# Alternator Coins

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Mathematicians have long been fascinated with coin puzzles. The simplest one is formulated like this:

> You are given $N$ coins that look identical, but one is fake and is lighter than the real coins, all of which weigh the same. What is the fewest number of weighings on a balance scale that guarantees finding the fake coin?

This puzzle first appeared in 1945. Since then there have been many generalizations (see R. K. Guy and R. J. Nowakowski, Coin-weighing problems, *Amer. Math. Monthly* **102** no. 2 [1995] 164–167).

For example, a 2015 generalization inspired this article. A *chameleon coin* can mimic a fake or real coin. It has a mind of its own and can choose how to behave at any weighing. It is impossible to find chameleon coins among real coins as they can pretend to be real all the time. An interesting question to ask is: Given $N$ coins containing one chameleon and one fake coin, find two coins one of which is guaranteed to be fake. (See T. Khovanova, K. Knop, and O. Polubasov, Chameleon coins, *https://arxiv.org/abs/1512.07338.*)

We can draw a parallel between coin puzzles and logic puzzles. Real coins are similar to truth-tellers, and fake coins are similar to liars. Many logic puzzles include *normal* people: people who sometimes tell the truth and sometimes lie. Thus, a chameleon coin is an analogue of a normal person. Some logic puzzles have *alternators*: people who alternate between telling the truth and lying. Although it is impossible to identify a normal person—who can behave consistently as a truth-teller—we can identify an alternating person by asking for the sum of two and two—twice.

So, it is natural to introduce the analogue of an alternator to coin puzzles: An *alternator coin* can mimic a fake or a real coin, but it switches its behavior each time it is on the scale. Unlike the chameleon, the alternator can always be found.

## Alternators

In this article we pose the natural question: Suppose we have one alternator among $N$ coins and a balance scale. What is the fewest number of weighings that guarantees finding the alternator coin? We denote the smallest number of weighings $a(N)$.

We say that the alternator is in *state f* or *state r* if the next time it is on the scale it behaves as a fake coin or real coin, respectively. If we do not know the state, we say it is in *state a.*

We can simplify our problem by assuming that we know the state of the alternator coin in advance. Let

$f(N)$ denote the smallest number of weighings that guarantees finding the alternator if we know it starts in state *f*. Likewise, $r(N)$ is the smallest number of weighings that guarantees finding the alternator if we know it starts in state $r$. When calculating $a(N)$, we assume it starts in state $a$.

## One Fake Coin

Here we remind the readers of the standard analysis of the puzzle with one fake coin.

Suppose there is a strategy that is guaranteed to find a fake coin in $w$ weighings. Then there is a sequence of weighings that will find the fake coin. And we can represent the sequence of weighings as a string of three letters: E (the pans are equal weights), L (the left pan is lighter), and R (the right pan is lighter). The same string cannot correspond to two different coins; that is, if $i \neq j$, the string yielding the conclusion that the $i$th coin is fake must be different than the one yielding the $j$th coin is fake. So, the number of coins that can be processed in $w$ weighings is not more than $3^w$. In other words, if we have $N$ coins, no strategy can guarantee finding the fake coin in fewer than $\lceil \log_3 N \rceil$ weighings. (Recall that $\lceil x \rceil$ is the least integer greater than or equal to $x$.)

On the other hand, we can produce a strategy that finds the fake coin out of $N$ coins in $\lceil \log_3 N \rceil$ weighings. For example, if there are $3^w$ coins, divide them into three piles of $3^{w-1}$ coins. Put two piles on the scale, and if it doesn't balance, the fake coin is on the lighter pan. If it balances, the fake coin is in the pile not on the scale. So, with each weighing we make the pile containing the fake coin one third the size. Using this algorithm we can find the fake coin in $w$ weighings. The same idea works if the number of coins is not a power of three. We leave the details to the reader.

## Small Examples

The alternator is trickier to find than a fake coin, so we expect to need more weighings. We could, for instance, do the same thing as if looking for a fake coin, but perform every weighing twice. If the alternator participates in two consecutive weighings, it has to act as fake in one of them. But we can do better. Let's try some small examples. These will illustrate our general strategy.

When there are two or three coins, we can perform only one type of weighing: Compare one coin to another. We can find the alternator coin in two weighings by comparing the first and the second coin twice. If one of these coins is the alternator, it will reveal itself. If not, which can happen only if there are three coins, then the third coin is the alternator. So, $f(2) = f(3) = 1$ and $r(2) = r(3) = a(2) = a(3) = 2$.

Now, suppose there are four or five coins. If the coin is in state *f*, weigh coin 1 versus coin 2 and then 3 versus 4 if necessary. If it is in states $r$ or $a$, compare coins 1 and 2 versus 3 and 4. If the scale doesn't balance, then the alternator is in the lighter pan; we can find it in two weighings. If the scale balances, compare 1 to 2, then 3 to 4. If the alternator is among the first four coins, it will reveal itself. If all the weighings balance, then coin 5 is the alternator. Therefore, $a(4) = r(4) = a(5) = r(5) = 3$ and $f(4) = f(5) = 2$.

We know that state $a$ provides us less information than the *f* and $r$ states. So, $a(N) \geq r(N)$ and $a(N) \geq f(N)$. But in these examples, $a(N) = r(N) = f(N) + 1$. Is this always true? To

maintain the suspense, let us hold back the answer to this question.

## A Lower Bound

The *Jacobsthal numbers* are defined as $J_k = \frac{1}{3}(2^k - (-1)^k)$, and they satisfy the recurrence relation $J_{k+1} = J_k + 2J_{k-1}$. The first several terms, starting with $k = 0$, are 0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341,... We can see from the sequence that each term is one more or one less than twice the previous term. Indeed, it is straightforward to prove that $J_k = 2J_{k-1} - (-1)^k$. This sequence is A001045 in the *On-Line Encyclopedia of Integer Sequences* (*https://oeis.org/A001045*).

The Jacobsthal numbers are the coolest numbers you have never heard about. For example, $J_k$ is the number of ways to tile a 3-by-$(k-1)$ rectangle with 1-by-1 and 2-by-2 square tiles. It is also the number of ways to tile a 2-by-$(k-1)$ rectangle with 1-by-2 dominoes and 2-by-2 squares. Moreover, the product of two successive Jacobsthal numbers is a triangular number. We leave it to the reader to prove these properties. But we digress. We do not need these cool properties for our future progress.

The Jacobsthal numbers are intimately related to our coin puzzle. We will use them to produce a better bound.

**Theorem.** If a strategy is guaranteed to find a state-$f$ alternator coin in $N$ coins using $w$ weighings, then $N \leq J_{w+2}$. If the alternator is in state $r$ or $a$, then $N \leq J_{w+1}$.

*Proof.* Suppose there is a strategy for finding the alternator. As before, this gives a unique string of length at most $w$ of letters E, L, and R to each coin. The key observation is that L and R cannot follow each other; they must be followed by an E. Indeed, after an L or R, the alternator coin will be in state $r$. So, regardless of whether it participates in the next weighing, the scale will balance. (Of course, we only ever put equal numbers of coins in both pans.)

It is possible to have a string of length less than $w$ point to a coin; that is, some coins might be found faster than others. But if a shorter string points to a coin, then all the strings that have this same beginning substring point to the same coin. Thus, to find the theoretical maximum, we should count all the strings of length $w$.

Let $s(n)$ denote the number of such strings of length $n$. We have one string (an empty one) of length zero and three strings of length one. The number of strings of length $k$ can be calculated as follows. If the string starts with E, then it can be followed by any such string of length $k-1$. If it starts with L or R, it must be followed by E and then any such string of length $k-2$. Therefore, $s(k) = s(k-1) + 2s(k-2)$. This is the same recurrence relation as for the Jacobsthal numbers. The initial terms are the Jacobsthal numbers shifted two places, so $s(k) = J_{k+2}$. What remains to note is that if the alternator is in state $r$, the first letter of the string must be E.

Thus, the lower bound for the number of weighings increases after each Jacobsthal number. In the next section, we will see that the bound is precise.

## An Optimal Strategy

Next, we give a strategy that can find the alternator in the number of weighings provided by the bound above.

**Theorem.** Suppose we have $N$ coins with one alternator coin. If we know the alternator is in state $f$ and $N \leq J_{w+2}$ or if the alternator is in state $r$ or $a$ and $N \leq J_{w+1}$, then we can find it in $w$ weighings.

*Proof.* This is a proof by strong induction on $k$, the index of the Jacobsthal number. We have already proved that when the alternator is in state $f$ and $N \leq 3 = J_3$ or it is in state $a$ or $r$ and $N \leq 1 = J_2$, we can find the coin in one weighing. And when it is in state $a$ or $r$ and $N \leq 3 = J_3$ we can find the coin in two weighings. These are the base cases.

Now, suppose that the result holds for every index up to some $k \geq 3$. That is, for every $i \leq k$, whenever $N \leq J_i$, an $f$-state alternator can be found in $i-2$ weighings and an $r$- or $a$-state alternator can be found in $i-1$ weighings.

Suppose $N \leq J_{k+1}$. We must show that we can find an $f$-state alternator in $k-1$ weighings and an $r$- or $a$-state alternator in $k$ weighings.

First suppose we know the alternator is in state $f$. If $N \leq J_k$, then by the induction hypothesis, we can find the alternator in $k-2$ weighings. So, assume that

$J_k < N \leq J_{k+1}$. Our strategy is to weigh two piles each containing $J_{k-1}$ coins and set aside the remaining $m$ coins. If the scale doesn't balance, then the alternator is in one of the piles on the scale, and it switches to state $r$. By our induction hypothesis, we can find this coin in $k-2$ weighings. If the scale balances, then the alternator is in the leftover pile and has state $f$. By the recurrence relation for the Jacobsthal numbers, $m \leq J_k$, and so, by our induction hypothesis, we can find the coin in $k-2$ weighings. Because we used one weighing at the beginning, the total number of weighings is at most $k-1$.

Next, suppose we know the alternator is in state $r$. If $N$ is even, weigh all the coins, switching the alternator to state $f$. We just showed how to find the alternator among these coins in $k-1$ more weighings. Therefore, we can find the alternator in $k$ weighings total.

If $N$ is odd, set aside one coin. Now we have an even number of coins that may or may not have an alternator in state $r$. If it does, our procedure will find it in $k$ weighings. If it does not, then every weighing will balance, and we will know that the remaining coin is the alternator.

Lastly, suppose we don't know the state of the alternator; that is, it is in state $a$. Divide the coins into two equal piles, setting aside one coin if needed. Because $N \leq J_{k+1} = 2J_k - (-1)^{k+1}$, these piles have at most $J_k$ coins. If the scale doesn't balance, then the alternator switches to state $r$ and, hence, can be found in $k-2$ weighings, for a total of $k-1$ weighings. If the scale balances and the alternator is on the scale, it is now in state $f$, and we can find it in $k-1$ weighings. If it is not on the scale, then after $k-1$ even balances, we conclude that the extra coin is the alternator.

Thus, bringing these two theorems together, we have the following lovely theorem about alternator coins.

**Theorem.** If $J_k < N \leq J_{k+1}$, then $f(N) = k-1$ and $a(N) = r(N) = k$. ■

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**Cornelia Van Cott**

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## Solutions to puzzles on page 2

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|---|---|---|---|---|---|---|---|---|
| 3 | 5 | 9 | 2 | 7 | 1 | 6 | 8 | 4 |
| 4 | 6 | 8 | 5 | 3 | 9 | 1 | 7 | 2 |
| 2 | 1 | 7 | 4 | 8 | 6 | 3 | 9 | 5 |
| 5 | 9 | 1 | 3 | 2 | 8 | 4 | 6 | 7 |
| 7 | 2 | 3 | 9 | 6 | 4 | 5 | 1 | 8 |
| 6 | 8 | 4 | 7 | 1 | 5 | 9 | 2 | 3 |
| 9 | 7 | 2 | 1 | 4 | 3 | 8 | 5 | 6 |
| 8 | 3 | 5 | 6 | 9 | 7 | 2 | 4 | 1 |
| 1 | 4 | 6 | 8 | 5 | 2 | 7 | 3 | 9 |

**Quotientdoku solution.**

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|---|---|---|---|---|---|---|---|---|
| 9 | 6 | 2 | 5 | 4 | 3 | 7 | 8 | 1 |
| 7 | 5 | 4 | 8 | 6 | 1 | 9 | 2 | 3 |
| 1 | 3 | 8 | 9 | 7 | 2 | 5 | 4 | 6 |
| 4 | 9 | 5 | 2 | 3 | 7 | 1 | 6 | 8 |
| 3 | 2 | 6 | 1 | 9 | 8 | 4 | 7 | 5 |
| 8 | 7 | 1 | 6 | 5 | 4 | 2 | 3 | 9 |
| 2 | 8 | 7 | 3 | 1 | 9 | 6 | 5 | 4 |
| 5 | 1 | 3 | 4 | 2 | 6 | 8 | 9 | 7 |
| 6 | 4 | 9 | 7 | 8 | 5 | 3 | 1 | 2 |

**Remainderdoku solution.**