with a bar 1-visibility representation equivalent to the one in Figure 1. For convenience, sightlines that pass through an additional bar are drawn thicker than the original sightlines.

2.2 Semi-bar visibility graphs

We also study semi-bar k -visibility graphs. A *semi-bar k -visibility graph* is a bar k -visibility graph where the left endpoints of all the bars have x -coordinates equal to 0.

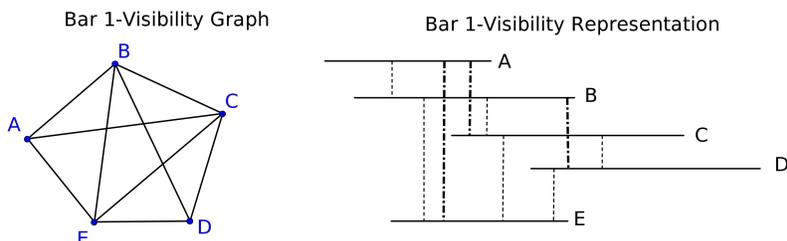


Figure 2: A bar 1-visibility graph and its bar 1-visibility representation.

These types of visibility graphs will be covered in Section 3, where we improve an old upper bound of $2k(9k - 1)$ on the thickness of bar k -visibility graphs, found by Dean *et al.* [2], to $6k$, by using a method similar to that developed by Felsner and Massow [4] for bounding the thickness of semi-bar k -visibility graphs. We also show that there exist bar k -visibility graphs with thickness at least $k + 1$. Afterwards, we prove an upper bound of $2k$ on the thickness of semi-bar k -visibility graphs, while we also show that there exist semi-bar visibility graphs with thickness at least $\lceil \frac{2}{3}(k + 1) \rceil$. Finally, we give a method to count the number of edges of a semi-bar k -visibility graph based on the structure of its semi-bar k -visibility representation.

This method is inspired by skyscraper problems. Skyscraper puzzles are grids of integers in which the integers correspond to the heights of semi-bars in a grid visibility representation. A skyscraper puzzle consists of an empty $n \times n$ grid with numbers written left or right of some rows and above or below some columns. The solver fills the grid with numbers between 1 and n representing heights of skyscrapers placed in each entry of the grid. The numbers are placed so that no two skyscrapers in the same column or same row have the same height. If there is a number m above a column in the empty grid, then the numbers $1, \dots, n$ must be placed in that column so that there are m numbers in the column which are greater than every number above them. If there is a number m below a column in the empty grid, then the numbers $1, \dots, n$ must be placed in that column so that there are m numbers in the column which are greater than every number below them. The restrictions for the numbers

in rows are defined analogously. Then each number m outside the grid corresponds to the number of visible skyscrapers in the row adjacent to m which are visible from the location of m if sightlines are parallel to the ground.

We consider a visibility representation based on skyscraper puzzles in which there is just a single column in which to place numbers, some of which may be the same, and there are possibly numbers above or below the column. Any such configuration corresponds to a semi-bar visibility graph. Any numbers above (resp. below) the column are the number of semi-bars which are longer than all semi-bars above (resp. below) them.

In Section 3 we show how to count the number of edges in any semi-bar visibility graph by using the numbers above and below the column in its skyscraper configuration. Furthermore we extend the skyscraper analogy to k -visibility graphs to show a similar result for semi-bar k -visibility representations.

2.3 Rectangle visibility graphs

A less restrictive version of the bar visibility graph is the *rectangle visibility graph*, whose vertices correspond to elements of a given set of rectangles with sides all either horizontal or vertical, and two vertices of the graph are adjacent whenever there is a vertical or horizontal sightline between their corresponding rectangles. Bose *et al.* [10] examined rectangle visibility graphs, and proved several statements about their structure. In this paper, we extend the definition to create the notion of *rectangle k -visibility graphs*, where sightlines are allowed to intersect at most k additional rectangles. We extend the results proven for bar and semi-bar k -visibility graphs to find an upper bound on the number of edges of rectangle k -visibility graphs, and show that the chromatic number is at most $12k + 12$ and the thickness is at most $12k$.

2.4 Arc and circle visibility graphs

An interesting extension of bar visibility graphs is the concept of *arc visibility graphs*, introduced by Hutchinson [5]. She defines a non-degenerate *cone* in the plane to be a 4-sided region of positive area with two opposite sides being arcs of circles concentric about the origin, and the other two sides being (possibly intersecting) radial line segments. Two concentric arcs a_1 and a_2 are then said to be *radially visible* if there exists a cone that intersects only these two arcs, and whose two circular ends are subsets of the two arcs. A graph is then called an *arc visibility graph* if its vertices can be represented by pairwise disjoint arcs of circles centered at the origin such that two vertices are adjacent in the graph if and only if their corresponding arcs are radially visible. *Circle visibility graphs* are defined in nearly the same way, with the difference that vertices can be represented as circles as well as arcs. Note that all arc visibility graphs are also circle visibility graphs. Figure 3 shows an arc visibility graph and its arc visibility representation.

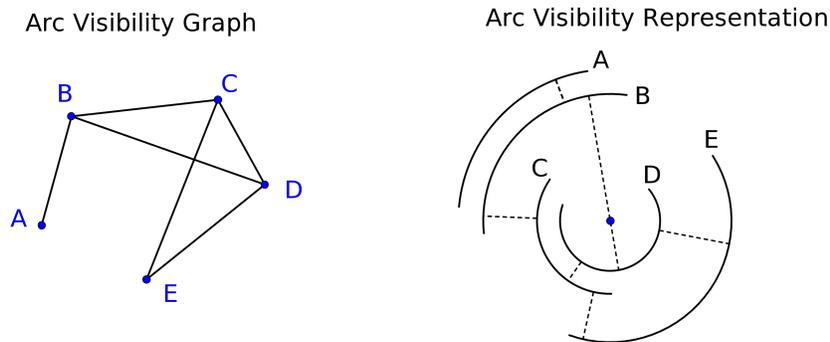


Figure 3: An arc visibility graph and its arc visibility representation.

We also examine *arc k -visibility graphs* and *circle k -visibility graphs*, where cones are allowed to see through k additional arcs and circles. Figure 4 shows the arc 1-visibility graph of the arc visibility representation shown in Figure 3.

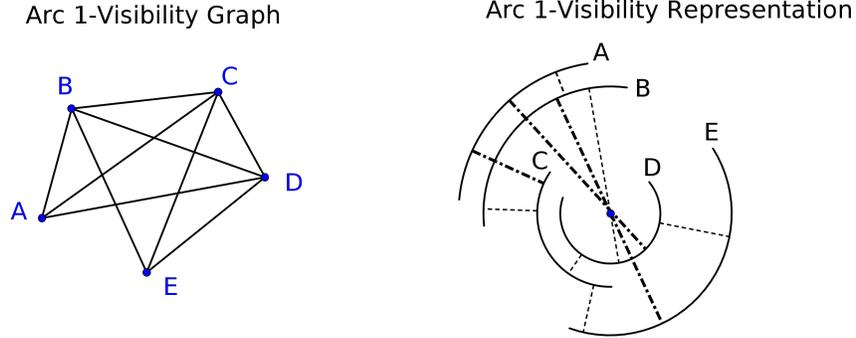


Figure 4: An arc 1-visibility graph and its arc 1-visibility representation.

Remark 2.2. *It immediately follows by definition that all bar k -visibility graphs are also arc k -visibility graphs, and it is a relatively simple task to generate an arc k -visibility representation from a bar k -visibility representation.*

When considering arc and circle k -visibility graphs, we make two assumptions, the first of which is analogous to the assumption made for bar k -visibility graphs in Section 1.1.

Assumption 2.3. *Each arc can be expressed as a set of polar coordinates $\{(r_i, \alpha) : \alpha_{i,1} \leq \alpha \leq \alpha_{i,2}\}$ for some positive r_i and some radian measures $\alpha_{i,1}$ and $\alpha_{i,2}$ in the interval $(-2\pi, 2\pi)$ with $0 < \alpha_{i,2} - \alpha_{i,1} < 2\pi$. We call the coordinates $(r_i, \alpha_{i,1})$ and $(r_i, \alpha_{i,2})$ the negative and positive endpoints of arc a_i , respectively. If two endpoints of two arcs have the same angular coordinate, then we can lengthen one slightly without deleting edges in the arc k -visibility graph, so we assume that no two arcs have endpoints with the same angular coordinate.*

Assumption 2.4. *If there are two arcs that are the same distance from the origin, then we can slightly increase the radius of one so that their radii are different without affecting the arc k -visibility graph. Therefore we also assume that no two arcs are the same distance away from the origin. We then label the arcs with a_1, a_2, \dots , and a_n , where a_i is given to the arc with the i th greatest radius.*

In Section 4 we establish an upper bound of $(k + 1)(3n - k - 2)$ on the number of edges, and an upper bound of $6k + 6$ on the chromatic number of circle k -visibility graphs.

2.5 Compact visibility graphs

One of the most general types of visibility graphs is the compact visibility graph. *Compact visibility graphs* are graphs that can be represented as a set of compact connected regions in the plane (a region in the plane is *compact* if it is both closed and bounded). The set of regions that generate a compact visibility graph is called the *compact visibility representation* of the graph. Here there are no directional restrictions on the sightlines. These types of visibility graphs are used to model physical problems, such as navigating a robot through a set of obstacles. Figure 5 shows several regions and the corresponding visibility graph.

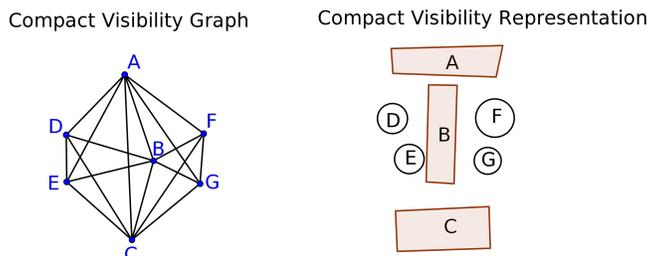


Figure 5: A compact visibility graph and its compact visibility representation.

The restriction that the regions must be convex can be added, to produce the notion of *convex compact visibility graphs*. Develin *et al.* [6] formulated restrictions on the structure of these two types of visibility graphs, and posed several open conjectures. In Section 5, we relate two of these conjectures by showing that they contradict, and we show that every n -partite graph of the form $K_{1,a_1,a_2,\dots,a_{n-1}}$ is a compact visibility graph. Finally, we classify compact visibility graphs with 6 or less vertices with the exception of one graph, with a comprehensive list of graphs and representations in Appendix A.

3 Bar, Semi-Bar, and Rectangle k -Visibility Graphs

3.1 Bar k -visibility graphs

Dean *et al.* [2] showed that the thickness of any bar k -visibility graph G_k is at most $2k(9k-1)$ by coloring the edges based on a vertex-coloring of G_{k-1} . Using a different construction, we achieve a bound strictly less than $2k(9k-1)$. We make the use of the following definition:

Definition 3.1. The *underlying bar i -visibility graph* G_i of S is the graph with bar i -visibility representation S .

We are now equipped to place an upper bound on the thickness of bar k -visibility graphs:

Theorem 3.2. *If G_k is a bar k -visibility graph with $k \geq 1$, then $\Theta(G_k) \leq 6k$.*

Proof. Consider a bar k -visibility representation S of some bar k -visibility graph. We now construct a bar k -visibility graph from S . Define a *one-bend drawing* of a graph as Felsner and Massow did in [4]: a drawing in the plane in which each edge is a polyline with at most one bend. We first create a one-bend drawing of G_k . First widen the bars so that they are rectangles, while keeping their lengths constant. Each vertex v in the graph now corresponds to a rectangle R_v . Next draw each vertex on the midpoint of the left side of each rectangle. Then take the leftmost endpoint, say u , of each edge $e = \{u, v\}$. This exists by Assumption 1.4. Then project v orthogonally onto the nearest side of R_u , and call this projection v' . Note that the line between v and v' is a sightline between R_u and R_v . Take v_ϵ on the side of R_u containing v' so that the length of $\overline{v'v_\epsilon}$ is small enough so that $\overline{v_\epsilon v}$ does not intersect any other drawn line segment. Let e be the union of the two line segments $\overline{uv_\epsilon}$ and $\overline{v_\epsilon v}$. Figure 6 shows the construction of such an edge. Note that if two edges are adjacent, then by definition they will not intersect anywhere other than their common endpoint.

Now take a vertex-coloring of G_{k-1} of S . In our one-bend drawing, color each edge using the color of its left-most vertex. We make use of Lemma 2.3:

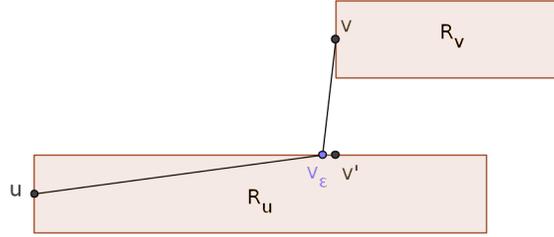


Figure 6: Construction of a one-bend edge.

Lemma 3.3. *If any two edges e and f with endpoints (u, v) and (w, x) respectively intersect, then they have different colors.*

Proof. Without loss of generality assume that u is to the left of v and w is to the left of x . It is clear that if e and f intersect, then the intersection point is either that of uv_ϵ and $x_\epsilon x$, or $v_\epsilon v$ and wx_ϵ . Without loss of generality, assume that the former occurs. Then the sightline between x and its orthogonal projection onto R_w passes through R_u . This means that u and w must have been connected in G_{k-1} , and it follows that u and w were assigned different colors. This shows that e and f were assigned different colors. \triangle

The only colors we used were the ones used in the vertex-coloring of G_{k-1} , so $\Theta(G_k)$ is bounded by $\chi(G_{k-1})$. This is shown in Corollary 9 by Dean *et al.* [2] to be at most $6k$, so it follows that $\Theta(G_k) \leq 6k$. \square

This bound is better since it is not quadratic, but Felsner and Massow [4] showed that for every bar 1-visibility graph G_1 , $\Theta(G_1) \leq 4$, so this upper bound is not tight.

We now establish a lower bound on the maximal thickness of bar k -visibility graphs.

Theorem 3.4. *There exist bar k -visibility graphs with thickness at least $k + 1$ for all $k \geq 0$.*

Proof. Consider m planar subgraphs of a bar k -visibility graph G_k with n vertices. It is a well known fact that the number of edges in a planar graph is at most six less than three times the number of vertices, so it follows that the number of edges in G_k is at most $m(3n - 6)$. Hartke *et al.* [7] showed that if G_k has least $2k + 2$ vertices, G_k has at most $(k + 1)(3n - 4k - 6)$

edges. Dean *et al.* [2] showed that this bound is sharp, so we consider a bar k -visibility graph with $(k+1)(3n-4k-6)$ edges. Therefore $m(3n-6) \geq (k+1)(3n-4k-6)$. It then follows that $\Theta(G_k) \geq (k+1)\frac{3n-4k-6}{3n-6}$. As n increases, the right hand side of the inequality grows larger than k , so for sufficiently large n we have that $\Theta(G_k) \geq k+1$. \square

3.2 Semi-bar k -visibility graphs

We can use the technique of bounding the thickness of bar k -visibility graphs in Theorem 2.2 to produce an upper bound of $2k+1$ on the thickness of semi-bar k -visibility graphs. However, we achieve a tighter bound of $2k$ by using a different coloring scheme.

Theorem 3.5. *If G is a semi-bar k -visibility graph with $k \geq 1$, then $\Theta(G) \leq 2k$.*

Proof. We use a construction analogous to that in Theorem 2.1, but have each edge start at the right vertices. The crucial step of this proof is Lemma 2.6:

Lemma 3.6. *Given a bar B in a semi-bar k -visibility representation, there are at most $2k-1$ longer bars such that the edges starting from those bars cross B .*

Proof. If there are $k+1$ or more longer bars on each side with edges transversing B , then the $(k+1)^{\text{th}}$ bar would traverse $k+1$ bars (the first k bars and then B) to get to a smaller bar on the other side of B and thus transversing B . Therefore, there are at most k bars on each side. Assume that there are k longer bars on each side. Now consider the top-most and bottom-most bars B_t and B_b respectively among those k bars. From our assumption, there must be bars b_t and b_b on the upper and lower sides of B respectively such that they are shorter than B . Assume without loss of generality that b_t is shorter than b_b . This implies that B_t must cross b_t to reach b_b , which shows that B_t traverses $k+1$ bars. This is a contradiction, which implies that we cannot have k longer bars on each side of B with edges transversing B . This completes the proof of Lemma 2.6. \triangle

We now start coloring the bars in decreasing order of length. Clearly we can color every edge emanating from the first $2k - 1$ bars with $2k - 1$ colors. Now for every next bar, edges colored with at most $2k - 1$ colors will traverse this bar, so we color this bar with the $(2k)^{\text{th}}$ remaining color. From our original construction, intersections will only happen within bars, so this coloring produces $2k$ planar subgraphs. This completes the proof of Theorem 2.5. \square

We now prove a lower bound on the maximal thickness of semi-bar k -visibility graphs.

Theorem 3.7. *There exist semi-bar k -visibility graphs with thickness at least $\lceil \frac{2}{3}(k + 1) \rceil$ for all $k \geq 0$.*

The proof of Theorem 2.7 is analogous to the proof of Theorem 2.4; the only difference is the upper bound on the number of edges. Hartke *et al.* [7] proved a sharp upper bound of $(k + 1)(2n - 2k - 3)$ on the number of edges.

3.3 Counting edges in semi-bar k -visibility graphs

Here we devise a method of counting the number of edges in a semi-bar k -visibility graph.

Let $G = (V, E)$ be a semi-bar visibility graph with n vertices. Then G has some semi-bar visibility representation $S_G = \{s_v\}_{v \in V}$ of disjoint horizontal segments with left endpoints on the y -axis (semi-bars) such that for all $a, b \in V$, $\{a, b\} \in E$ if and only if all semi-bars between s_a and s_b are shorter than both s_a and s_b .

Let the function $A(S)$ be the number of semi-bars in S which are taller than all semi-bars above them, and $U(S)$ be the number of semi-bars in S which are taller than all semi-bars under them. These are analogous to the numbers above and below each column in skyscraper puzzles. For each $s \in S$ let $a(s) = 1$ if s is taller than all semi-bars above it and let $a(s) = 0$ otherwise. Let $u(s) = 1$ if s is taller than all semi-bars under it and let $u(s) = 0$ otherwise. Then $A(S) = \sum_{s \in S} a(s)$ and $U(S) = \sum_{s \in S} u(s)$.

Lemma 3.8. *If S_G is any semi-bar visibility representation of G and all semi-bars in S_G have different lengths, then the number of edges in G is $2n - A(S_G) - U(S_G)$.*

Proof. Pick an arbitrary semi-bar visibility representation S_G of G . For each $v \in V$, count how many edges in E include v and some w for which s_w is taller than s_v . Then each v contributes $2 - a(s_v) - u(s_v)$ edges, so there are $2n - A(S_G) - U(S_G)$ total edges. \square

Call an unordered pair of semi-bars $\{s_a, s_b\}$ a bridge if s_a is the same height as s_b and all semi-bars between s_a and s_b are shorter than s_a . A semi-bar can be contained in at most two bridges. Let $Br(S)$ denote the number of bridges in S .

Lemma 3.9. *If S_G is any semi-bar visibility representation of G , then the number of edges in G is $2n - A(S_G) - U(S_G) - Br(S_G)$.*

Proof. Pick an arbitrary semi-bar visibility representation S_G of G . For each $v \in V$, count how many edges in E include v and some w for which s_w is at least as tall as s_v . Then each v contributes $2 - a(s_v) - u(s_v)$ edges, but the edge $\{a, b\}$ is double counted whenever $\{s_a, s_b\}$ is a bridge. So there are $2n - A(S_G) - U(S_G) - Br(S_G)$ total edges. \square

We now extend this notion to semi-bar k -visibility graph. Let $G_k = (V, E)$ be a semi-bar k -visibility graph. Then G_k has a semi-bar k -visibility representation $S_{G_k} = \{s_v\}_{v \in V}$ of disjoint horizontal semi-bars with left endpoints on the y -axis such that for all $a, b \in V$, $\{a, b\} \in E$ if and only if all but at most k semi-bars between s_a and s_b are shorter than both s_a and s_b . Define $\{s_a, s_b\}$ to be a j -visibility edge if all but exactly j semi-bars between s_a and s_b are shorter than both s_a and s_b .

Let the function $A_j(S)$ be the number of semi-bars in S which are taller than all but at most j semi-bars above them, and $U_j(S)$ be the number of semi-bars in S which are taller than all but at most j semi-bars under them. For each $s \in S$ let $a_j(s) = 1$ if s is taller than all but at most j semi-bars above it and let $a_j(s) = 0$ otherwise. Let $u_j(s) = 1$ if s is taller than

all but at most j semi-bars under it and let $u_j(s) = 0$ otherwise. Then $A_j(S) = \sum_{s \in S} a_j(s)$ and $U_j(S) = \sum_{s \in S} u_j(s)$.

Lemma 3.10. *If S_{G_k} is any semi-bar k -visibility representation of G_k and all semi-bars in S_{G_k} have different lengths, then the number of edges in G_k is*

$$2(k+1)n - \sum_{j=0}^k (A_j(S_{G_k}) + U_j(S_{G_k})).$$

Proof. Pick an arbitrary semi-bar k -visibility representation S_{G_k} of G_k . Fix $j \leq k$, and for each $v \in V$, count how many j -visibility edges include s_v and some s_w for which s_w is taller than s_v . Then each v contributes $2 - a_j(s_v) - u_j(s_v)$ j -visibility edges, so there are $2n - A_j(S_{G_k}) - U_j(S_{G_k})$ total j -visibility edges in S_{G_k} . Summing over all j from 0 to k there are $2(k+1)n - \sum_{j=0}^k (RL_j(S_{G_k}) + LR_j(S_{G_k}))$ total edges in G_k . \square

Call an unordered pair of semi-bars $\{s_a, s_b\}$ a j -bridge if s_a is the same height as s_b and all but exactly j semi-bars between s_a and s_b are shorter than s_a . A semi-bar can be contained in at most two j -bridges for each j . Let $Br_j(S)$ denote the number of j -bridges in S .

Lemma 3.11. *If S_{G_k} is any semi-bar k -visibility representation of G_k , then the number of edges in G_k is*

$$2(k+1)n - \sum_{j=0}^k (A_j(S_{G_k}) + U_j(S_{G_k}) + Br_j(S_{G_k})).$$

Proof. Pick an arbitrary semi-bar k -visibility representation S_{G_k} of G_k . Fix $j \leq k$, and for each $v \in V$, count how many j -visibility edges include s_v and some s_w for which s_w is at least as tall as s_v . Then each v contributes $2 - a_j(s_v) - u_j(s_v)$ j -visibility edges, but the j -visibility edge between s_a and s_b is double counted whenever $\{s_a, s_b\}$ is a j -bridge. So there are $2n - A_j(S_{G_k}) - U_j(S_{G_k}) - Br_j(S_{G_k})$ total j -visibility edges in S_{G_k} . Then there are $2(k+1)n - \sum_{j=0}^k (A_j(S_{G_k}) + U_j(S_{G_k}) + Br_j(S_{G_k}))$ total edges in G_k . \square

Since Felsner and Massow showed a tight upper bound of $(k + 1)(2n - 2k - 3)$ on the number of edges in semi-bar k -visibility graphs with $n \geq 2k + 2$ vertices, then the Lemma 3.11 implies the next corollary.

Corollary 3.12. *If S_{G_k} is any semi-bar k -visibility representation of G , then*

$$\sum_{j=0}^k (A_j(S_{G_k}) + U_j(S_{G_k}) + Br_j(S_{G_k})) \geq (k + 1)(2k + 3).$$

3.4 Rectangle k -visibility graphs

Some bounds on parameters of rectangle k -visibility graphs are immediate corollaries of the results about bar k -visibility graphs. If G_k is a rectangle k -visibility graph with $n \geq 2k + 2$ vertices, then there are at most $(k + 1)(3n - 4k - 6)$ horizontal visibility segments and $(k + 1)(3n - 4k - 6)$ vertical visibility segments in any rectangle k -visibility representation of G , by the upper bound on edges in bar k -visibility graphs [7]. Therefore G_k has at most $2(k + 1)(3n - 4k - 6)$ edges and the chromatic number of G_k is at most $12k + 12$.

As with bar k -visibility graphs, this yields an upper bound on the thickness of G_k . Observe that making all rectangle sides in the representation lie on different lines does not decrease the maximum possible thickness of G_k .

Lemma 3.13. *If G_k is a rectangle k -visibility graph, then $\Theta(G_k) \leq 12k$.*

Proof. Let R_{G_k} be a rectangle k -visibility representation of G_k . We first alter R_{G_k} so that we can use Theorem 3.2. Consider the k -visibility representation L_{G_k} obtained from R_{G_k} by removing the top and rightmost sides of all rectangles and allowing visibility lines between two L -shapes u and v to be vertical or horizontal segments passing through u , v , and at most k other L -shapes. Then G_k is also the k -visibility graph of L_{G_k} .

Modify L_{G_k} by changing each L -shape s in L_{G_k} to a region r_s consisting of two nondegenerate rectangles. The endpoints of the segments in s are the midpoints of the short sides of the

rectangles in r_s and the width of r_s is small enough that the resulting L -shaped regions are disjoint, the k -visibility graph of L_{G_k} is the same, and no sides of any regions are on the same line.

Draw the vertex for each region r_s in L_{G_k} on the bottom left corner of s , where short sides of the rectangles in r_s intersect at their midpoints. With $6k$ colors make a one-edge drawing as in Theorem 3.2 connecting the vertices with horizontal k -visibility lines so that no crossing edges have the same color. Do the same for pairs of vertices with vertical k -visibility lines using $6k$ other colors. Then the resulting drawing uses $12k$ colors and no pair of crossing edges have the same color. \square

4 Arc/Circle k -Visibility Graphs

Upper bounds on the number of edges and the chromatic number of bar k -visibility graphs were found by Dean *et al.* [2], and several of the methods used can be applied to arc k -visibility graphs and circle k -visibility graphs. Here we set upper bounds on these properties for these types of graphs.

We first bound the number of edges of arc k -visibility graphs. Consider an edge $\{u, v\}$ in the visibility graph, and let U and V be their corresponding arcs. Let $\ell(\{u, v\})$ denote the radial line segment between U and V whose angular coordinate is the infimum of the (possibly negative) angular coordinates of all lines of sight between U and V . If $\ell(\{u, v\})$ contains the negative endpoint of U (respectively V) then we call $\{u, v\}$ a *negative edge* of U (respectively V). If $\ell(\{u, v\})$ does not contain the negative endpoint of U or V , then it must be a k -visibility edge, and it must contain the positive endpoint of some arc B that blocks the k -visibility between U and V after that point. We call it a *positive edge* of B .

Remark 4.1. *By definition, there are at most $k + 1$ positive edges and at most $2k + 2$ negative edges corresponding to each arc. Therefore, there are at most $(3k + 3)n$ edges in an*

arc k -visibility graph with n vertices.

Theorem 4.2. *In a circle k -visibility graph with $n \geq 2k + 2$ vertices, there are at most $(k + 1)(3n - k - 2)$ edges.*

Proof. We begin by proving the statement for arc k -visibility graphs. Remark 3.1 gives us an upper bound of $3n(k + 1)$ edges. However, arcs a_1, a_2, \dots, a_{k+1} have at most $k + 1, k + 2, \dots, 2k + 1$ negative edges respectively and $0, 1, \dots, k$ positive edges respectively. Therefore the bound on edges can be improved to

$$(3k + 3)n - 2 \sum_{i=1}^{k+1} i = (k + 1)(3n - k - 2).$$

Now consider circle k -visibility graphs. Note that circles do not have any positive or negative edges associated with them. This implies that having circles in the circle k -visibility representation does not increase the upper bound on the edges. Thus there are at most $(k + 1)(3n - k - 2)$ edges in a circle k -visibility graph with n vertices. \square

Remark 4.3. *Note that letting $k = 0$ produces an upper bound on the number of edges that a circle visibility graph can have.*

These upper bounds on the number of edges of arc and circle k -visibility graphs immediately give an upper bound on the chromatic number of these two types of graphs.

Corollary 4.4. *If G is a circle k -visibility graph, then $\chi(G) \leq 6k + 6$.*

The proof is nearly identical to the proof of Corollary 9 in a paper by Dean *et al.* [2].

We showed in Section 3 that the thickness of bar k -visibility graphs is bounded by their chromatic numbers. Bar k -visibility graphs are all arc k -visibility graphs, which are in turn circle k -visibility graphs. We are then tempted to claim that the maximal thickness of all circle k -visibility graphs is at most $6k$, but a similar proof does not hold, for arcs can see through the center of the arc k -visibility representation. We have not found a bound yet.

5 Compact Visibility Graphs

5.1 Two open problems

Several conjectures were presented by Develin *et al.* [6] on the structure of compact visibility graphs. One of note is the following:

Conjecture 5.1 (Develin *et al.* [6]). *A planar graph G is a compact visibility graph if and only if it has a plane drawing such that for all internal faces F of G , the subgraph G_F induced by the vertices of F is a compact visibility graph.*

This conjecture is quite significant, for if true it enables us to generate infinitely many and arbitrarily large compact visibility graphs without having to find the sets of regions corresponding to them. However, Develin *et al.* [6] posed another problem which conflicts with this conjecture. While considering compact visibility graphs with regions in higher dimensions, they asked the following question:

Question 5.2 (Develin *et al.* [6]). *Is it true that all graphs representable as compact visibility graphs in \mathbb{R}^3 are also representable as compact visibility graphs in \mathbb{R}^2 ?*

We construct a planar graph that is representable in three dimensions that has no planar drawing with the property given in Conjecture 4.1. Consider a sphere and a very small slice C . Now take four extremely thin ellipsoids and place them around C . Make the ellipsoids thin enough so that opposite ellipsoids cannot see each other over C , and wide enough so that each ellipsoid can see the other two. Iterating this once produces a wheel. Reiterating this several times produces a graph similar to that in Figure 7. The central vertex corresponds to the sphere, and the other vertices correspond to the ellipsoids.

It is not hard to see from the geometry of this graph that it cannot be drawn such that none of the 4-cycles are faces, and the graphs induced on these faces are simply 4-cycles,

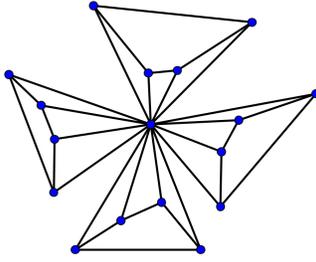


Figure 7: Graph that disproves Conjecture 4.1 if Question 4.2 is true.

which are not compact visibility graphs by Theorem A.1. It then follows that Conjecture 4.1 and Question 4.2 cannot both be true.

5.2 Complete n -partite compact visibility graphs

Inspired by the compact visibility representation of $K_{1,2,3}$ shown in Figure 8, we classify this special class of n -partite graphs as compact visibility graphs.

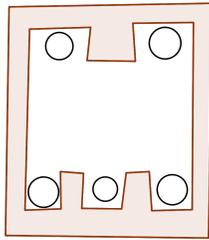


Figure 8: Compact visibility representation of $K_{1,2,3}$.

Theorem 5.3. *The complete n -partite graph $K_{1,a_1,a_2,\dots,a_{n-1}}$ is a compact visibility graph.*

The construction in Figure 8 can easily be generalized to generate representations for every such graph by making the surrounding region a polygon with $n - 1$ sides.

5.3 Classification of compact visibility graphs

The work that Develin *et al.* [6] have done on compact visibility graphs has provided tools to identify graphs that are not compact visibility graphs. Though these tools most likely do not identify all graphs that are not compact visibility graphs, they do identify most such small

graphs. They then enable us to classify most graphs with six or fewer vertices. Appendix A gives a nearly comprehensive list of every compact visibility graph with six vertices or less, excluding the graph depicted in Figure 9.

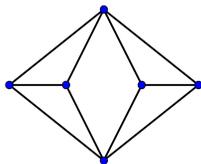


Figure 9: The only unclassified graph with six or less vertices.

After some experimentation, it seems unlikely that this graph could be a compact visibility graph. If it is in fact a compact visibility graph, then it falsifies Conjecture 4.1.

6 Concluding Remarks

We have improved a quadratic upper bound on the thickness of bar k -visibility graphs to a linear upper bound, and we have placed a similar upper bound on the thickness of semi-bar k -visibility graphs. These bounds are not yet sharp, and we have added linear lower bounds on the upper bounds, so we are inspired to pose the following question:

Question 6.1. *Is there a linear function $f(k)$ such that $\Theta(G_k) \leq f(k)$ for all bar k -visibility graphs G_k , while for all $k \geq 0$ there exist such graphs such that $\Theta(G_k) = f(k)$? Do similar functions exist for semi-bar, rectangle, or circle k -visibility graphs?*

We have also demonstrated a formula for counting the number of edges in a semi-bar k -visibility graph by using information about the structure of its k -visibility representation. It would be interesting to determine if similar formulas exist for bar or rectangle k -visibility graphs.

In addition, we have placed upper bounds on the number of edges in and the chromatic number of circle k -visibility graphs. We are then tempted to ask:

Question 6.2. *What is the largest possible number of edges in a circle k -visibility graph G_k with n vertices? What is the largest possible value of $\chi(G_k)$?*

Figure 9 begs the final question:

Question 6.3. *Is the graph in Figure 9 a compact visibility graph?*

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A Appendix of Compact Visibility Graphs

The classification of compact visibility graphs is made easier by Theorem A.1.

Theorem A.1 (Develin et al. [6]). *If G is a compact visibility graph, then every edge of G is either a cut-edge or part of a triangle.*

This theorem is valid for compact visibility graphs that have concave regions as well as convex regions, and it directly shows that the 48 graphs listed in Figure 10 are not compact visibility graphs.

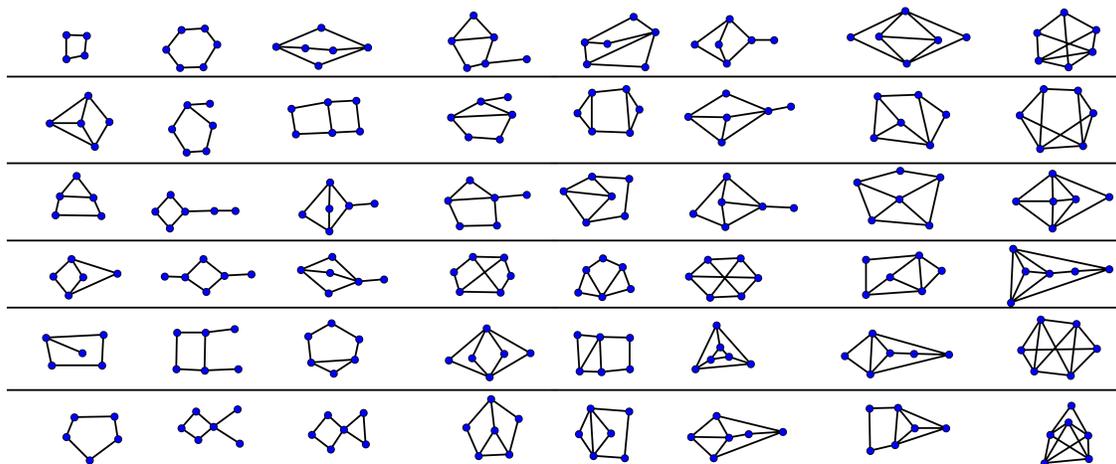


Figure 10: Table of graphs eliminated by Theorem A.1.

Develin et al. [6] also gave a method of constructing an infinite family of compact visibility graphs in the proof of Theorem A.2.

Theorem A.2 (Develin et al. [6]). *If G has a plane drawing such that all of the internal faces are triangles, then G is a compact visibility graph.*

It follows that the 24 graphs given in Figure 11 are compact visibility graphs.

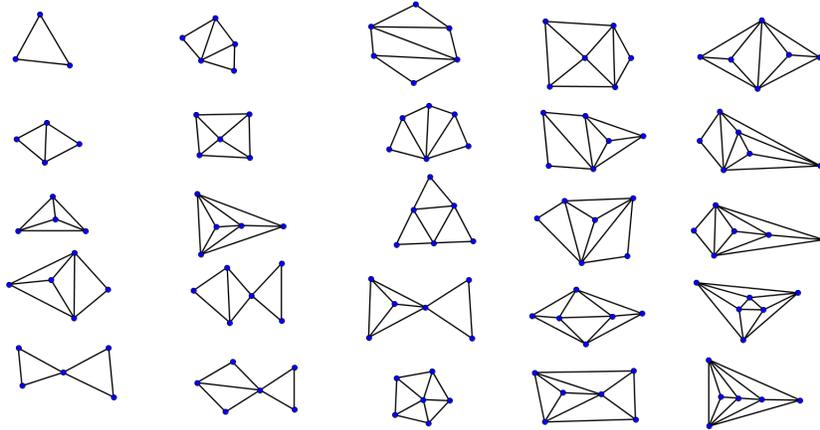


Figure 11: Compact visibility graphs given by Theorem A.2.

In addition, if a graph G is a visibility graph, then if we attach a leaf to a vertex v , the region corresponding to v can be hollowed so that the region corresponding to the new leaf can be inserted inside it. It follows that if a vertex u has a degree of 1 and $G - \{u\}$ is a compact visibility graph, then G is also a compact visibility graph. Therefore the graphs in Figure 12 are compact visibility graphs:

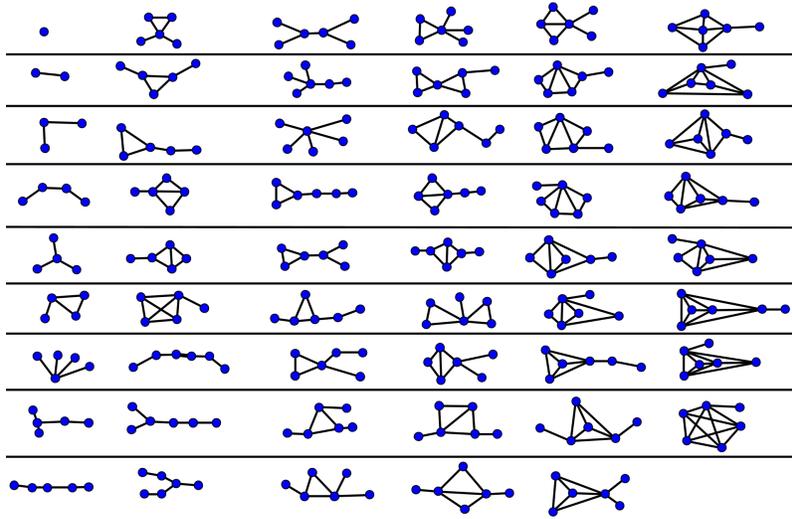


Figure 12: Table of compact visibility graphs with leaves.

This leaves the 16 graphs in Figure 13, and the unclassified graph in Figure 9. The compact visibility representations are given with the other 16 graphs.

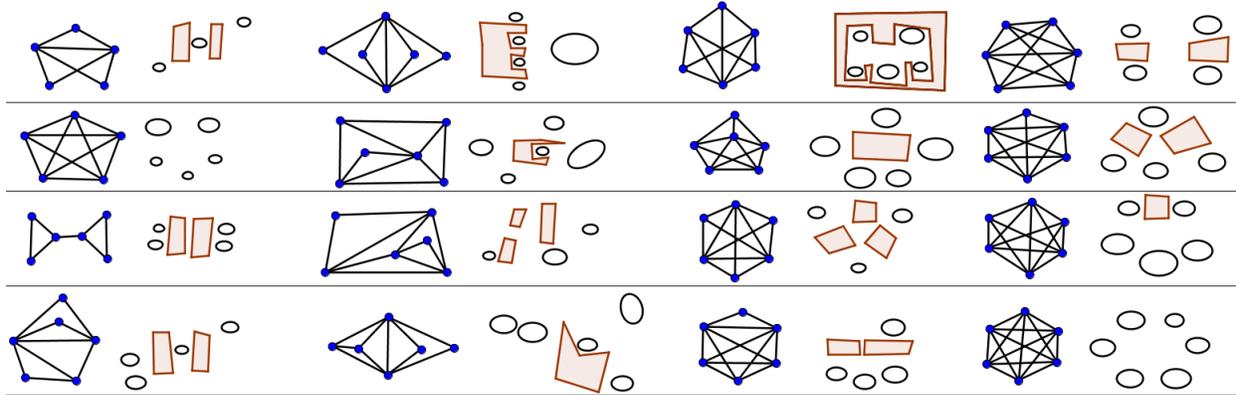


Figure 13: 16 remaining compact visibility graphs.