PRIMES STEP 2017 Entrance Exam
Part 1. General Questions. Logic.

Your name:

Problem 1. 1 point. The professor is watching across a field how the son of the professor's father is fighting with the father of the professor's son. How is this possible?

Answer:

Problem 2. 1 point. Bob spent the first Tuesday of the month hiking in the Himalayas. He spent the Tuesday after the first Monday of the same month at a conference in Seattle. In the next month he spent the first Tuesday on a cruise in the Black sea. He spent the Tuesday after the first Monday of that month celebrating his birthday in Boston. What day of the year is his birthday?

Answer:

Problem 3. 1 point. I can predict the score of every basketball game before it starts. How?

Answer:

Problem 4. 1 point. There are four silver coins marked 1, 2, 3, and 5. They are supposed to weigh the number of grams that is written on them. One of the coins is fake and is lighter than it should be. Find the fake coin using a balance scale not more than twice. Explain.

Solution:

Problem 5. 1 point. Can a power of 2 have the same number of of all the digits: zeros, ones, twos, ..., nines? Explain.

Solution:

Problem 6. 1 point. The King hated his Prime Minister for being too smart. He decided to fire the Prime Minister, but wanted to pretend that the Prime Minister has a choice. He called the Prime Minister and told him to chose one of the two boxes. The note in the chosen box will decide the fate of the Prime Minister: to be fired or to stay on the job. The Prime Minister guessed that both notes said "FIRE". What can he do to save his job?

Answer:

Problem 7. 2 points. There are 12 people in the room. Some of them are liars and some truth-tellers. The first person said, "There are no truth-tellers here." The second person said, "There is no more than 1 truth-teller here." The third person said, "There are no more than 2 truth-tellers here." And so on. The 12-th person said, "There are no more than 11 truth-tellers here." How many truth-tellers are in the room? Explain.

Solution:

PRIMES STEP 2017 Entrance Exam
Part 2. Number Theory. Combinatorics. Probability. Algebra. Geometry.

Problem 8. 1 point. Two two-digit prime numbers are reversals of each other. Their difference is a square. What are the numbers?

Answer:

Problem 9. 1 points. A teacher claims that he has 100 students in his class consisting of 24 boys and 32 girls. What base does the teacher use for his integers? Explain.

Solution:

Problem 10. 1 point. I am buying seven balloons. The store sells white, green, blue, and red balloons. In how many different ways can I buy my seven balloons?

Answer:

Problem 11. 2 points. I have a regular deck of 52 cards. I picked 4 cards at random. What is the probability that I have an even number of aces? Give a formula, not the exact number.

Answer:

Problem 12. 2 points. In a certain country the Congress consists of 30 congress-people. Every pair of them are either friends or enemies with each other. It is known that every congress-person has exactly 6 friends in Congress. The Congress has many committees. In fact, every 3 congress-persons form a committee. Find the number of committees such that all three members are pairwise friends or all three members are pairwise enemies. Explain.

Solution:

Problem 13. 1 point. If $x + \frac{1}{16x} = 1$, find the value of $64x^3 + \frac{1}{64x^3}$.

Solution:

Problem 14. 1 point. Find the remainder of the polynomial $P(x) = x^{81} + x^{27} + x^9 + x^3 + x$ when dividing by $x + 1$. Explain.

Solution:

Problem 15. 2 points. In the acute triangle ABC, three segments are drawn from vertex A until the intersection with the side BC so that AD is the bisector, AH is the altitude, AM is the median. Given that angle B is bigger than C, in what order are points B, C, D, H, and M are placed on the line BC? Explain.

Solution:

PRIMES STEP 2017 Entrance Exam
Part 3. Advanced Topics. Free Fibonacci Sequences.

The Fibonacci sequence (0, 1, 1, 2, 3, 5, 8, . . .) is defined recursively: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. There are many other sequences that are defined with the same recursion, but different initial numbers. The most famous among them is Lucas sequence: 2, 1, 3, 4, 7, 11, ...

John H. Conway loves playing with the Fibonacci rule. He invented many sequences by tweaking the recursion rule. Today we consider one of his rules. The rule for n-free Fibonacci numbers is the following: sum two previous integers and divide the result by the largest power of n possible. For example, if the sum is 96, and $n = 4$, we find that 96 is divisible by 4^2, but not 4^3. So we divide by 16 to get 6.

Let us look at 5-free Fibonacci numbers. We start with 0, 1. The next number, 1, is not divisible by 5 so we keep it. Then 2 and 3 follow as is. When we get to the next number, 5, we need to divide it by 5, to get 1. For the next number we sum up 3 and 1, and keep 4. For the next number we get $1 + 4 = 5$. We need to divide by 5 to get back to 1 again. The resulting sequence is: 0, 1, 2, 3, 1, 4, 1, 1, 2, 3, 1, 4, 1, and so on.

Let us try Lucas numbers as a start: 2, 1, 3, 4, 7, 11, and so on. No number is divisible by 5 yet.

Problem 16. 1 point. Prove that Lucas numbers are never divisible by 5.

solution:

That means that 5-free Fibonacci sequence that starts with 2, 1 is the same sequence as Lucas sequence. In the next exercise we explore free Fibonacci sequences for $n = 2$. Assume that your starting integers are always non-negative.

Problem 17. 2 points. Write 2-free Fibonacci numbers, that is a 2-free Fibonacci sequence that starts with 0, 1. Prove the pattern. What happens if you start with other integers? Prove.

Solution: